Research Article

Disconnected Forbidden Subgraphs, Toughness and Hamilton Cycles

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In 1974, Goodman and Hedetniemi proved that every 2-connected $(K_{1,3}, K_{1,3} + e)$ -free graph is hamiltonian. This result gave rise many other conditions for Hamilton cycles concerning various pairs and triples of forbidden connected subgraphs under additional connectivity conditions. In this paper we investigate analogous problems when forbidden subgraphs are disconnected which affects more global structures in graphs such as tough structures instead of traditional connectivity structures. In 1997, it was proved that a single forbidden connected subgraph R in 2-connected graphs can create only a trivial class of hamiltonian graphs (complete graphs) with $R = P_3$. In this paper we prove that a single forbidden subgraph R can create a non trivial class of hamiltonian graphs if R is disconnected: (*1) every $(K_1 \cup P_2)$ -free graph either is hamiltonian or belongs to a well defined class of non hamiltonian graphs; (*2) every 1-tough $(K_1 \cup P_3)$ -free graph is hamiltonian. We conjecture that every 1-tough $(K_1 \cup P_4)$ -free graph is hamiltonian and every 1-tough P_4 -free graph is hamiltonian.

1. Introduction

Only finite undirected graphs without loops or multiple edges are considered. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We write $G[S]$ for the subgraph of G induced by S . The neighborhood of a vertex $x \in V(G)$ will be denoted by $N(x)$. We use n to denote the number of vertices (order) of G . The independence number of G , denoted by α , is the maximum size of a mutually nonadjacent vertices in G . If $H_1, \dots, H_t$ ($t \geq 1$) are graphs then a graph G is said to be $(H_1, \dots, H_t)$ -free if G contains no copy of any of the graphs $H_1, \dots, H_t$ as an induced subgraph; the graphs $H_1, \dots, H_t$ will be also referred to in this context as forbidden subgraphs. We denote by P_i and C_i the path and the cycle on i vertices. Further, K_n denotes the complete graph of order n , and $K_{a,b}$ denotes the complete bipartite graph with partite sets of cardinalities a and b . Specifically, the graph $K_{1,3}$ will be called a claw. The graph $K_{1,3} + e$ is obtained from $K_{1,3}$ by adding an edge. Let $N_{i,j,k}$ be the graph which is obtained by identifying each vertex of a triangle with an end vertex of one of three vertex-disjoint paths of lengths i, j, k . Let $s(G)$ denote

the number of components of a graph G . A graph G is t -tough if $|S| \geq ts(G \setminus S)$ for every subset S of the vertex set $V(G)$ with $s(G \setminus S) > 1$. The toughness of G , denoted $\tau(G)$, is the maximum value of t for which G is t -tough (taking $\tau(K_n) = \infty$ for all $n \geq 1$). A good reference for any undefined terms is [1].

The first sufficient condition for hamiltonicity of a graph in terms of forbidden subgraphs is by Goodman and Hedetniemi [2].

Theorem A (see [2]). *Every 2-connected $(K_{1,3}, K_{1,3} + e)$ -free graph is hamiltonian.*

This result gave rise to many other hamiltonicity conditions for various pairs and triples of forbidden connected subgraphs under additional connectivity conditions.

In 1997, Faudree and Gould [3] proved that a single forbidden connected subgraph R in 2-connected graphs can create only a trivial class (complete graphs) of hamiltonian graphs with $R = P_3$.

Theorem B (see [3]). *Let R be a connected graph and let G be a 2-connected graph. Then G is R -free implies G is hamiltonian if and only if $R = P_3$.*

The following theorem presents the forbidden pairs version of Theorem B.

Theorem C (see [3]). *Let R and S be connected graphs ($R, S \neq P_3$) and G be a 2-connected graph of order $n \geq 10$. Then G is (R, S) -free implies G is Hamiltonian if and only if $R = K_{1,3}$ and S is one of the graphs:*

$$P_4, P_5, P_6, N_{0,0,0}, N_{0,0,1}, N_{0,0,2}, N_{0,1,1}, N_{0,0,3}, N_{0,1,2}, N_{1,1,1}. \quad (1)$$

The forbidden triple version of Theorems B and C still is not known.

In this paper we investigate analogous problems when forbidden subgraphs are disconnected which affects more global structures in graphs such as tough structures instead of traditional connectivity structures.

First of all, we prove that a single forbidden subgraph R can create a non trivial class of hamiltonian graphs if R is disconnected. As starting point, we consider $K_1 \cup K_1$ and $K_1 \cup K_1 \cup K_1$ as disconnected subgraphs consisting of two and three isolated vertices, respectively. In other words, $K_1 \cup K_1$ is a compliment of K_2 and $K_1 \cup K_1 \cup K_1$ is a complement of K_3 (a triangle). The first proposition follows immediately.

Proposition 1. *Every $(K_1 \cup K_1)$ -free graph is complete and therefore is hamiltonian.*

The next proposition shows that $(K_1 \cup K_1 \cup K_1)$ -free graphs are hamiltonian if and only if they are 2-connected.

Proposition 2. *Every 2-connected $(K_1 \cup K_1 \cup K_1)$ -free graph is hamiltonian.*

Observe that $K_1 \cup P_2$ is the minimal forbidden disconnected subgraph containing at least one edge. To describe the hamiltonian graphs with forbidden subgraph $(K_1 \cup P_2)$, we need the following recursive definition.

Definition 3. We say that $G \in \mathbb{N}$ if and only if either $V(G)$ is independent set of vertices or G is complete graph or there is a bipartition $V(G) = V_1 \cup V_2$ such that

- (1) $N(v) = V_2$ for each $v \in V_1$,
- (2) $G[V_2] \in \mathbb{N}$.

By the Definition, if G is a complete bipartite graph then $G \in \mathbb{N}$.

Theorem 4. *If G is a $(K_1 \cup P_2)$ -free graph then either G is Hamiltonian or $G \in \mathbb{N}$ with $\alpha(G) > n/2$.*

If G is a 1-tough graph then by the definition, $\alpha(G) \leq n/2$. Hence, the following holds from Theorem 4 immediately.

Corollary 5. *Every 1-tough $(K_1 \cup P_2)$ -free graph is hamiltonian.*

This result can be slightly improved.

Theorem 6. *Every 1-tough $(K_1 \cup P_3)$ -free graph is hamiltonian.*

Theorem 6 is sharp in some respects. Indeed, $K_{2,3}$ is a nonhamiltonian $(K_1 \cup P_3)$ -free (even $(K_1 \cup P_2)$ -free) graph with $\tau(K_{2,3}) = 2/3$ and $\kappa(K_{2,3}) = 2$, implying that the condition “ G is 1-tough” in Theorem 6 cannot be removed or replaced by “ G is 2-connected.” Now form a graph, denoted by H_1 , by adding a new vertex x_7 to $C_6 = x_1x_2x_3x_4x_5x_6x_1$ and new edges $x_7x_1, x_7x_4, x_2x_6, x_2x_4, x_4x_6$. Since H_1 is a nonhamiltonian $(K_2 \cup P_3)$ -free graph with $\tau(H_1) = 1$, we can claim that the condition “ G is $(K_1 \cup P_3)$ -free” in Theorem 6 cannot be relaxed to “ G is $(K_2 \cup P_3)$ -free.” Finally, H_1 is a $(K_1 \cup K_{1,3})$ -free graph and hence the condition “ G is $(K_1 \cup P_3)$ -free” in Theorem 6 cannot be relaxed to “ G is $(K_1 \cup K_{1,3})$ -free.” So, Theorem 6 is best possible in many respects. The condition “ G is $(K_1 \cup P_3)$ -free” in Theorem 6 perhaps can be relaxed to “ G is $(K_1 \cup P_4)$ -free.”

Conjecture 7. *Every 1-tough $(K_1 \cup P_4)$ -free graph is hamiltonian.*

Conjecture 7 (if true) is best possible in all respects. Indeed, the above constructed graph H_1 shows that the condition “ G is $(K_1 \cup P_4)$ -free” in Conjecture 7 cannot be replaced by “ G is $(K_1 \cup P_5)$ -free.”

Moreover, the following seems reasonable as well.

Conjecture 8. *Every 1-tough P_4 -free graph is hamiltonian.*

For more than one tough graphs, we conjecture the following.

Conjecture 9. *Every $(K_1 \cup P_5)$ -free graph with $\tau > 1$ is hamiltonian.*

The Petersen graph shows that the condition “ G is $(K_1 \cup P_5)$ -free” in Conjecture 9 cannot be replaced by “ G is $(K_1 \cup P_6)$ -free”.

The next conjecture concerns $(K_2 \cup K_2)$ -free graphs.

Conjecture 10. *Every $(K_2 \cup K_2)$ -free graph with $\tau > 1$ is hamiltonian.*

The graph H_1 (see the graph examples concerning the sharpness of Theorem 6) shows that the condition $\tau > 1$ in Conjecture 10 can not be replaced by $\tau = 1$. Further, the Petersen graph shows that the condition “ G is $(K_2 \cup K_2)$ -free” in Conjecture 10 can not be replaced by “ G is $(K_2 \cup K_3)$ -free”.

The next conjecture is based on a forbidden subgraph $K_1 \cup K_{1,3}$ containing the claw $K_{1,3}$ as a substructure.

Conjecture 11. *Every $(K_1 \cup K_{1,3})$ -free graph with $\tau > 4/3$ is hamiltonian.*

The Petersen graph shows that the condition $\tau > 4/3$ in Conjecture 11 can not be replaced by $\tau = 4/3$.

2. Additional Notation and Preliminaries

For S a subset of $V(G)$, we denote by $G \setminus S$ the subgraph of G induced by $V(G) \setminus S$. For a subgraph H of G we use $G \setminus H$

short for $G \setminus V(H)$. Furthermore, for a subgraph H of G and $x \in V(G)$, we define $N_H(x) = N(x) \cap V(H)$.

A simple cycle (or just a cycle) C of length $t \geq 2$ is a sequence $v_1 v_2 \dots v_t v_1$ of distinct vertices $v_1, \dots, v_t$ with $v_i v_{i+1} \in E(G)$ for each $i \in \{1, \dots, t\}$, where $v_{t+1} = v_1$. When $t = 2$, the cycle $C = v_1 v_2 v_1$ on two vertices v_1, v_2 coincides with the edge $v_1 v_2$. A graph G is hamiltonian if G contains a Hamilton cycle, that is, a cycle of length n . A cycle C in G is dominating if $G \setminus C$ is edgeless.

Paths and cycles in a graph G are considered as subgraphs of G . If Q is a path or a cycle, then the length of Q , denoted by $|Q|$, is $|E(Q)|$. We write Q with a given orientation by $\vec{Q}$. For $x, y \in V(Q)$, we denote by $x\vec{Q}y$ the subpath of Q in the chosen direction from x to y . The expression $Q = x\vec{Q}y$ means that x, y are the endvertices of Q . For $x \in V(C)$, we denote the h th successor and the h th predecessor of x on $\vec{C}$ by x^{+h} and x^{-h} , respectively. We abbreviate x^{+1} by x^+ . For each $X \subset V(C)$, we define $X^{+h} = \{x^{+h} \mid x \in X\}$.

To prove Proposition 2 and Theorems 4, 6, we need the following useful lemma.

Lemma 12. *Let G be a 2-connected graph, $\vec{C}$ a longest cycle in G with a given orientation and $x \in V(G \setminus C)$. Let $\vec{P}_1, \vec{P}_2$ be two internally disjoint paths with given orientations starting at x and terminating on C at different vertices u, v . Then $u^+ v^+ \notin E(G)$.*

3. Proofs

Proof of Lemma 12. Assume the contrary, that is $u^+ v^+ \in E(G)$. Then

$$u\vec{P}_1 x \vec{P}_2 v \vec{C} u^+ v^+ \vec{C} u \quad (2)$$

is longer than C , contradicting the fact that C is a longest cycle in G . Hence, $u^+ v^+ \notin E(G)$. Lemma 12 is proved. $\square$

Proof of Proposition 2. Since G is 2-connected graph, it contains a cycle. Let $\vec{C}$ be a longest cycle in G with a given orientation. If $V(G \setminus C) = \emptyset$ then C is a Hamilton cycle and we are done. Let $V(G \setminus C) \neq \emptyset$ and let $x \in V(G \setminus C)$. Since G is 2-connected, there are two internally disjoint paths P_1, P_2 with given orientations starting at x and terminating on C at different vertices u, v . If $xu^+ \in E(G)$ then $u\vec{P}_1 xu^+ \vec{C} u$ is longer than C , contradicting the fact that C is a longest cycle. Hence, $xu^+ \notin E(G)$. Analogously, $xv^+ \notin E(G)$. Furthermore, by Lemma 12, $u^+ v^+ \notin E(G)$, implying that $\{x, u^+, v^+\}$ is an independent set of vertices which contradicts the fact that G is $(K_1 \cup K_1 \cup K_1)$ -free graph. This final contradiction proves that G is hamiltonian. Proposition 2 is proved. $\square$

Proof of Theorem 4. Let G be a $(K_1 \cup P_2)$ -free graph. If $V(G)$ is independent set of vertices, then by the definition, $G \in \mathcal{N}$ with $\alpha(G) = n > n/2$ and we are done. Let G contain at least one edge. Next, if G is disconnected, then clearly G contains $K_1 \cup P_2$ as an induced subgraph, contradicting the hypothesis. Let G be connected. Further, if G is a tree, then clearly G either is a star (i.e., G is a complete bipartite graph and hence $G \in \mathcal{N}$

with $\alpha(G) = n - 1 > n$) or contains $K_1 \cup P_2$ as an induced subgraph, again contradicting the hypothesis. Now assume that G is not a tree and let $C = v_1 v_2 \dots v_t v_1$ be a longest cycle in G with a given orientation. If $V(G \setminus C) = \emptyset$, then C is a Hamilton cycle and we are done. Let $V(G \setminus C) \neq \emptyset$. Since G is connected, we have $xy \in E(G)$ for some $x \in V(G \setminus C)$ and $y \in V(C)$. Assume without loss of generality that $y = v_1$. Since C is a longest cycle, we have $xv_2 \notin E(G)$. If $xv_3 \notin E(G)$, then x and $v_2 v_3$ form an induced subgraph $K_1 \cup P_2$, contradicting the hypothesis. Hence $xv_3 \in E(G)$. By a similar argument, $xv_i \in E(G)$ for each $i = 1, 3, 5, \dots, t - 1$ and $xv_i \notin E(G)$ for each $i = 2, 4, 6, \dots, t$ where t is even. Further, since C is a longest cycle, by Lemma 12, $\{x, v_2, v_4, \dots, v_t\}$ is an independent set of vertices. Moreover, for each $u, v \in \{x, v_2, v_4, \dots, v_t\}$, there is no path connecting u and v and passing through $V(G) \setminus (V(C) \cup \{x\})$.

To prove that $N(x) = \{v_1, v_3, \dots, v_{t-1}\}$, assume the contrary, that is $xz \in E(G)$ for some $z \in V(G) \setminus (V(C) \cup \{x\})$. Recalling that C is a longest cycle, we conclude that $zv_2 \notin E(G)$ and $zv_3 \notin E(G)$. Therefore, z and $v_2 v_3$ form a disconnected subgraph $K_1 \cup P_2$ which contradicts the fact that G is $(K_1 \cup P_2)$ -free. Hence $N(x) \subseteq V(C)$. But then it is not hard to see that $V(G \setminus C)$ is an independent set of vertices.

Now let us prove that $N(v) = \{v_1, v_3, \dots, v_{t-1}\}$ for each $v \in V(G \setminus C)$. Assume that $v \neq x$. Since $V(G \setminus C)$ is an independent set of vertices, we have $N(v) \subseteq V(C)$. if $v_2 v \in E(G)$, then as above $v_4 v \in E(G)$, implying that

$$v_1 x v_3 v_2 v v_4 \vec{C} v_1 \quad (3)$$

is longer than C , a contradiction. Let $v_2 v \notin E(G)$. But then (as above) $v_1 v \in E(G)$, implying that $N(v) = \{v_1, v_3, \dots, v_{t-1}\}$. Put

$$V_1 = V(G \setminus C) \cup \{v_2, v_4, \dots, v_t\}, \quad V_2 = \{v_1, v_3, \dots, v_{t-1}\}. \quad (4)$$

Clearly $V(G) = V_1 \cup V_2$. Since V_1 is an independent set of vertices and $|V_1| > |V_2|$, we can state that $\alpha(G) \geq |V_1| > n/2$ and thus G is not hamiltonian. It remains to prove that $G \in \mathcal{N}$. If V_2 is independent or $G[V_2]$ is complete, then $G \in \mathcal{N}$ and we are done. Otherwise denote by V_3 a largest independent subset in V_2 and put $V_4 = V_2 \setminus V_3$. Clearly, $|V_3| \geq 2$ and $V_4 \neq \emptyset$. Let w_2 be any vertex in V_4 . Since V_3 is a largest independent set, there exists a vertex $w_1 \in V_3$ such that $w_1 w_2 \in E(G)$. Let $w_3 \in V_3 \setminus \{w_1\}$. If $w_2 w_3 \notin E(G)$, then w_3 and $w_1 w_2$ form an induced subgraph $K_1 \cup P_2$, contradicting the hypothesis. Hence $w_2 w_3 \in E(G)$ implying that $w_2 v \in E(G)$ for each $v \in V_3$. This means also that each vertex in V_4 is adjacent to all vertices in V_3 . Applying the same arguments to V_4 instead of V_2 , we conclude that $G \in \mathcal{N}$. $\square$

Proof of Theorem 6. Let G be a 1-tough $(K_1 \cup P_3)$ -free graph. In particular, G is 2-connected graph and therefore contains a cycle. Let C be a longest cycle in G with a given orientation. If $V(G \setminus C) = \emptyset$ then C is a Hamilton cycle and we are done. Let $V(G \setminus C) \neq \emptyset$ and let H be a connected component of $G \setminus C$ of maximum order. Denote by $\xi_1, \dots, \xi_s$ the elements of $N_C(H)$ occurring on C in a consecutive order. Since G is 1-tough, we have $s \geq 2$. Set

$$I_i = \xi_i \vec{C} \xi_{i+1}, \quad I_i^* = \xi_i^+ \vec{C} \xi_{i+1}^- \quad (i = 1, 2, \dots, s), \quad (5)$$

where $\xi_{s+1} = \xi_1$. Since C is a longest cycle in G , we have $|V(I_i)| \geq 3$ and $|V(I_i^*)| \geq 1$ for each $i \in \{1, 2, \dots, s\}$. The segments $I_1, I_2, \dots, I_s$ are called elementary segments on C induced by $N_C(H)$. We call a path $L = z\vec{L}w$ an intermediate path between two distinct elementary segments I_a and I_b if

$$z \in V(I_a^*), w \in V(I_b^*), \quad V(L) \cap V(C \cup H) = \{z, w\}. \quad (6)$$

Define $Y(I_{i_1}, I_{i_2}, \dots, I_{i_t})$ to be the set of all intermediate paths between elementary segments $I_{i_1}, I_{i_2}, \dots, I_{i_t}$. If $Y(I_1, I_2, \dots, I_s) = \emptyset$, then $G \setminus \{\xi_1, \dots, \xi_s\}$ has at least $s + 1$ connected components, contradicting the fact that G is 1-tough. Otherwise $Y(I_a, I_b) \neq \emptyset$ for some distinct $a, b \in \{1, \dots, s\}$. Choose a path $L = x\vec{L}y$ in $Y(I_a, I_b)$ such that $x \in V(I_a^*)$ and $y \in V(I_b^*)$. If $|V(L)| \geq 3$, then each vertex $v \in V(H)$ with the subpath of L of length 2 forms an induced subgraph $K_1 \cup P_3$, contradicting the hypothesis. Let $|V(L)| = 2$, that is, $L = xy$ and hence $Y(I_1, I_2, \dots, I_s) \subseteq E(G)$. Put

$$Q = \xi_a^+ \vec{C} xy \overleftarrow{C} \xi_b^+. \quad (7)$$

Assume without loss of generality that L is chosen from $Y(I_a, I_b)$ such that $|V(Q)|$ is minimum. Since C is a longest cycle in G , by Lemma 12, $\{\xi_1, \dots, \xi_s\}^+$ is an independent set of vertices, implying that either $x \neq \xi_a^+$ or $y \neq \xi_b^+$, say $x \neq \xi_a^+$. Since $|V(Q)|$ is minimum, we have $x^-y \notin E(G)$, that is, x^-xy forms an induced subgraph P_3 . But then each vertex $v \in V(H)$ with x^-xy forms an induced $K_1 \cup P_3$, again contradicting the hypothesis. Theorem 6 is proved. $\square$

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