Research Article

Common Fixed Point Theorems for $\alpha$-$\psi$-Contractive Type Mappings

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Recently, Samet et al. (2012) introduced the notion of $\alpha$-$\psi$-contractive type mappings. They established some fixed point theorems for these mappings in complete metric spaces. In this paper, we introduce the notion of a coupled $\alpha$-$\psi$-contractive mapping and give a common fixed point result about the mapping. Also, we give a result of common fixed points of some coupled self-maps on complete metric spaces satisfying a contractive condition.

1. Introduction

We know fixed point theory has many applications and was extended by several authors from different views (see, e.g., [1–3]). Recently, Samet et al. introduced the notion of $\alpha$-$\psi$-contractive type mappings [3]. Denote with $\Psi$ the family of nondecreasing functions $\psi : ]0, \infty[ \rightarrow [0, \infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where $\psi^n$ is the $n$th iterate of $\psi$. It is known that $\psi(t) < t$ for all $t > 0$ and $\psi \in \Psi$ [3]. Let $(X, d)$ be a metric space, $T$ a self map on $X$, $\psi \in \Psi$ and $\alpha : X \times X \rightarrow [0, \infty)$. Then, $T$ is called a $\alpha$-$\psi$-contractive mapping whenever $\alpha(x,y)d(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$.

Also, we say that $T$ is $\alpha$-admissible whenever $\alpha(x,y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$ for all $x, y \in X$ [3]. Also, we say that $X$ has the property $(B_\alpha)$ if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 1$ and $x_n \rightarrow x$, then $\alpha(x_n, x) \geq 1$ for all $n \geq 1$. Let $(X, d)$ be a complete metric space and let $T$ a $\alpha$-admissible $\alpha$-$\psi$-contractive mapping on $X$. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. If $T$ is continuous or $X$ has the property $(B_\alpha)$, then $T$ has a fixed point (see [3]; Theorems 2.1 and 2.2). Finally, we say that $X$ has the property $(H_\alpha)$ whenever for each $x, y \in X$ there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$. If $X$ has the property $(H_\alpha)$ in the Theorems 2.1 and 2.2, then $X$ has a unique fixed point ([3]; Theorem 2.3). It is considerable that the results of Samet et al. generalize similar ordered results in the literature (see the results of the third section [3]). The aim of this paper is introducing the notion of generalized coupled $\alpha$-$\psi$-contractive mappings and give a common fixed point result about the mappings.

Definition 1. Let $f$ the family of functions $f : [0, \infty) \rightarrow \mathbb{R}$ satisfy:

(i) $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, +\infty)$;

(ii) $f$ is continuous;

(iii) $f$ is nondecreasing on $[0, +\infty)$;

(iv) $f(t_1 + t_2) \leq f(t_1) + f(t_2)$ for all $t_1, t_2 \in (0, +\infty)$.

Definition 2. Let $\Psi$ the family of functions $\psi : [0, +\infty) \rightarrow [0, +\infty)$ satisfy

$(\psi_1)$ $\psi$ is nondecreasing;

$(\psi_2)$ $\sum_{n=1}^{+\infty} \psi^n(t) < \infty$, for all $t \in (0, +\infty)$.

These functions are known in the literature as $(c)$-comparison functions. It is easily proved that if $\psi$ is a $(c)$-comparison function, then $\psi(t) < t$ for all $t > 0$.

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Definition 3. Let \((X, d)\) be a metric space, and let \(T, S : X \to X\) with given coupled mappings. Let \(\alpha : X \times X \to [0, +\infty)\), \(f \in f, \psi \in \Psi\), and let
\[
m(Ax, By) = \max \left\{ d(x, y), d(x, Ax), d(y, By), \right\},
\]
for all coupled mappings \(A, B : X \to X\) and \(x, y \in X\). One says that \(T, S\) are generalized coupled \(\alpha, \psi\)-contractive mappings whenever
\[
\begin{align*}
\alpha(x, y) f(d(Tx, Sy)) &\leq \psi(f(m(Tx, Sy))), \\
\alpha(x, y) f(d(Sx, Ty)) &\leq \psi(f(m(Sx, Ty))),
\end{align*}
\]
for all \(x, y \in X\).

Definition 4. Let \(T, S : X \to X\), and let \(\alpha : X \times X \to [0, +\infty)\). One says that \(T, S\) are coupled \(\alpha\)-admissible if
\[
\alpha(x, y) \geq 1 \implies \alpha(Tx, Sy) \geq 1 \quad \text{or} \quad \alpha(Sx, Ty) \geq 1
\]
for all \(x, y \in X\).

Definition 5. Let \((X, d)\) be a complete metric space. For two subsets \(A, B\) of \(X\), one marks \(A \preceq B\), if for all \(a \in A\), there exists \(b \in B\) such that \(a \preceq b\).

Definition 6. A partial metric on a nonempty set \(X\) is a function \(\rho : X \times X \to \mathbb{R}^+\) such that for all \(x, y, z \in X\):
\[
\begin{align*}
(\rho_1) &\quad x = y \implies \rho(x, x) = \rho(x, y) = \rho(y, y); \\
(\rho_2) &\quad \rho(x, x) \leq \rho(x, y); \\
(\rho_3) &\quad \rho(x, y) = \rho(y, x); \\
(\rho_4) &\quad \rho(x, y) \leq \rho(x, z) + \rho(z, y) - \rho(z, z).
\end{align*}
\]

A partial metric space is a pair \((X, \rho)\) such that \(X\) is a nonempty set, and \(\rho\) is a partial metric on \(X\). It is clear that if \(\rho(x, y) = 0\), then from \((\rho_1)\) and \((\rho_2)\), \(x = y\). But if \(x = y\), \(\rho(x, y)\) may not be 0. A basic example of a partial metric is the pair \((\mathbb{R}^+, \rho)\), where \(\rho(x, y) = \max\{x, y\}\) for all \(x, y \in \mathbb{R}^+\). If \(\rho\) is a partial metric on \(X\), then the function \(\rho^* : X \times X \to \mathbb{R}^+\) given by \(\rho^*(x, y) = 2\rho(x, y) - \rho(x, x) - \rho(y, y)\) is a metric on \(X\).

Example 7. Let \(X = \mathbb{R}\) endowed with the standard metric \(d(x, y) = |x - y|\) for all \(x, y \in \mathbb{R}\). Define the coupled mappings \(T, S : X \to X\) by
\[
\begin{align*}
Tx &= \begin{cases} 
2x^2, & \text{if } 0 \leq x \leq 1, \\
x + 1, & \text{if } x > 1, \\
0, & \text{if } x < 0,
\end{cases} \\
Sx &= \begin{cases} 
x - 1, & \text{if } x > 1, \\
x^3, & \text{if } 0 \leq x \leq 1, \\
0, & \text{if } x < 0.
\end{cases}
\]

We define the mapping \(\alpha : X \times X \to [0, +\infty)\) by
\[
\alpha(x, y) = \begin{cases} 
1, & \text{if } x, y \in [0, 1], \\
0, & \text{otherwise}.
\end{cases}
\]

Similarly, \(\alpha(x, y) \geq 1 \implies \alpha(Sx, Ty) \geq 1\). This shows that \(T, S\) are coupled \(\alpha\)-admissible.

Lemma 8. Let \((X, d)\) be a metric space. Suppose that \(T, S : X \to X\) are generalized coupled \(\alpha, \psi\)-contractive mappings. Then, \(\mathcal{F}(T) \subseteq \mathcal{F}(S)\).

Proof. We first show that any fixed point of \(T\) is also a fixed point of \(S\) and conversely. Define \(\rho(x, y) = 1\) for all \(x, y \in X\). Since \(\mathcal{F}(T) \neq \mathcal{F}(S)\), we may assume there exists \(x^* \in X\) such that \(x^* \in \mathcal{F}(T)\), but \(x^* \notin \mathcal{F}(S)\). Since \(d(x^*, Sx^*) > 0\), we have
\[
m(Tx^*, Sx^*) = \max \left\{ d(x^*, x^*), d(x^*, Tx^*), d(x^*, Sx^*) \right\},
\]
for all \(x, y \in X\).

\[
\begin{align*}
\alpha(x, y) &\leq \alpha(Sx, Ty), \\
\alpha(x, y) f(d(Tx, Sy)) &\leq \psi(f(m(Tx, Sy))), \\
\alpha(x, y) f(d(Sx, Ty)) &\leq \psi(f(m(Sx, Ty))).
\end{align*}
\]

This contradiction establishes that \(\mathcal{F}(T) \subseteq \mathcal{F}(S)\). A similar argument establishes the reverse containment, and therefore \(\mathcal{F}(T) = \mathcal{F}(S)\). \(\square\)

2. Main Results

Now, we are ready to state and prove our main results.

Theorem 9. Let \((X, d)\) be a complete metric space. Suppose that \(T, S : X \to X\) are generalized coupled \(\alpha, \psi\)-contractive mappings and satisfy the following conditions:
\[
\begin{align*}
(i) &\quad T, S \text{ are coupled } \alpha\text{-admissible}; \\
(ii) &\quad \text{there exists } x_0 \in X \text{ such that } \alpha(x_0, Tx_0) \geq 1; \text{ or} \\
(iii) &\quad \text{there exists } x_0 \in X \text{ such that } \alpha(x_0, Tx_0) \geq 1; \text{ or} \\
(iv) &\quad T \text{ or } S \text{ is continuous.}
\end{align*}
\]
Then $T, S$ have common fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ converges to the fixed point of $T, S$.

**Proof.** By Lemma 8, we have $\mathcal{F}(T) = \mathcal{F}(S)$. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Define the sequence $\{x_n\}$ in $X$ by $x_{2n+1} = Tx_{2n}$ and $x_{2n+2} = Sx_{2n+1}$ for all $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \geq 1$, then $x^* = x_n$, are a common fixed point for $T, S$. So, we can assume that $x_{2n} \neq Tx_{2n}$ and $x_{2n+1} \neq Sx_{2n+1}$ for all $n \in \mathbb{N}_0$. Since $T, S$ are coupled $\alpha$-admissible, we have

\[
\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq 1 \implies \alpha(Tx_0, Sx_1) = \alpha(x_1, x_2) \geq 1 \implies \alpha(Sx_1, Tx_2) = \alpha(x_2, x_3) \geq 1.
\]

Inductively, we have

\[
\alpha(x_{2n}, x_{2n+1}) \geq 1,
\alpha(x_{2n+1}, x_{2n+2}) \geq 1,
\]

for all $n \in \mathbb{N}_0$. We obtain

\[
f\left(d(x_{2n+1}, x_{2n+2})\right) = f\left(d(Tx_{2n}, Sx_{2n+1})\right) \\
\leq \alpha(x_{2n}, x_{2n+1}) f\left(d(Tx_{2n}, Sx_{2n+1})\right) \\
\leq \psi(f(m(Tx_{2n}, Sx_{2n+1}))).
\]

(8)

Now,

\[
m(Tx_{2n}, Sx_{2n+1}) \\
= \max \left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1}), \frac{1}{2} \left[d(x_{2n}, Sx_{2n+1}) + d(x_{2n+1}, Tx_{2n})\right]\right\} \\
= \max \left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{1}{2} \left[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+1})\right]\right\} \\
\leq \max \left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{1}{2} \left[d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + 0\right]\right\} \\
= \max \left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\right\}
\]

(9)

we have

\[
f\left(d(x_n, x_{n+1})\right) \leq \psi(f(m(Tx_{2n}, Sx_{2n+1}))).
\]

(10)

which is a contradiction since $d(x_{2n+1}, x_{2n+2}) > 0$. Thus,

\[
f\left(d(x_{2n+1}, x_{2n+2})\right) \leq \psi(f(d(x_{2n}, x_{2n+1}))).
\]

(14)

Similarly, if

\[
f\left(d(x_{2n}, x_{2n+1})\right) \leq \psi(f(d(x_{2n-1}, x_{2n}))),
\]

(15)

we have

\[
f\left(d(x_n, x_{n+1})\right) \leq \psi(f(d(x_n-1, x_n))),
\]

(16)

for all $n \in \mathbb{N}_0$. By induction, we get

\[
f\left(d(x_n, x_{n+1})\right) \leq \psi(f(d(x_{n-2}, x_{n-1}))) \\
\leq \cdots \leq \psi^n(f(d(x_0, x_1)))
\]

(17)

for all $n \in \mathbb{N}_0$. Fix $\varepsilon > 0$, and let $n(\varepsilon) \in \mathbb{N}_0$ such that

\[
\sum_{n=n(\varepsilon)}^{\infty} \psi^n(f(d(x_0, x_1))) < \varepsilon.
\]

(18)

Let $n, m \in \mathbb{N}_0$ with $m > n > n(\varepsilon)$. Using the triangle inequality, we obtain

\[
f\left(d(x_n, x_m)\right) \leq \sum_{k=n}^{m-1} f\left(d(x_k, x_{k+1})\right) \\
\leq \sum_{k=n}^{m-1} \psi^k(f(d(x_0, x_1))) \\
\leq \sum_{n=n(\varepsilon)}^{\infty} \psi^n(f(d(x_0, x_1))) < \varepsilon.
\]

Thus we proved that $\{x_n\}$ is a Cauchy sequence in the metric space $(X, d)$.

Since $(X, d)$ is a complete metric space, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to +\infty$. From the continuity of $T$, it follows that $x_{2n+1} = Tx_{2n} \to Tx^*$ as $n \to +\infty$, then $x^* = Tx^*$. Similarly if $S$ is continuous, we have $x^* = Sx^*$.

**Corollary 10.** Let $(X, d)$ be a complete metric space. Suppose that $T : X \to X$ is a generalized $\alpha$-$\psi$-contractive mapping and satisfies the following conditions:

(i) $T$ is $\alpha$-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$; or

(iii) $T$ is continuous.

Then, $T$ has a fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{n+1} = Tx_n$ converges to the fixed point of $T$. 


Example 11. Let \( X = \mathbb{R} \) endowed with the standard metric \( d(x, y) = |x - y| \) for all \( x, y \in \mathbb{R} \). Define the coupled mappings \( T, S : X \to X \) by

\[
T(x) = \begin{cases} 
2x - \frac{3}{2}, & \text{if } x > 1, \\
\frac{x}{2}, & \text{if } 0 \leq x \leq 1, \\
0, & \text{if } x < 0,
\end{cases}
\]

\[
S(x) = \begin{cases} 
2x - \frac{5}{3}, & \text{if } x > 1, \\
\frac{x}{2}, & \text{if } 0 \leq x \leq 1, \\
0, & \text{if } x < 0.
\end{cases}
\]

We define the mapping \( \alpha : X \times X \to [0, +\infty) \) by

\[
\alpha(x, y) = \begin{cases} 
1, & \text{if } x, y \in [0, 1], \\
0, & \text{otherwise}.
\end{cases}
\]

If \( f(t) = t \) and \( \psi(t) = (1/2)t \) for all \( t \geq 0 \), we have

\[
\alpha(x, y) f(d(Tx, Sy)) = f(d(Tx, Sy)), \quad \text{if } x, y \in [0, 1],
\]

\[
\alpha(x, y) f(d(Tx, Sy)) \leq f(d(Tx, Sy))
\]

\[
= \left| \frac{x - y}{2} \right| \leq \frac{1}{2} |x - y| \leq \psi(f(m(Tx, Sy))),
\]

for all \( x, y \in X \). Thus, \( T, S \) are generalized coupled \( \alpha \)-\( \psi \)-contractive mappings. Moreover, there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \). In fact, for \( x_0 = 0 \), we have \( \alpha(1, T1) = 1 \). Obviously, \( T \) is continuous, and so it remains to show that \( T, S \) are coupled \( \alpha \)-admissible. To do so, let \( x, y \in X \) such that \( \alpha(x, y) \geq 1 \). This implies that \( x, y \in [0, 1] \) by the definition of \( \alpha \). We then have \( Tx \in [0, 1], Sy \in [0, 1] \) and \( \alpha(Tx, Sy) = 1 \). Then, \( T, S \) are coupled \( \alpha \)-admissible. Now, all the hypotheses of Theorem 9 are satisfied.

Consequently, \( T, S \) have common fixed points. In this example, 0 is at least one common fixed point of \( T \) and \( S \).

Now, we omit the continuity hypothesis of \( T \) and \( S \).

Theorem 12. Let \((X, d)\) be a complete metric space. Suppose that \( T, S : X \to X \) are generalized coupled \( \alpha \)-\( \psi \)-contractive mappings and satisfy the following conditions:

(i) \( T, S \) are coupled \( \alpha \)-admissible;

(ii) there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \);

(iii) if \( \{x_n\} \) is a sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \) and \( x_n \to x \in X \) as \( n \to \infty \), then there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n(k)}, x) \geq 1 \) for all \( k \); or

(iii)* if \( \{x_n\} \) is a sequence in \( X \) such that \( \alpha(x_{n+1}, x_n) \geq 1 \) for all \( n \) and \( x_n \to x \in X \) as \( n \to \infty \), then there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( \alpha(x_n, x_{n(k)}) \geq 1 \) for all \( k \).

Then, \( T, S \) have common fixed point \( x^* \in X \). Further, for each \( x_0 \in X \), the iterated sequence \( \{x_n\} \) with \( x_{2n+1} = Tx_{2n} \) and \( x_{2n+2} = Sx_{2n+1} \) converges to the common fixed point of \( T, S \).

Proof. Following the proof of Theorem 9, we know that \( \{x_n\} \) is a Cauchy sequence in the complete metric space \((X, d)\). Then, there exists \( x^* \in X \) such that \( x_n \to x^* \) as \( n \to +\infty \). From Theorem 9 and condition (iii), there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n(k)}, x^*) \geq 1 \) for all \( k \).

Applying Theorem 9, for all \( k \), we get that

\[
f(d(x_{2n(k)+1}, Sx^*)) = f(d(Tx_{2n(k)}, Sx^*)) \leq \alpha(x_{2n(k)}, x^*) d(Tx_{2n(k)}, Sx^*) \]

\[
\leq \psi(f(m(Tx_{2n(k)}, Sx^*)�).
\]

On the other hand, we have

\[
m(Tx_{2n(k)}, Sx^*) = \max \left\{ d(x_{2n(k)}, x^*), d(Tx_{2n(k)}, Tx_{2n(k)}), d(x^*, Sx^*) \right\}.
\]

\[
\geq \frac{1}{2} [d(x_{2n(k)}, Sx^*) + d(x^*, Tx_{2n(k)})].
\]

Letting \( k \to +\infty \), in the above equality, we get that

\[
\lim_{k \to +\infty} m(Tx_{2n(k)}, Sx^*) = d(x^*, Sx^*). \tag{25}
\]

Suppose that \( d(x^*, Sx^*) > 0 \). From (25), for \( k \) large enough, we have \( m(Tx_{2n(k)}, Sx^*) > 0 \), which implies that

\[
\psi(f(m(Tx_{2n(k)}, Sx^*))) < f(m(Tx_{2n(k)}, Sx^*)�).
\]

Thus, from (23), we have \( f(d(x_{2n(k)+1}, Sx^*)) < m(Tx_{2n(k)}, Sx^*) \). Letting \( k \to \infty \) in the above inequality, and using (25), we obtain that \( f(d(x^*, Sx^*)) < f(d(x^*, Sx^*)) \) which is a contradiction. Thus, we have \( d(x^*, Sx^*) = 0 \); that is, \( x^* = Sx^* \). Similarly, it can be shown that \( x^* = Tx^* \). \( \square \)

Corollary 13. Let \((X, d)\) be a complete metric space. Suppose that \( T : X \to X \) is a generalized \( \alpha \)-\( \psi \)-contractive mapping and satisfies the following conditions:

(i) \( T \) is \( \alpha \)-admissible;

(ii) there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \); or

(iii)* there exists \( x_0 \in X \) such that \( \alpha(x_0, Tx_0) \geq 1 \);

(iii) if \( \{x_n\} \) is a sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \) and \( x_n \to x \in X \) as \( n \to \infty \), then there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n(k)}, x) \geq 1 \) for all \( k \); or

(iii)* if \( \{x_n\} \) is a sequence in \( X \) such that \( \alpha(x_{n+1}, x_n) \geq 1 \) for all \( n \) and \( x_n \to x \in X \) as \( n \to \infty \), then there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( \alpha(x, x_{n(k)}) \geq 1 \) for all \( k \).
Then, $T$ has a fixed point $x^* \in X$. Further, for each $x_0 \in X$, the iterated sequence $\{x_n\}$ with $x_{n+1} = Tx_n$ converges to the fixed point of $T$.

Example 14. Let $X = \mathbb{R}$ endowed with the standard metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Define the coupled mappings $T, S : X \rightarrow X$ by

$$ Tx = \begin{cases} \frac{x - \frac{3}{4}}{2}, & \text{if } x > 1, \\ \frac{x}{2}, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x < 0, \end{cases} $$

$$ Sx = \begin{cases} \frac{x - \frac{5}{6}}{2}, & \text{if } x > 1, \\ \frac{x}{2}, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x < 0. \end{cases} $$

We define the mapping $\alpha : X \times X \rightarrow [0, +\infty)$ by

$$ \alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases} $$

If $f(t) = t$ and $\psi(t) = (1/2)t$ for all $t \geq 0$, we have

$$ \alpha(x, y) = f(d(Tx, Sy)) = \begin{cases} f(d(Tx, Sy)), & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise,} \end{cases} $$

$$ = \frac{\frac{x - y}{2}}{2} = \frac{1}{2} |x - y| \leq \psi(f(m(Tx, Sy))), $$

for all $x, y \in X$. Thus, $T, S$ are generalized coupled $\alpha$-contraction mappings. Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, x_0) \geq \psi(f(m(Tx, Ty)))$ for all $x, y \in X$, $x \neq y$. Define $x_n \in X$ such that $x_{n+1} = Tx_n$ for all $n \geq 1$. By the definition of $\alpha$ we have, $x_{n+1} - x_n \in [0, 1]$ for all $n$, so $x \in [0, 1]$ and $\alpha(x_n, x) \geq 1$. It remains to show that $T, S$ are coupled $\alpha$-admissible. In doing so, let $x, y \in X$ such that $\alpha(x, y) \geq 1$. This implies that $x, y \in [0, 1]$ by the definition of $\alpha$. We have $Tx \in [0, 1], Sy \in [0, 1]$, and $\alpha(Tx, Ty) = 1$. Then $T, S$ are coupled $\alpha$-admissible. Now, all the hypotheses of Theorem 12 are satisfied. Consequently, $T, S$ have common fixed points. In this example, $0$ is at least one common fixed point of $T$ and $S$.

3. Fixed Point Theorems on Ordered Metric Space

Theorem 15. Let $(X, \leq, d)$ be a complete ordered metric space, $\psi \in \Psi$, $f \in \mathcal{F}$, and $T, S$ be coupled mappings on $X$ such that $f(d(Tx, Sy)) \leq \psi(f(m(Tx, Sy)))$ and $d(Sx, Ty) \leq \psi(f(m(Sx, Ty)))$ for all $x, y \in X$ with $x \leq y$. Suppose that there exists $x_0 \in X$ such that $x_0 \leq Tx_0$ or $(Tx_0 \leq x_0)$, and if $\{x_n\}$ is a sequence in $X$ such that $x_n \leq x_{n+1}$ or $(x_{n+1} \leq x_n)$ for all $n \geq 1$ and $x_n \rightarrow x$, then $x_n \leq x$ or $(x \leq x_n)$ for all $n \geq 1$. If $x \leq y$ implies $Tx \leq Sy$ or $Sx \leq Ty$ or $(Ty \leq Sy$ or $Sy \leq Tx)$, then $T, S$ have common fixed points.

Proof. Define $\alpha : X \times X \rightarrow [0, +\infty)$ by $\alpha(x, y) = 1$ whenever $x \leq y$, and define $\alpha(x, y) = 0$ whenever $x \neq y$. It is easy to check that $T, S$ are a coupled $\alpha$-admissible and generalized coupled $\alpha$-contraction mappings on $X$. Now, by using Theorem 9, $T, S$ have common fixed points.

Corollary 16. Let $(X, \leq, d)$ be a complete ordered metric space, $\psi \in \Psi$, $f \in \mathcal{F}$, and $T$ a mapping on $X$ such that $f(d(Tx, Ty)) \leq \psi(f(m(Tx, Ty)))$ for all $x, y \in X$ with $x \leq y$. Suppose that there exists $x_0 \in X$ such that $x_0 \leq Tx_0$ or $(Tx_0 \leq x_0)$ and if $\{x_n\}$ is a sequence in $X$ such that $x_n \leq x_{n+1} \leq x_n$ or $(x_{n+1} \leq x_n)$ for all $n \geq 1$ and $x_n \rightarrow x$, then $x_n \leq x$ or $(x \leq x_n)$ for all $n \geq 1$. If $x \leq y$ implies $Tx \leq Ty$ or $(Ty \leq Tx)$, then $T$ has a fixed point.

4. Fixed Point Theorems on Metric Spaces Endowed with Partial Metric

If we substitute a partial metric $\rho$ instead the metric $d$ in Theorem 9, it is easy to check that a similar result holds for the partial metric space case as follows. We define

$$ m^*(Ax, By) = \max \left\{ \rho(x, y), \rho(x, Ax) \right\}, $$

$$ = \frac{1}{2} \left[ \rho(x, By) + \rho(y, Ax) \right] $$

for all coupled mappings $A, B : X \rightarrow X$ and $x, y \in X$.

Theorem 17. Let $(X, \rho)$ be a complete partial metric space, $\alpha : X \times X \rightarrow [0, +\infty)$ a function, $\psi \in \Psi$, $f \in \mathcal{F}$, and $T, S$ self maps on $X$ such that

$$ \alpha(x, y) f(d(Tx, Sy)) \leq \psi(f(m^*(Tx, Sy))), $$

$$ \alpha(x, y) f(d(Sx, Ty)) \leq \psi(f(m^*(Sx, Ty))) $$

for all $x, y \in X$. Suppose that $T, S$ are coupled $\alpha$-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, x_0) \geq 1$ or $(\alpha(Tx_0, x_0) \geq 1)$. Assume that if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ or $(\alpha(x_{n+1}, x_n) \geq 1)$ for all $n$ and $x_n \rightarrow x$ as $n \rightarrow +\infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ or $(\alpha(x, x_{n(k)}) \geq 1)$ for all $k$. Then $T, S$ have common fixed points.

Corollary 18. Let $(X, \rho)$ be a complete partial metric space, $\alpha : X \times X \rightarrow [0, +\infty)$ a function, $\psi \in \Psi$, $f \in \mathcal{F}$, and $T$ a self map on $X$ such that

$$ \alpha(x, y) f(d(Tx, Ty)) \leq \psi(f(m^*(Tx, Ty))) $$

for all $x, y \in X$. Suppose that $T$ is $\alpha$-admissible and there exists $x_0 \in X$ such that $\alpha(x_0, x_0) \geq 1$ or $(\alpha(Tx_0, x_0) \geq 1)$. Assume that if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ or $(\alpha(x_{n+1}, x_n) \geq 1)$ for all $n$ and $x_n \rightarrow x$ as $n \rightarrow +\infty$, then
there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( \alpha(x_n(k), x) \geq 1 \) or \( \alpha(x, x_{n(k)}) \geq 1 \) for all \( k \). Then \( T \) has a fixed point.

Example 19. Let \( X = [0, +\infty) \) endowed with the partial metric \( \rho(x, y) = \max\{x, y\} \), for all \( x, y \in X \). Define the mapping \( T : X \to X \), by

\[
T x = \begin{cases} 
  \frac{2x - 3}{2}, & \text{if } x > 1, \\
  x^2, & \text{if } 0 \leq x \leq 1, \\
  0, & \text{if } x < 0.
\end{cases} \tag{33}
\]

We define the mapping \( \alpha : X \times X \to [0, +\infty) \) by

\[
\alpha(x, y) = \begin{cases} 
  1, & \text{if } x, y \in [0, 1], x \leq y, \\
  0, & \text{otherwise},
\end{cases}
\]

\[
a(x, y) \geq 1 \iff x, y \in [0, 1], \\
x \leq y \iff Tx, Ty \in [0, 1], \\
Tx \leq Ty \iff \alpha(Tx, Ty) \geq 1
\]

(34)

since \( T \) is \( \alpha \)-admissible. If \( \psi(t) = t^2/(1 + t) \), we have

\[
\alpha(x, y) \rho(Tx, Ty) \leq \rho \left( \frac{x^2 - y^2}{1 + x - y} \right)
\]

\[
= \max \left\{ \frac{x^2 - y^2}{1 + x - y}, \frac{y^2 - x^2}{1 + y - x} \right\}
\]

\[
= \frac{y^2}{1 + y} \leq y
\]

\[
= \rho(x, y).
\]

(35)

Suppose that \( \{x_n\} \) is a sequence in \( X \) such that \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \) and \( x_n \to x \) as \( n \to +\infty \); by definition of \( \alpha \), we have \( x_n, x_{n+1}, x \in [0, 1] \), on the other hand \( x_n \leq x_{n+1} \Rightarrow x_n \leq x \); then there exists a subsequence \( \{x_{n(k)}\} \) of \( \{x_n\} \) such that \( \alpha(x_{n(k)}, x) \geq 1 \) for all \( k \). There exists \( x_0 = 1 \) such that \( \alpha(x_0, Tx_0) = 1 \). This show that all conditions of Corollary 18 are satisfied, and so \( T \) has a fixed point in \( X \).

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