Research Article

On Compactness of Embeddings of Fourier-Lebesgue Spaces into Modulation Spaces

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It is shown that, for a certain range of parameters, embeddings of Fourier-Lebesgue $L_a^p \cap \mathcal{FL}_b^q$ spaces into modulation spaces $M_{\alpha,\beta}^{r,s}$ are compact.

1. Introduction

In [1], Galperin and Gröchenig studied the question of how changing the requirements on smoothness and decay of f and $\hat{f}$ affects the lower bound in the uncertainty principle. They derived a class of uncertainty principles in the form

$$\|V_g f\|_{L_{\alpha,\beta}^{r,s}} \leq C \left(\|f\|_{L_a^p} + \|\hat{f}\|_{L_b^q} \right) \quad (1)$$

and partially characterized the range of the parameters p, q, r, s, a, b, α , and β for which (1) holds. Here $\hat{f}$ is the Fourier transform of f normalized as $\mathcal{F}f(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \omega} dx$ and the quantities $\|f\|_{L_a^p} = \|(1 + |x|)^a f\|_{L^p}$ and $\|\hat{f}\|_{L_b^q} = \|(1 + |\omega|)^b \hat{f}\|_{L^q}$ are used as measures of the concentration of f in time and frequency, respectively. For a fixed $g \in \mathcal{S}(\mathbb{R}^d)$, a so-called window function, the STFT of a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ with respect to g is defined by

$$V_g f(x, \omega) = \int_{\mathbb{R}^d} f(t) \overline{g(t-x)} e^{-2\pi i \omega t} dt = \langle f, M_\omega T_x \rangle, \quad (2)$$

where the translation and modulation operators are defined by $T_x g(t) = g(t-x)$ and $M_\omega g(t) = e^{2\pi i \omega t} g(t)$. To measure

the joint time-frequency concentration of a function f , the mixed weighted Lebesgue norm

$$\|V_g f\|_{L_{\alpha,\beta}^{r,s}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_g f(x, \omega)|^r (1 + |x|)^{ar} dx \right)^{s/r} \times (1 + |\omega|)^{\beta s} d\omega \right)^{1/s} \quad (3)$$

is imposed on the short-time Fourier transform (STFT) of f . The theory of mixed-norm Lebesgue spaces is developed in [2].

The uncertainty principles of form (1) are equivalent to embeddings of Fourier-Lebesgue spaces into modulation spaces. For a fixed $g \in \mathcal{S}(\mathbb{R}^d)$, define $M_{\alpha,\beta}^{r,s}$ by the (quasi-)norm $\|f\|_{M_{\alpha,\beta}^{r,s}} = \|V_g f\|_{L_{\alpha,\beta}^{r,s}}$, and then the uncertainty principle (1) is equivalent to the embedding

$$L_a^p \cap \mathcal{FL}_b^q \hookrightarrow M_{\alpha,\beta}^{r,s}, \quad (4)$$

where $\mathcal{FL}_b^q$ denotes the space of the tempered distributions f whose Fourier transform $\hat{f}$ is in L_b^q .

In this paper, we show that embeddings (4) are compact. We prove the following theorem.

Theorem 1. Let $0 < r, s \leq 2$, $\alpha, \beta \geq 0$, and $1 \leq p, q \leq \infty$. Suppose that $0 < r \leq p \leq \infty$ and $0 < s \leq q \leq \infty$.

If

$$\begin{aligned} & \left(\frac{a-\alpha}{d} + \frac{1}{p} - \frac{1}{r} \right) \left(\frac{b-\beta}{d} + \frac{1}{q} - \frac{1}{s} \right) \\ & > \max \left\{ \left(\frac{1}{r} - \frac{1}{q'} + \frac{\alpha}{d} \right), \left(\frac{1}{r} - \frac{1}{2} + \frac{\alpha}{d} \right) \right\} \\ & \quad \times \max \left\{ \left(\frac{1}{s} - \frac{1}{p'} + \frac{\beta}{d} \right), \left(\frac{1}{s} - \frac{1}{2} + \frac{\beta}{d} \right) \right\}, \end{aligned} \quad (5)$$

with all factors being nonnegative, then $L_a^p \cap \mathcal{FL}_b^q$ is compactly embedded in $M_{\alpha,\beta}^{r,s}$.

The factors on the left side of (5) are quite natural. When they are both positive, $L_a^p \hookrightarrow L_\alpha^r$ and $\mathcal{FL}_b^q \hookrightarrow \mathcal{FL}_\beta^s$. Since this is not enough to guarantee that $L_a^p \cap \mathcal{FL}_b^q \hookrightarrow M_{\alpha,\beta}^{r,s}$ (because the modulation space norm measures the decay of f in time and in frequency *simultaneously*, whereas the Lebesgue space norms of f and $\hat{f}$ treat time and frequency as separate inputs), the term on the right side of (5) indicates the exact measure of additional decay that has to be imposed on f and $\hat{f}$. However, the strict inequality in (5) implies some *excessive* decay, which results in tightness of the STFT on sets bounded in $L_a^p \cap \mathcal{FL}_b^q$.

It is interesting to compare Lemma 6 and Theorem 1 with the results obtained in [3, Theorem 3.2], which is concerned with compactness of embeddings into modulation space $M_s^{p,q}$. Whereas the weights $m_s(x, \omega) = (1 + |x| + |\omega|)^s$ used in [3] assume the same rate of decay of the short-time Fourier transform in the time and frequency variables, the weights used in this paper differentiate between these two rates. Thus, the result proved in [3] is not directly applicable to our case.

Our result relies on the following criterion of compactness in modulation spaces in terms of tightness of the STFT.

Theorem 2 (see [4, Theorem 5]). *Assume that $g \in \mathcal{S}(\mathbb{R}^d)$, $0 < r, s < \infty$, and S is a closed and bounded subset of $M_{\alpha,\beta}^{r,s}$. Then S is compact in $M_{\alpha,\beta}^{r,s}$ if and only if, for all $\epsilon > 0$, there exists a compact set $U \subset \mathbb{R}^{2d}$, such that*

$$\sup_{f \in S} \|\chi_{U^c} \cdot V_g f\|_{L_{\alpha,\beta}^{r,s}} < \epsilon. \quad (6)$$

Remark. (1) In [4], Theorem 2 was proved in the context of the co-orbit spaces (with more general weight functions) for the case $1 \leq r, s < \infty$. However, the same argument works for $M_{\alpha,\beta}^{r,s}$, $0 < r, s < \infty$.

(2) For the theory of modulation spaces we refer to [5, Chapter 11-12], [6] and to the original literature [7–9].

(3) It is shown in [1] that condition (5) is optimal. If the inequality is reversed, $L_a^p \cap \mathcal{FL}_b^q$ is not embedded in $M_{\alpha,\beta}^{r,s}$.

2. Definitions and Preliminary Results

We first provide the necessary definitions and tools. Our notation and definitions are consistent with those in [5].

2.1. Weights and Mixed Norm Spaces. To alleviate notation, we write $\langle x \rangle = 1 + |x|$. We need the following lemma for weighted mixed norm spaces.

Lemma 3 (Hölder's inequality). *Let $p \geq r$ and $q \geq s$. Write $t = r(p/r)' = rp/(p-r)$ and $u = s(q/s)' = sq/(q-s)$. Then*

$$\|F \cdot H\|_{L_{\alpha,\beta}^{r,s}} \leq \|F\|_{L_{a,b}^{p,q}} \cdot \|H\|_{L_{\alpha-a,\beta-b}^{t,u}}, \quad (7)$$

whenever the right-hand side is finite.

Proof. We write the left-hand side as

$$\begin{aligned} \|F \cdot H\|_{L_{\alpha,\beta}^{r,s}} &= \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x, \omega)|^r \langle x \rangle^{ar} \right. \right. \\ & \quad \times |H(x, \omega)|^r \langle x \rangle^{(a-a)r} dx \Big)^{s/r} \\ & \quad \times \langle \omega \rangle^{bs} \langle \omega \rangle^{(\beta-b)s} d\omega \Big)^{1/s}. \end{aligned} \quad (8)$$

Next apply Hölder's inequality with exponents p/r and $(p/r)'$ to the integral in dx and with exponents q/s and $(q/s)'$ to the integral in $d\omega$. This yields

$$\begin{aligned} \|F \cdot H\|_{L_{\alpha,\beta}^{r,s}} &\leq \left(\int_{\mathbb{R}^d} \|F(\cdot, \omega)\|_{L_a^p}^s \langle \omega \rangle^{bs} \right. \\ & \quad \cdot \|H(\cdot, \omega)\|_{L_{\alpha-a}^t}^s \langle \omega \rangle^{(\beta-b)s} d\omega \Big)^{1/s} \\ &\leq \left(\int_{\mathbb{R}^d} \|F(\cdot, \omega)\|_{L_a^p}^q \langle \omega \rangle^{bq} d\omega \right)^{1/q} \\ & \quad \cdot \left(\int_{\mathbb{R}^d} \|H(\cdot, \omega)\|_{L_{\alpha-a}^t}^u \langle \omega \rangle^{(\beta-b)u} d\omega \right)^{1/u} \\ &= \|F\|_{L_{a,b}^{p,q}} \cdot \|H\|_{L_{\alpha-a,\beta-b}^{t,u}}, \end{aligned} \quad (9)$$

as desired. $\square$

We will also use the following elementary embedding.

Lemma 4. *Suppose that $1 \leq r < p \leq \infty$. Then $L_a^p \hookrightarrow L_\alpha^r$ if and only if $(a-\alpha)/d + 1/p - 1/r > 0$.*

The following technical lemma about weighted mixed norms of certain characteristic functions is instrumental for the main embedding result.

Lemma 5 (see [1]). *Let $\alpha, \beta \in \mathbb{R}$, $0 < r, s \leq \infty$, and $\sigma > 0$. Define $A_\sigma = \{(x, \omega) \in \mathbb{R}^{2d} : |x| \geq |\omega|^{1/\sigma}\}$ and $B_\sigma = \mathbb{R}^{2d} \setminus A_\sigma = \{(x, \omega) \in \mathbb{R}^{2d} : |x| < |\omega|^{1/\sigma}\}$.*

Then $\chi_{B_\sigma} \in L_{\alpha,\beta}^{r,s}$ provided that

$$\frac{1}{\sigma} \max \left\{ \frac{\alpha}{d} + \frac{1}{r}, 0 \right\} + \frac{\beta}{d} + \frac{1}{s} < 0. \quad (10)$$

Furthermore, $\chi_{A_\sigma} \in L_{\alpha,\beta}^{r,s}$ provided that

$$\begin{aligned} \frac{\alpha}{d} + \frac{1}{r} &< 0, \\ \frac{1}{\sigma} \left(\frac{\alpha}{d} + \frac{1}{r} \right) + \frac{\beta}{d} + \frac{1}{s} &< 0. \end{aligned} \quad (11)$$

3. Proof of the Main Result

In order to prove Theorem 1, we first establish compactness of certain embeddings between modulation spaces.

Lemma 6. Assume that $0 < r, s < \infty$, $\alpha, \beta \geq 0$, and $p, q' \in [r, s']$. If

$$\begin{aligned} &\left(\frac{a-\alpha}{d} + \frac{1}{p} - \frac{1}{r} \right) \left(\frac{b-\beta}{d} + \frac{1}{q} - \frac{1}{s} \right) \\ &> \left(\frac{\alpha}{d} - \frac{1}{q'} + \frac{1}{r} \right) \left(\frac{\beta}{d} - \frac{1}{p'} + \frac{1}{s} \right) \end{aligned} \quad (12)$$

with all factors nonnegative, then $M_{a,0}^{p,p'} \cap M_{0,b}^{q',q}$ is compactly embedded in $M_{\alpha,\beta}^{r,s}$.

Proof. We split the time-frequency plane into the two regions $A_\sigma = \{(x, \omega) \in \mathbb{R}^{2d} : |x| \geq |\omega|^{1/\sigma}\}$ and $B_\sigma = \mathbb{R}^{2d} \setminus A_\sigma$ for some $\sigma > 0$ to be determined later, and we estimate the modulation space (quasi-)norm of f accordingly by

$$\begin{aligned} \|f\|_{M_{\alpha,\beta}^{r,s}} &= \|V_g f\|_{L_{\alpha,\beta}^{r,s}} \\ &\leq C \left(\|V_g f \cdot \chi_{A_\sigma}\|_{L_{\alpha,\beta}^{r,s}} + \|V_g f \cdot \chi_{B_\sigma}\|_{L_{\alpha,\beta}^{r,s}} \right). \end{aligned} \quad (13)$$

We then apply Hölder's inequality (Lemma 3) to each term and use Lemma 5. Writing $t = r(p/r)' = rp/(p-r)$ and $u = s(p'/s)' = sp'/(p'-s)$, we obtain that

$$\begin{aligned} \|V_g f \cdot \chi_{A_\sigma}\|_{L_{\alpha,\beta}^{r,s}} &\leq \|V_g f\|_{L_{a,0}^{p,p'}} \cdot \|\chi_{A_\sigma}\|_{L_{\alpha-a,\beta}^{t,u}} \\ &= \|f\|_{M_{a,0}^{p,p'}} \cdot \|\chi_{A_\sigma}\|_{L_{\alpha-a,\beta}^{t,u}}. \end{aligned} \quad (14)$$

Lemma 5 implies that $\chi_{A_\sigma} \in L_{\alpha-a,\beta}^{t,u}$, whenever

$$\begin{aligned} \frac{\alpha-a}{d} + \frac{1}{t} &= \frac{\alpha-a}{d} + \frac{1}{r} - \frac{1}{p} < 0, \\ \frac{1}{\sigma} \left(\frac{\alpha-a}{d} + \frac{1}{t} \right) + \frac{\beta}{d} + \frac{1}{u} &< 0. \end{aligned} \quad (15)$$

Equivalently,

$$\frac{a-\alpha}{d} + \frac{1}{p} - \frac{1}{r} > \sigma \left(\frac{\beta}{d} - \frac{1}{p'} + \frac{1}{s} \right) \geq 0. \quad (16)$$

Similarly, we obtain for the second term that

$$\|V_g f \cdot \chi_{B_\sigma}\|_{L_{\alpha,\beta}^{r,s}} \leq \|V_g f\|_{L_{0,b}^{q',q}} \cdot \|\chi_{B_\sigma}\|_{L_{\alpha,\beta-b}^{\tilde{t},\tilde{u}}}, \quad (17)$$

where $\tilde{t} = r(q'/r)' = rq'/(q'-r)$ and $\tilde{u} = s(q/s)' = sq/(q-s)$. By Lemma 5 we have $\chi_{B_\sigma} \in L_{\alpha,\beta-b}^{\tilde{t},\tilde{u}}$ provided that

$$\frac{1}{\sigma} \max \left\{ \frac{\alpha}{d} + \frac{1}{\tilde{t}}, 0 \right\} + \frac{\beta-b}{d} + \frac{1}{\tilde{u}} < 0, \quad (18)$$

or equivalently,

$$\begin{aligned} \frac{b-\beta}{d} + \frac{1}{q} - \frac{1}{s} &> \frac{1}{\sigma} \max \left\{ \frac{\alpha}{d} - \frac{1}{q'} + \frac{1}{r}, 0 \right\} \\ &= \frac{1}{\sigma} \left(\frac{\alpha}{d} - \frac{1}{q'} + \frac{1}{r} \right) \geq 0. \end{aligned} \quad (19)$$

Finally, if (12) holds, then there exists $\sigma > 0$ so that both (16) and (19) and all factors are positive. Hence, $\chi_{A_\sigma} \in L_{\alpha-a,\beta}^{t,u}$ and $\chi_{B_\sigma} \in L_{\alpha,\beta-b}^{\tilde{t},\tilde{u}}$. It follows that, for a given $\epsilon > 0$, there exist compact sets $U_1, U_2 \subset \mathbb{R}^{2d}$, such that

$$\|\chi_{A_\sigma} \cdot \chi_{U_1}\|_{L_{\alpha-a,\beta}^{t,u}} < \frac{\epsilon}{2C}, \quad \|\chi_{B_\sigma} \cdot \chi_{U_2}\|_{L_{\alpha,\beta-b}^{\tilde{t},\tilde{u}}} < \frac{\epsilon}{2C}. \quad (20)$$

Define $U = U_1 \cup U_2$. The combination of (14), (17), and (20) yields that

$$\sup_{f \in \text{Ball}(M_{a,0}^{p,p'} \cap M_{0,b}^{q',q})} \|\chi_U \cdot V_g f\|_{L_{\alpha,\beta}^{r,s}} < \epsilon. \quad (21)$$

Therefore, by Theorem 2, the embedding $M_{a,0}^{p,p'} \cap M_{0,b}^{q',q} \hookrightarrow M_{\alpha,\beta}^{r,s}$ is compact. $\square$

The next lemma relates weighted L^p spaces to modulation spaces and can be considered a version of the Hausdorff-Young inequality for the STFT. For more general inequalities see [10, Section 4.2.1].

Lemma 7 (see [1, 11]). Suppose that $1 \leq p \leq 2$. Then

$$\begin{aligned} \text{(a)} \quad L_a^p &\hookrightarrow M_{a,0}^{p,p'} \text{ and} \\ \|V_g f\|_{L_{a,0}^{p,p'}} &\leq \|f\|_{L_a^p} \|g\|_{L_{|a|}^p}, \end{aligned} \quad (22)$$

where $C = C(p, a)$ is independent of f and g ,

$$\begin{aligned} \text{(b)} \quad \mathcal{F} L_a^p &\hookrightarrow M_{0,a}^{p',p} \text{ and} \\ \|V_g f\|_{L_{0,a}^{p',p}} &\leq \|\widehat{f}\|_{L_a^p} \|\widehat{g}\|_{L_{|a|}^p}, \end{aligned} \quad (23)$$

where $C = C(p, a)$ is independent of f and g .

The combination of Lemmas 6 and 7 leads to Theorem 1.

Proof of Theorem 1. Following the proof of the main result in [1], we distinguish several cases.

Case 1. If $r \leq p \leq 2$ and $s \leq q \leq 2$, then $L_a^p \hookrightarrow M_{a,0}^{p,p'}$ and $\mathcal{F} L_b^q \hookrightarrow M_{0,b}^{q',q}$ by Lemma 7. As a consequence of (5), Lemma 6 is applicable and thus $M_{a,0}^{p,p'} \cap M_{0,b}^{q',q}$ is compactly

embedded in $M_{\alpha,\beta}^{r,s}$. Combining the above, we obtain that $L_a^p \cap \mathcal{F}L_b^q$ is compactly embedded in $L_{\alpha,\beta}^{r,s}$.

Case 2. $p > 2, s \leq q \leq 2$.

By continuity there exists $c > 0$ such that

$$\begin{aligned} & \left(\frac{a-\alpha}{d} + \frac{1}{p} - \frac{1}{r} \right) \left(\frac{b-\beta}{d} + \frac{1}{q} - \frac{1}{s} \right) \\ & > \left(\frac{c-\alpha}{d} + \frac{1}{2} - \frac{1}{r} \right) \left(\frac{b-\beta}{d} + \frac{1}{q} - \frac{1}{s} \right) \\ & > \left(\frac{1}{r} - \frac{1}{q'} + \frac{\alpha}{d} \right) \left(\frac{1}{s} - \frac{1}{2} + \frac{\beta}{d} \right). \end{aligned} \quad (24)$$

The first inequality in (24) implies that $(a-c)/d + 1/p - 1/2 > 0$ and thus $L_a^p \hookrightarrow L_c^2$ by Lemma 4. In view of (5) Case 1 is now applicable with L_c^2 instead of L_a^p , and we obtain the compact embedding

$$L_a^p \cap \mathcal{F}L_b^q \hookrightarrow L_c^2 \cap \mathcal{F}L_b^q \hookrightarrow M_{\alpha,\beta}^{r,s}. \quad (25)$$

Case 3. $q > 2, r \leq p \leq 2$ is similar.

Case 4. $p, q > 2$.

By continuity we may choose $c > 0$ and $\gamma > 0$, so that the inequalities $(a-c)/d + 1/p - 1/2 > 0$, $(b-\gamma)/d + 1/q - 1/2 > 0$, and

$$\begin{aligned} & \left(\frac{a-\alpha}{d} + \frac{1}{p} - \frac{1}{r} \right) \left(\frac{b-\beta}{d} + \frac{1}{q} - \frac{1}{s} \right) \\ & > \left(\frac{c-\alpha}{d} + \frac{1}{2} - \frac{1}{r} \right) \left(\frac{\gamma-\beta}{d} + \frac{1}{2} - \frac{1}{s} \right) \\ & > \left(\frac{1}{r} - \frac{1}{2} + \frac{\alpha}{d} \right) \left(\frac{1}{s} - \frac{1}{2} + \frac{\beta}{d} \right) \end{aligned} \quad (26)$$

hold simultaneously. It follows that $L_a^p \hookrightarrow L_c^2$ and $\mathcal{F}L_b^q \hookrightarrow \mathcal{F}L_\gamma^2$ by Lemma 4 and that $L_c^2 \cap \mathcal{F}L_\gamma^2$ is compactly embedded in $M_{\alpha,\beta}^{r,s}$ by Lemmas 6 and 7. Therefore $L_a^p \cap \mathcal{F}L_b^q$ is compactly embedded in $M_{\alpha,\beta}^{r,s}$, as desired. The theorem is proved completely. $\square$

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