Research Article

On a Subclass of Meromorphic Functions Defined by Hilbert Space Operator

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In this paper, we define a new operator on the class of meromorphic functions and define a subclass using Hilbert space operator. Coefficient estimate, distortion bounds, extreme points, radii of starlikeness, and convexity are obtained.

1. Introduction

Let Σ denote the class of meromorphic functions

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (1)$$

defined on $U = \{z \in \mathbb{C} : 0 < |z| < 1\}$. For $f \in \Sigma$ given by (1) and $g \in \Sigma$ given by

$$g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n, \quad (2)$$

the Hadamard product (or convolution) [1] of f and g is defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n, \quad z \in U. \quad (3)$$

Many subclasses of meromorphic functions have been defined and studied in the past. In particular, the subclasses $M_{l,m}^{\alpha_1}(\lambda, \alpha)$, $0 \leq \alpha < 1$ and $0 \leq \lambda < 1$ [2], and $W(\alpha, \beta)$, $0 \leq \alpha < 1$ and $0 < \beta \leq 1$ [3], are considered by researchers. Let H be a complex Hilbert space and $L(H)$ denote the algebra of all bounded linear operators on H . For a complex-valued function f analytic in a domain E of the complex plane containing the spectrum $\sigma(T)$ of the bounded linear operator T ,

let $f(T)$ denote the operator on H defined by the Riesz-Dunford integral [4]

$$f(T) = \frac{1}{2\pi i} \int_C (zI - T)^{-1} f(z) dz, \quad (4)$$

where I is the identity operator on H and C is a positively oriented simple closed rectifiable closed contour containing the spectrum $\sigma(T)$ in the interior domain [5]. The operator $f(T)$ can also be defined by the following series:

$$f(T) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} T^n \quad (5)$$

which converges in the norm topology. In this paper, we introduce a subclass $M_p(\alpha, \lambda, T)$ of Σ defined using Hilbert space operator and prove a necessary and sufficient condition for the function to belong to this class, the distortion theorem, radius of starlikeness, and convexity. In [6], Atshan and Buti had defined an operator acting on analytic functions in terms of a definite integral. We modify their operator for meromorphic functions as follows.

Lemma 1. For $f \in \Sigma$ given by (1), $0 \leq \mu \leq 1$, and $0 \leq \theta \leq 1$, if the operator $S_{\mu}^{\theta} : \Sigma \rightarrow \Sigma$ is defined by

$$S_{\mu}^{\theta} f(z) = \frac{1}{(1-\mu)^{\theta+1} \Gamma(\theta+1)} \int_0^{\infty} t^{\theta+1} e^{-(t/(1-\mu))} f(zt) dt, \quad (6)$$

then

$$S_\mu^\theta f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} L(n, \theta, \mu) a_n z^n, \quad (7)$$

where $L(n, \theta, \mu) = (1 - \mu)^{n+1} \Gamma(n + \theta + 2) / \Gamma(\theta + 1)$.

Denote by Σ_P the class of all functions $f \in \Sigma$ with $a_n \geq 0$.

Definition 2. For $0 \leq \lambda < 1$, $0 \leq \alpha < 1$, a function $f \in \Sigma_P$ is in the class $M_P(\alpha, \lambda, T)$ if

$$\begin{aligned} & \left\| T(S_\mu^\theta f(T))' - \left\{ (\lambda - 1) S_\mu^\theta f(T) + \lambda T(S_\mu^\theta f(T))' \right\} \right\| \\ & < \left\| T(S_\mu^\theta f(T))' \right\| \\ & + (1 - 2\alpha) \left\{ (\lambda - 1) S_\mu^\theta f(T) + \lambda T(S_\mu^\theta f(T))' \right\} \end{aligned} \quad (8)$$

for all operators T with $\|T\| < 1$ and $T \neq 0$, 0 being the zero operator on H .

2. Coefficient Bounds

Theorem 3. A meromorphic function $f \in \Sigma_P$ given by (1) is in the class $M_P(\alpha, \lambda, T)$ if and only if

$$\sum_{n=1}^{\infty} [n + \alpha - \alpha\lambda(n + 1)] L(n, \theta, \mu) a_n \leq 1 - \alpha. \quad (9)$$

The result is sharp for $f(z) = 1/z + ((1 - \alpha)/[n + \alpha - \alpha\lambda(n + 1)]L(n, \theta, \mu))z^n$.

Proof. Let $f(T) = T^{-1} + \sum_{n=1}^{\infty} a_n T^n$. Assume that (9) holds. Then,

$$\begin{aligned} & \left\| T(S_\mu^\theta f(T))' - \left\{ (\lambda - 1) S_\mu^\theta f(T) + \lambda T(S_\mu^\theta f(T))' \right\} \right\| \\ & - \left\| T(S_\mu^\theta f(T))' \right\| \\ & + (1 - 2\alpha) \left\{ (\lambda - 1) S_\mu^\theta f(T) + \lambda T(S_\mu^\theta f(T))' \right\} \\ & = \left\| \sum_{n=1}^{\infty} [(n + 1) - \lambda(n + 1)] L(n, \mu, \theta) a_n T^n \right\| \\ & - \left\| 2T^{-1}(1 - \alpha) \right\| \\ & - \sum_{n=1}^{\infty} [n + (1 - 2\alpha)(\lambda - 1) + \lambda(1 - 2\alpha)n] \\ & \quad \times L(n, \mu, \theta) a_n T^n \end{aligned} \quad (10)$$

$$\leq \sum_{n=1}^{\infty} [n + \alpha - \alpha\lambda(n + 1)] L(n, \theta, \mu) a_n - (1 - \alpha)$$

≤ 0 , by (9),

and hence $f \in M_P(\alpha, \lambda, T)$.

Conversely, let

$$\begin{aligned} & \left\| T(S_\mu^\theta f(T))' - \left\{ (\lambda - 1) S_\mu^\theta f(T) + \lambda T(S_\mu^\theta f(T))' \right\} \right\| \\ & < \left\| T(S_\mu^\theta f(T))' \right\| \\ & + (1 - 2\alpha) \left\{ (\lambda - 1) S_\mu^\theta f(T) + \lambda T(S_\mu^\theta f(T))' \right\}. \end{aligned} \quad (11)$$

This implies that

$$\begin{aligned} & \left\| \sum_{n=1}^{\infty} [(n + 1)(1 - \lambda)] L(n, \theta, \mu) a_n T^{n+1} \right\| \\ & \leq \left\| 2(1 - \alpha) - \sum_{n=1}^{\infty} [n + (1 - 2\alpha)(\lambda - 1) + \lambda(1 - 2\alpha)n] \right. \\ & \quad \times L(n, \theta, \mu) a_n T^{n+1} \left. \right\|. \end{aligned} \quad (12)$$

Choose $T = eI$, $0 < e < 1$. Then, $\sum_{n=1}^{\infty} [(n + 1)(1 - \lambda)]L(n, \theta, \mu)a_n e^n / (2e(1 - \alpha) - \sum_{n=1}^{\infty} [n + (1 - 2\alpha)(\lambda - 1) + \lambda(1 - 2\alpha)n]L(n, \theta, \mu)a_n e^n) \leq 1$. As $e \rightarrow 1$, we obtain (9). $\square$

Corollary 4. If f of the form (1) is in $M_P(\alpha, \lambda, T)$, then

$$a_n \leq \frac{1 - \alpha}{[n + \alpha - \alpha\lambda(n + 1)] L(n, \theta, \mu)}. \quad (13)$$

The result is sharp for the function $f(z) = 1/z + ((1 - \alpha)/[n + \alpha - \alpha\lambda(n + 1)]L(n, \theta, \mu))z^n$.

Theorem 5. Let $f_0(z) = 1/z$ and $f_n(z) = 1/z + ((1 - \alpha)/[n + \alpha - \alpha\lambda(n + 1)]L(n, \theta, \mu))z^n$, $n \geq 1$, $0 \leq \alpha < 1$, $0 \leq \lambda < 1$. Then, f is in the class $M_P(\alpha, \lambda, T)$ if and only if it can be expressed in the form $f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z)$, where $\mu_n \geq 0$ ($n = 0, 1, 2, \dots$) and $\sum_{n=0}^{\infty} \mu_n = 1$.

Proof. Suppose that $f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z) = 1/z + \sum_{n=1}^{\infty} \mu_n ((1 - \alpha)/[n + \alpha - \alpha\lambda(n + 1)]L(n, \theta, \mu))z^n$; then, we have

$$\begin{aligned} & \sum_{n=1}^{\infty} [n + \alpha - \alpha\lambda(n + 1)] L(n, \theta, \mu) \\ & \quad \times \frac{\mu_n (1 - \alpha)}{[n + \alpha - \alpha\lambda(n + 1)] L(n, \theta, \mu)} \\ & = \sum_{n=1}^{\infty} \mu_n = 1 - \mu_0 < 1. \end{aligned} \quad (14)$$

By Theorem 3, $f \in M_P(\alpha, \lambda, T)$. Conversely, assume that f is in the class $M_P(\alpha, \lambda, T)$; then, by Corollary 4,

$$a_n \leq \frac{1 - \alpha}{[n + \alpha - \alpha\lambda(n + 1)] L(n, \theta, \mu)}. \quad (15)$$

Set $\mu_n = ([n + \alpha - \alpha\lambda(n + 1)]L(n, \theta, \mu)/(1 - \alpha))a_n$, $n = 1, 2, \dots$ and $\mu_0 = 1 - \sum_{n=1}^{\infty} \mu_n$. Then, $f(z) = \sum_{n=0}^{\infty} \mu_n f_n(z) \in M_P(\alpha, \lambda, T)$. $\square$

3. Distortion Bounds

In this section, we obtain growth and distortion bounds for the class $M_P(\alpha, \lambda, T)$.

Theorem 6. If $f \in M_P(\alpha, \lambda, T)$, then

$$\begin{aligned} \|f(T)\| &\geq \frac{1}{\|T\|} - \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)(1-\mu)^2(\theta+1)(\theta+2)} \|T\|, \\ \|f(T)\| &\leq \frac{1}{\|T\|} + \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)(1-\mu)^2(\theta+1)(\theta+2)} \|T\|. \end{aligned} \quad (16)$$

The result is sharp for $f(z) = 1/z + ((1-\alpha)/(1+\alpha-2\alpha\lambda)(1-\mu)^2(\theta+1)(\theta+2))z$.

Proof. By Theorem 3,

$$\begin{aligned} (1-\alpha-2\alpha\lambda)L(1, \theta, \mu) \sum_{n=2}^{\infty} a_n \\ \leq \sum_{n=1}^{\infty} [n+\alpha-\alpha\lambda(n+1)] L(n, \theta, \mu) a_n \leq 1-\alpha. \end{aligned} \quad (17)$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} a_n &\leq \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)L(1, \theta, \mu)} \\ &= \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)(1-\mu)^2(\theta+1)(\theta+2)}. \end{aligned} \quad (18)$$

Also, if $f(T) = T^{-1} + \sum_{n=1}^{\infty} a_n T^n$, then

$$\frac{1}{\|T\|} - \sum_{n=1}^{\infty} a_n \|T\|^n \leq \|f(T)\| \leq \frac{1}{\|T\|} + \sum_{n=1}^{\infty} a_n \|T\|^n. \quad (19)$$

Since $\|T\| < 1$, the above inequality becomes

$$\frac{1}{\|T\|} - \|T\| \sum_{n=1}^{\infty} a_n \leq \|f(T)\| \leq \frac{1}{\|T\|} + \|T\| \sum_{n=1}^{\infty} a_n. \quad (20)$$

Using (18), we get the result. $\square$

Theorem 7. If $f \in M_P(\alpha, \lambda, T)$, then

$$\begin{aligned} \|f'(T)\| &\geq \frac{1}{\|T\|^2} - \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)(1-\mu)^2(\theta+1)(\theta+2)}, \\ \|f'(T)\| &\leq \frac{1}{\|T\|^2} + \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)(1-\mu)^2(\theta+1)(\theta+2)}. \end{aligned} \quad (21)$$

The result is sharp for $f(z) = 1/z + ((1-\alpha)/(1+\alpha-2\alpha\lambda)(1-\mu)^2(\theta+1)(\theta+2))z$.

4. Radii Results

We now find the radius of meromorphically starlikeness and convexity for functions f in the class $M_P(\alpha, \lambda, T)$.

Theorem 8. Let $f \in M_P(\alpha, \lambda, T)$. Then, f is meromorphically starlike of order δ , ($0 \leq \delta < 1$) in $|z| < r_1$, where

$$r_1 = \inf_{n \geq 1} \left[\frac{(1-\delta)(n+\alpha-\alpha\lambda(n+1))L(n, \theta, \mu)}{(1-\alpha)(n+2-\delta)} \right]^{1/(n+1)}, \quad (22)$$

$n \geq 1$.

Proof. Let $f(T) = T^{-1} + \sum_{n=1}^{\infty} a_n T^n$. Since $f \in M_P(\alpha, \lambda, T)$ is meromorphically starlike of order δ ,

$$\left\| -\frac{Tf'(T)}{f(T)} - 1 \right\| \leq 1-\delta. \quad (23)$$

Substituting for f , the above inequality becomes

$$\sum_{n=1}^{\infty} \left(\frac{n+2-\delta}{1-\delta} \right) \|T\|^{n+1} a_n \leq 1. \quad (24)$$

By Theorem 3,

$$\sum_{n=1}^{\infty} \frac{[n+\alpha-\alpha\lambda(n+1)]L(n, \theta, \mu)}{1-\alpha} a_n \leq 1. \quad (25)$$

Thus, (24) will be true if

$$\left(\frac{n+2-\delta}{1-\delta} \right) \|T\|^{n+1} \leq \frac{[n+\alpha-\alpha\lambda(n+1)]L(n, \theta, \mu)}{(1-\alpha)}, \quad (26)$$

that is,

$$\|T\| \leq \left[\frac{(1-\delta)(n+\alpha-\alpha\lambda(n+1))L(n, \theta, \mu)}{(1-\alpha)(n+2-\delta)} \right]^{1/(n+1)}. \quad (27)$$

$\square$

Theorem 9. Let $f \in M_P(\alpha, \lambda, T)$. Then, f is meromorphically convex of order δ , ($0 \leq \delta < 1$) in $|z| < r_2$, where

$$r_2 = \inf_{n \geq 1} \left[\frac{(1-\delta)(n+\alpha-\alpha\lambda(n+1))L(n, \theta, \mu)}{(1-\alpha)n(n+2-\delta)} \right]^{1/(n+1)}. \quad (28)$$

Theorem 10. The class $M_P(\alpha, \lambda, T)$ is closed under convex combination.

Proof. Let $f(z) = T^{-1} + \sum_{n=0}^{\infty} a_n T^n$ and $g(z) = T^{-1} + \sum_{n=0}^{\infty} b_n T^n \in M_P(\alpha, \lambda, T)$. Then, by Theorem 3,

$$\begin{aligned} \sum_{n=1}^{\infty} [n+\alpha-\alpha\lambda(n+1)]L(n, \theta, \mu) a_n &\leq 1-\alpha, \\ \sum_{n=1}^{\infty} [n+\alpha-\alpha\lambda(n+1)]L(n, \theta, \mu) b_n &\leq 1-\alpha. \end{aligned} \quad (29)$$

Define $h(T) = \lambda f(T) + (1 - \lambda)g(T)$. Then, $h(T) = T^{-1} + \sum_{n=1}^{\infty} [\lambda a_n + (1 - \lambda)b_n]T^n$. Now,

$$\begin{aligned} & \sum_{n=1}^{\infty} [n + \alpha - \alpha\lambda(n+1)] L(n, \theta, \mu) [\lambda a_n + (1 - \lambda)b_n] \\ &= \lambda \sum_{n=1}^{\infty} [n + \alpha - \alpha\lambda(n+1)] L(n, \theta, \mu) a_n \\ & \quad + (1 - \lambda) \sum_{n=1}^{\infty} [n + \alpha - \alpha\lambda(n+1)] L(n, \theta, \mu) b_n \\ &\leq \lambda(1 - \alpha) + (1 - \lambda)(1 - \alpha) \quad (\text{using (30)}) \\ &= 1 - \alpha. \end{aligned} \quad (30)$$

Thus, $h \in M_P(\alpha, \lambda, T)$. $\square$

5. Hadamard Product

Theorem 11. *The Hadamard product $f * g$ of two functions f and g in $M_P(\alpha, \lambda, T)$ belongs to the class $M_P(\xi, \lambda, T)$, where $\xi = 1 - 2(1 - \alpha)^2(1 - \lambda)/(1 - \alpha)^2(1 - 2\lambda) + (1 - \alpha - 2\alpha\lambda)^2(1 - \mu)^2(\theta + 1)(\theta + 2)$.*

Proof. It suffices to prove that

$$\sum_{n=1}^{\infty} \frac{[n + \xi - \xi\lambda(n+1)] L(n, \theta, \mu)}{1 - \xi} a_n b_n \leq 1, \quad (31)$$

where

$$\begin{aligned} \xi &= 1 - (2(1 - \alpha)^2(1 - \lambda)) \\ &\quad \times ((1 - \alpha)^2(1 - 2\lambda) + (1 - \alpha - 2\alpha\lambda)^2 \\ &\quad \times (1 - \mu)^2(\theta + 1)(\theta + 2))^{-1}. \end{aligned} \quad (32)$$

Since $f, g \in M_P(\alpha, \lambda, T)$

$$\begin{aligned} & \frac{\sum_{n=1}^{\infty} [n + \alpha - \alpha\lambda(n+1)] L(n, \theta, \mu)}{1 - \alpha} a_n \leq 1, \\ & \frac{\sum_{n=1}^{\infty} [n + \alpha - \alpha\lambda(n+1)] L(n, \theta, \mu)}{1 - \alpha} b_n \leq 1. \end{aligned} \quad (33)$$

By Cauchy-Schwartz inequality,

$$\frac{\sum_{n=1}^{\infty} [n + \alpha - \alpha\lambda(n+1)] L(n, \theta, \mu)}{1 - \alpha} \sqrt{a_n b_n} \leq 1. \quad (34)$$

We have to find the largest ξ such that

$$\begin{aligned} & \frac{[n + \xi - \xi\lambda(n+1)] L(n, \theta, \mu)}{1 - \xi} a_n b_n \\ &\leq \frac{[n + \alpha - \alpha\lambda(n+1)] L(n, \theta, \mu)}{1 - \alpha} \sqrt{a_n b_n}, \end{aligned} \quad (35)$$

that is,

$$\sqrt{a_n b_n} \leq \frac{1 - \alpha}{[n + \alpha - \alpha\lambda(n+1)] L(n, \theta, \mu)}. \quad (36)$$

It is enough to find the largest ξ such that

$$\frac{1 - \alpha}{[n + \alpha - \alpha\lambda(n+1)] L(n, \theta, \mu)} \leq \frac{(1 - \xi)[n + \alpha - \alpha\lambda(n+1)]}{(1 - \alpha)[n + \xi - \xi\lambda(n+1)]} \quad (37)$$

which yields

$$\begin{aligned} \xi &\leq ((n + \alpha - \alpha\lambda(n+1))^2 L(n, \theta, \mu) - (1 - \alpha)^2 n) \\ &\quad \times ((1 - \alpha)^2(1 - \lambda(n+1)) \\ &\quad + (n + \alpha - \alpha\lambda(n+1))^2 L(n, \theta, \mu))^{-1}. \end{aligned} \quad (38)$$

It follows from (38) that

$$\begin{aligned} \xi &\leq 1 - (1 - \alpha)^2(n+1)(1 - \lambda) \\ &\quad \times ((1 - \alpha)^2(1 - \lambda(n+1)) \\ &\quad + (n + \alpha - \alpha\lambda(n+1))^2 L(n, \theta, \mu))^{-1}. \end{aligned} \quad (39)$$

Let

$$\begin{aligned} \delta(n) &= (1 - \alpha)^2(n+1)(1 - \lambda) \\ &\quad \times ((1 - \alpha)^2(1 - \lambda(n+1)) \\ &\quad + (n + \alpha - \alpha\lambda(n+1))^2 L(n, \theta, \mu))^{-1}. \end{aligned} \quad (40)$$

Clearly $\delta(n)$ is an increasing function of n ($n \geq 1$). Letting $n = 1$, we prove the assertion. $\square$

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