Conference Paper

Applied Quantum Field Theory to General Diffusion-Reaction Phenomena

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Diffusion-reaction phenomena are generally described by parabolic differential equations (PDEs), and I am interested in those possessing solutions that fail at large time. A sophisticated method to study the large-time behavior is the Renormalization Group usually encountered in Particles-Physics and Critical Phenomena. In this paper, I review the application of such an approach. In particular, attention is paid to Quantum Field Theory techniques used for the extraction of the asymptotic solutions to PDEs. Finally, I extend discussion to the fractional-time PDEs and with noise.

1. Introduction

Diffusion-reaction equations are generally PDEs; that is, they contain a simple derivative with respect to time and spatial derivatives of arbitrary order [1, 2]. In fact, these equations model a large class of natural systems. Among these, I can quote thermal transfer with absorption, flow in porous media, pressure of a gas in the chock-waves problem, pressure of a liquid in a porous medium, growth of an interface, diffusion-reaction phenomena, or diffusion fronts.

PDEs are very often nonlinear and may rarely be solved analytically. When an analytic solution exists, it can be obtained transforming the original equation to a linear one that is easy to solve. Usually, use is made of Lie-algebra techniques [3–6] if this equation possesses a certain symmetry.

Two fundamental questions to ask are the study of the stability of the solution of PDEs and its behavior at large time. The most sophisticated method for the study of the time-asymptotic behavior is the Renormalization Group (RG) techniques. In the past, this approach has been successfully used, first in High-Energy Physics [7, 8], and second, in Critical Phenomena [9–11]. For the first domain, RG has been very useful to studying the infrared behavior and the

asymptotic freedom. For the second, RG was introduced in a series of brilliant papers by Wilson [12, 13], for the extraction of the critical behavior of those systems exhibiting a second order phase transition.

Since few years, it turned out that RG could be applied to investigate the time-asymptotic behavior of the solution of PDEs [14–22]. The fundamental hypothesis is that the solution asymptotically behaves as $t^{-\alpha/2} f(x/\sqrt{t})$, where the exponent α and the scaling function $f(x)$ depend generally on the nature of the problem. Here, x is the spatial variable and t is time. Therefore, searching the asymptotic behavior necessitates the knowledge of these two quantities. To this end, one first constructs a renormalization transformation. The fixed point of this transformation is nothing else but the scaling function $f(x)$.

Recently, use was made of Quantum Field Theory (QFT) for the study of the time-asymptotic behavior of solutions of PDEs [23]. This is precisely the purpose of this paper. A special attention will be paid to PDEs with noise and time-fractional PDEs.

The remaining of presentation proceeds as follows. In Section 2, we briefly recall the obtaining of the more general

diffusion-reaction equations. Then, we place the renormalization theory in its historical context, in Section 3. The important notion of self-similarity is recalled in Section 4. Application of real-space RG for the determination of the time-asymptotic behavior of solutions of PDEs is described in Section 5. Field-theoretical approach to large-time behavior is the aim of Section 6. Finally, some concluding remarks are drawn in the last section.

2. Diffusion-Reaction Equations

The diffusion mechanism models the motion of individuals in an environment or in a medium. These individuals may be small, as particles in physics, bacteria, molecules and cells, or very large objects like humans, animals, insects, organisms, plants, or certain types of events (epidemics, rumors...).

I assume that individuals reside in some region, Ω , which is an open domain of the Euclidean space $\mathbf{R}^n$, with $n \geq 1$. In particular, I shall be concerned with $n = 1, 2$, and 3 cases. But the formalism presented here applies for any space dimensionality.

The basic mathematical variable I consider is the *population density function*: $P(x, t)$, where $x \in \Omega$ is the position and t is time. This function is defined as the number of particles per unit length (for $n = 1$), per unit area (for $n = 2$), or per unit volume (for $n = 3$). For example, the human population density is often expressed as the number of individuals per square kilometer.

But as for other mathematical models, I assume that the function $P(x, t)$ has good mathematical properties, such as continuity and differentiability. This is more reasonable, when one considers a population with high number of individuals.

Technically, one defines the population density function $P(x, t)$ as follows: let x be a point of habitat Ω , and let $\{O_n \subset \Omega\}_{n=1}^{\infty}$ be a sequence of spatial regions surrounding the point x . Here, the subdomains $\{O_n\}$ are chosen so that the spatial measures $\{|O_n|\}$ (length, area, volume, or mathematically a Lebesgue measure) tend to zero, as $n \rightarrow \infty$, and $O_n \supset O_{n+1}$; then

$$P(x, t) = \lim_{n \rightarrow \infty} \frac{\text{Number of individuals in } O_n \text{ at time } t}{|O_n|}, \quad (1)$$

if the limit exists. It is clear that the total population in any region O of Ω , at time t , is

$$\int_{\Omega} P(x, t) dx. \quad (2)$$

The question of interest is how the function $P(x, t)$ changes, when varying time t and position x . In fact, the population can change in two ways: one is that individual particles can move in their field; and the second is that they can give birth to new individuals, or kill existing individuals for physical, chemical, or biological reasons. These two different phenomena can be separately modeled.

Now, the question is how particles can move? Generally, this is a high-complicated process that may be attributed to

several reasons. Usually, the population moves from high-density regions to low-density ones. This fact is similar to many physical phenomena, such as heat transfer (from the hottest place to the coldest), or a chemical dilution in water.

The motion of $P(x, t)$ is called *population density flow* that is a vector. This flow points to the direction of rapid decay of $P(x, t)$, which is a negative gradient of $P(x, t)$. This is the *Fick law*, mathematically traduced by

$$\mathbf{J}(x, t) = -D(x) \nabla_x P(x, t), \quad (3)$$

where $\mathbf{J}(x, t)$ is the flow of $P(x, t)$, $D(x)$ is the diffusion coefficient at point x , and ∇_x is the gradient operator: $\nabla_x f(x) = (\partial f / \partial x_1, \dots, \partial f / \partial x_n)$.

On the other hand, the number of individuals at any point may change, because of some reasons as the birth, death, hunting, or chemical reactions. I suppose that the change rate of the population density function is $f(x, t; P)$, termed *reaction rate*. Now, the aim is the obtaining of a differential equation solved by the population density, by using the so-called *balance law*. I choose an arbitrary region $O \subset \Omega$, where the population is $\int_O P(x, t) dx$. Then, the change rate of the total population is

$$\frac{d}{dt} \int_O P(x, t) dx. \quad (4)$$

The net growth of the population in domain O is

$$\int_O f(x, t; P(x, t)) dx, \quad (5)$$

and the total flow is

$$\int_{\partial O} \mathbf{J}(x, t) \cdot \mathbf{n}(x, t) dS, \quad (6)$$

where ∂O is the boundary of domain O and $\mathbf{n}(x, t)$ is the exterior normal direction at point x . Therefore, the balance law implies

$$\begin{aligned} \frac{d}{dt} \int_O P(x, t) dx \\ = - \int_{\partial O} \mathbf{J}(x, t) \cdot \mathbf{n}(x, t) dS + \int_O f(x, t; P(x, t)) dx. \end{aligned} \quad (7)$$

From the standard divergence theorem, I obtain

$$\int_{\partial O} \mathbf{J}(x, t) \cdot \mathbf{n}(x, t) dS = \int_O \text{div } \mathbf{J}(x, t) dx. \quad (8)$$

Combining formulae (3), (7), and (8) and interchanging the order of integration and derivation yields

$$\begin{aligned} \int_O \frac{\partial P(x, t)}{\partial t} dx \\ = \int_O \{ \text{div} [D(x) \nabla_x P(x, t)] + f(x, t; P(x, t)) \} dx. \end{aligned} \quad (9)$$

Since the choice of region O is arbitrary, then, we have the following differential equation:

$$\frac{\partial P(x, t)}{\partial t} = \text{div} [D(x) \nabla_x P(x, t)] + f(x, t; P(x, t)), \quad (10)$$

at any point x and any time t . The above equation is called *diffusion-reaction equation*. Here, $\text{div}[D(x)\nabla_x P(x,t)]$ accounts for the diffusion term governing the motion of individuals, and $f(x,t;P(x,t))$ for the reaction term that describes the birth/death or the reaction producing inside habitat or reactor.

The diffusion coefficient $D(x)$ is not generally constant, since the environment is usually heterogenous. But when the diffusion region is approximately homogenous, one can suppose that $D(x) = D$, at any point x . Therefore, (10) becomes

$$\frac{\partial P(x,t)}{\partial t} = D\Delta P(x,t) + f(x,t;P(x,t)), \quad (11)$$

where $\Delta = \sum_{i=1}^n \partial^2/\partial x_i^2$ is the Laplace operator.

3. Brief Overview of the Renormalization Theory

The renormalization is an old concept that first appeared in elementary physics. So, for vacuum, the dielectric constant is ϵ_0 , but for a dielectric medium (glasses, plastic materials...), this constant is rather $\epsilon = \epsilon_0 \times \epsilon_r$, where ϵ_r is the relative permittivity. It is then a multiplicative renormalization of the dielectric constant. In solids, for example, the free electrons, ensuring the electric conduction, interact with the underlying crystalline structure. To simplify the study, one can replace the real mass m of electrons by an effective mass $m^* = m \times \hat{m}$, where the factor $\hat{m}$ reflects the crystalline structure-electrons interactions.

Actually, the renormalization appears in QFT [7–11]. Such a theory is constructed with an action, $S[\varphi]$, which is a functional of field φ , and function of microscopic parameters, namely, the mass of particles, m , and the coupling constant, g . According to the diffusion (in High-Energy Physics language), one may have many fields (then many masses and coupling constants). Mathematically, a field $\varphi(x^0, \vec{x})$ is a temperate distribution. Here, $x^\mu = (x^0 = ct, \vec{x})$ represents a 4-vector, where $\vec{x} \in \mathbb{R}^3$ is the 3-vector, t is the time variable, and $c = 3 \times 10^8$ m/s is the light speed in vacuum. The set of all 4-vectors constitutes the so-called Minkowski space-time $\mathbf{M}_d$ ($d = 4$), which is a differential manifold.

For φ^4 -theory, for instance, the associated action reads

$$S[\varphi] = S_0[\varphi, m^2] + S_{\text{int}}[\varphi, g], \quad (12)$$

with

$$S_0[\varphi, m^2] = \frac{1}{2} \int d^d x \left[(\partial_\mu \varphi) (\partial^\mu \varphi) + m^2 \varphi^2 \right] (x), \quad (13)$$

$$S_{\text{int}}[\varphi, g] = \frac{g}{4!} \int d^d x \varphi^4(x).$$

Here, the free action $S_0[\varphi, m^2]$ is quadratic in field φ and its derivatives, and $S_{\text{int}}[\varphi, g]$ accounts for the interaction action, which is nonlinear in this field.

The theory has been developed through the Green functions, $G^{(N)}(x_1, \dots, x_N, m^2, g)$, which are the vacuum average of a product of field φ ,

$$G^{(N)}(x_1, \dots, x_N, m^2, g) = \langle 0 | T \varphi(x_1) \cdots \varphi(x_N) | 0 \rangle, \quad (14)$$

$$x_i \in \mathbf{M}_d,$$

where T is the *chronological order operation*. Thanks to these functions, one calculates a basic quantity, which is the S-matrix allowing the deduction of the experimentally measured cross section [7, 8]. These Green functions cannot be calculated exactly and are developed in series in the coupling constants g 's and appear short-distances (or *ultraviolet*) divergences that first necessitate a regularization procedure (introducing some cut-off). There are many ways to regularize the theory, but the most commode regularization is the dimensional one [24–26], with the regulator $\epsilon = 4 - d$, where d is the space dimensionality. Within the framework of this regularization, short-distances divergences are poles in ϵ . Hence, regularized (or bare) Green functions present as Laurent series in ϵ -variable and entire series in the bare coupling constant g .

It is not sufficient to regularize the theory but this must be renormalized removing all short-distances divergences. It has been shown [7, 8] that QFT is renormalized at any order of the perturbation series in the coupling constant. This renormalization theorem stipulates that there exists a renormalized square mass, m_R^2 , a renormalized coupling constant, g_R , and a renormalization factor, Z , which are functions of bare parameters (m^2, g) and regulator ϵ , such as the bare Green function $G^{(N)}$ is directly proportional to the renormalized one, $G_R^{(N)}$,

$$G^{(N)}(x_1, \dots, x_N, m^2, g, \epsilon) \quad (15)$$

$$= Z^{N/2}(m^2, g, \epsilon) G_R^{(N)}(x_1, \dots, x_N, m_R^2, g_R, \epsilon),$$

$$\varphi = Z^{1/2} \varphi_R. \quad (16)$$

The function $G_R^{(N)}$ is finite in the limit $\epsilon \rightarrow 0$, at fixed renormalized parameters m_R^2 and g_R . Therefore, a part of divergences has been absorbed redefining the bare parameters, and the remaining divergent part has been multiplicatively factorized (factor $Z^{N/2}$). One can say that a renormalization is a *change* of parameters.

From the multiplicative renormalization (15), one can derive the RG-equation or Callan-Symanzik equation [7, 8], satisfied by the renormalized Green function $G_R^{(N)}$. The resolution of this equation informs on the infrared (or short-distances) of Green functions, that is, when $m^2 \rightarrow 0$.

Thereafter, Wilson was extended RG originally known in High-Energy Physics to Critical Phenomena [9–11]. In this case, the order parameter plays the role of φ -field, and the shift to the critical temperature $t = T - T_c$, the role of square mass m^2 , and the correlation functions are the analog of Green functions $G^{(N)}$. Here, T_c accounts for the critical temperature. Thus, the approach of the transition ($T \rightarrow T_c$),

for Critical Phenomena, corresponds to the infrared limit ($m^2 \rightarrow 0$), for QFT.

For PDEs of interest, these various limits are the analog of the asymptotic limit $t \rightarrow \infty$ of the solution.

4. Notion of Self-Similarity

I consider a given property, $u(x_1, \dots, x_n, t)$, which depends on the point $(x_1, \dots, x_n, t) \in \mathbf{R}^{n+1}$. For dynamic systems, t is time and $(x_1, \dots, x_n)$ are the freedom degrees. For magnetic materials, for example, t is the distance from the critical temperature T_c and x_1 is the magnetic field H .

I assume that there exists some domain $\mathcal{D} \subset \mathbf{R}^{n+1}$, such as the restriction of property u to $\mathcal{D}$, denoted by $u^* = u|_{\mathcal{D}}$, satisfies the *generalized homogeneity property* (scale invariance or self-similarity)

$$u^*(L^{a_1}x_1, \dots, L^{a_n}x_n, L^at) = L^p u^*(x_1, \dots, x_n, t). \quad (17)$$

Here, the scale dilatation L is an arbitrary integer and p is the homogeneity degree. The exponents $(a_1, \dots, a_n, a)$ are real numbers. The above definition is equivalent to the standard Euler formula

$$\left(\sum_{i=1}^n a_i x_i \frac{\partial}{\partial x_i} + at \frac{\partial}{\partial t} - p \right) u^*(x_1, \dots, x_n, t) = 0. \quad (18)$$

I rewrite now relation (17) as

$$u^*(x_1, \dots, x_n, t) = L^{-p} u^*(L^{a_1}x_1, \dots, L^{a_n}x_n, L^at). \quad (19)$$

Of course, the RHS of this equality must not depend on the scale L . Then, I decide to choose this scale, such that $L^at = 1$ or $L = t^{-1/a}$. With this choice, relation (19) becomes

$$u^*(x_1, \dots, x_n, t) = t^{p/a} f^*(x_1 t^{-a_1/a}, \dots, x_n t^{-a_n/a}), \quad (20a)$$

with the notation

$$f^*(x_1 t^{-a_1/a}, \dots, x_n t^{-a_n/a}) = u^*(x_1 t^{-a_1/a}, \dots, x_n t^{-a_n/a}, 1). \quad (20b)$$

It is straightforward to check that the function f^* is scale invariant, called *scaling function*. Therefore, the solution of the functional equation (19) reads

$$u^*(x_1, \dots, x_n, t) = t^{-\alpha/2} f^*(x_1 t^{-\Delta_1}, \dots, x_n t^{-\Delta_n}), \quad (21)$$

with the *principal* exponent

$$\alpha = -\frac{2p}{a}, \quad (22)$$

and *crossover* exponents

$$\Delta_i = \frac{a_i}{a}, \quad (1 \leq i \leq n). \quad (23)$$

For dynamic systems, the self-similarity relation (21) constitutes the time-asymptotic behavior. Thus, the knowledge of this behavior necessitates that of exponents α and Δ_i and the scaling function f^* . The universality, when it is present, means that exponents α and Δ_i , and scaling function f^* are independent of the details of the problem; that is, they are the same for a large class of phenomena of different nature.

5. Asymptotic Behavior from Real-Space RG

I am interested in the following PDE, solved by the dynamic variable $u(x, t)$,

$$\partial_t u = \partial^2 u + F(u, \partial u, \partial^2 u), \quad (d = 1). \quad (24)$$

I consider here the one-dimensional problem ($d = 1$), but the analysis I present may be extended to any space dimensionality d . In the above equation, ∂^2 represents the Laplace operator, and the “reaction” F is a nonlinear function of solution u and its first and second derivatives ∂u and $\partial^2 u$.

5.1. $F = 0$ Case. In this situation, the solution is trivial and is given by the Gaussian

$$u_0(x, t) = (4\pi t)^{-1/2} e^{-x^2/4t}. \quad (25)$$

Such a solution is exactly self-similar; that is,

$$u_0(Lx, L^2t) = L^{-1} u_0(x, t). \quad (26)$$

This corresponds to the quantities

$$\begin{aligned} a_1 &= 1, & a &= 2, & p &= -1, \\ \Delta &= \frac{a_1}{a} = \frac{1}{2}, & \alpha &= -\frac{2p}{a} = 1. \end{aligned} \quad (27)$$

On the other hand, solution (25) rewrites on the scaling form

$$\begin{aligned} u_0(x, t) &= t^{-1/2} f_0^*(xt^{-1/2}), \\ f_0^*(y) &= (4\pi)^{-1/2} e^{-y^2}. \end{aligned} \quad (28)$$

5.2. $F \neq 0$ Case. Generally, in this case, the asymptotic solution cannot be determined in an exact way. I propose that, in the limit $t \rightarrow \infty$, the solution of (24) has the asymptotic form

$$u(x, t) \sim t^{-\alpha/2} f^*(xt^{-1/2}), \quad t \rightarrow \infty. \quad (29)$$

Therefore, asymptotically, the solution $u(x, t)$ has the scaling-invariance property

$$u(Lx, L^2t) \sim L^{-\alpha} u(x, t), \quad t \rightarrow \infty. \quad (30)$$

The aim is then the search of this asymptotic solution applying real-space RG. This is precisely the objective of the next paragraph.

5.3. Renormalization Transformations. I start by the introduction of the function

$$u_L(x, t) = L^\alpha u(Lx, L^2t). \quad (31)$$

Here, $u(Lx, L^2t)$ is the solution at scale L . It is easy to see that

$$u_L(x, t) \rightarrow t^{-\alpha/2} f^*(xt^{-1/2}), \quad L \rightarrow \infty, \quad (32)$$

and that u_L is solution to the following PDE:

$$\partial_t u_L = \partial^2 u_L + F_L(u_L, \partial u_L, \partial \partial u_L), \quad (33)$$

with the nonlinearity at scale L

$$\begin{aligned} F_L(u_L, \partial u_L, \partial \partial u_L) \\ = L^{2+\alpha} F(L^{-\alpha} u_L, L^{1-\alpha} \partial u_L, L^{-2-\alpha} \partial \partial u_L). \end{aligned} \quad (34)$$

I start with the initial data: $f(x) = u(x, 1)$. It is the solution at the initial time $t = 1$. The set of all initial data, $\mathcal{S}$, is a Banach-space. By definition [15–17], a renormalization transformation (RT) is the application of $\mathcal{R}_L : \mathcal{S} \rightarrow \mathcal{S}$, such that

$$(\mathcal{R}_L f)(x) = u_L(x, 1) = L^\alpha u(Lx, L^2). \quad (35)$$

Thus, RT depends on scale L and the form of the nonlinearity F .

It is easy to convince that RT fulfils the semigroup property

$$\mathcal{R}_L \circ \mathcal{R}_{L'} = \mathcal{R}_{LL'}. \quad (36)$$

RG is then the set of all possible $\mathcal{R}_L$'s, with $L > 1$. If I apply n times a RT at scale L , then, I shall obtain

$$u(x, t) \sim L^{n\alpha} u(L^n x, L^{2n} t), \quad L \rightarrow \infty. \quad (37)$$

Taking into account the definition (35) of RT gives

$$\begin{aligned} u(x, t) &\sim t^{-\alpha/2} (\mathcal{R}_{L^n} f)(xt^{-1/2}) \\ &\longleftrightarrow u(x, t) \sim t^{-\alpha/2} f^*(xt^{-1/2}). \end{aligned} \quad (38)$$

This result must be compared to the asymptotic form (20a), and one finds that

$$\mathcal{R}_{L^n} f \rightarrow f^*, \quad L \rightarrow \infty. \quad (39)$$

The scaling function f^* then constitutes a *fixed point* of RT. In this limit, the nonlinearity F_L , at scale L , goes to the fixed value F^* . Therefore, it is sufficient to obtain the asymptotic solution u^* replacing in the initial PDE the nonlinearity F by F^* . One then writes $F = F^* + \delta F$, where the deformation δF informs on corrections to the leading asymptotic behavior.

Universality means the independence of exponent α and scaling function f^* on the initial data and the nature of the problem.

The next step consists in constructing some relevance criterion for the search of the time-asymptotic behavior of solution. To this end, we first recall the Gaussian fixed point.

5.4. Gaussian Fixed Point. In the absence of any reaction ($F = 0$), the solution is defined in relation (25). The initial data and the corresponding RT are given by

$$\begin{aligned} f_0(x) &= u_0(x, 1) = (4\pi)^{-1/2} e^{-x^2/4} = f_0^*(x), \\ (\mathcal{R}_L f_0)(x) &= u_L(x, 1) = Lu_0(Lx, L^2) = f_0(x), \quad (\alpha = 1). \end{aligned} \quad (40)$$

Then, the renormalization transformation $\mathcal{R}_L$ is *exact* and possesses a line of fixed points, namely, the multiples of $f_0^*(x)$, with

$$f_0^*(x) = (4\pi)^{-1/2} e^{-x^2/4}. \quad (41)$$

5.5. Relevance Criterion. I start from a general nonlinearity, which is a function that maps $\mathbb{C}^3$ into $\mathbb{C}$. I assume that this function is analytic in the vicinity of origin $(0, 0, 0)$. It will be sufficient to make a reasoning about only one monomial, $F(u, \partial u, \partial \partial u) = u^m (\partial u)^n (\partial \partial u)^p$. After a scaling transformation, one obtains

$$F \rightarrow F_L = L^{-d_F} F, \quad (42)$$

with the exponent

$$d_F = m + 2n + 3p - 3. \quad (43)$$

In fact, the analysis shall depend on the sign of this exponent, and one has the criterion

$$\begin{aligned} d_F > 0, & \quad \text{Irrelevant monomial,} \\ d_F = 0, & \quad \text{Marginal monomial,} \\ d_F < 0, & \quad \text{Relevant monomial.} \end{aligned} \quad (44)$$

5.5.1. Example 1: Heat Equation with Absorption. The associated PDE is [15–17]

$$\partial_t u = \partial^2 u - u^m. \quad (45)$$

The corresponding exponent is $d_F = m - 3$. Therefore, one has the criterion

$$\begin{aligned} m > 3, & \quad \text{Irrelevant monomial,} \\ m = 3, & \quad \text{Marginal monomial,} \\ m < 3, & \quad \text{Relevant monomial.} \end{aligned} \quad (46)$$

It should be noted that the general solution of (45) is exactly self-similar,

$$u(x, t) = t^{-1/(m-1)} f^*(xt^{-1/2}), \quad \alpha = \frac{1}{m-1}, \quad (47)$$

where the nontrivial fixed point f^* solves an ordinary differential equation,

$$f'' + \frac{1}{2} x f' + \frac{f}{p-1} - f^m = 0. \quad (48)$$

It has been shown that, asymptotically, one has [15–17]

$$f^*(x) \sim |x|^{-2/(m-1)}, \quad x \rightarrow \infty. \quad (49)$$

5.5.2. Example 2: Burgers Equation. This equation is such that

$$\partial_t u = \partial^2 u + (\partial_x u)^2, \quad d_F = 1. \quad (50)$$

The nonlinearity $(\partial_x u)^2$ is then irrelevant. Although this equation is nonlinear, it possesses, however, an exact solution (using the Hopf transformation) [15–17], which is

$$u(x, t) = At^{-1/2} f^*(xt^{-1/2}), \quad f^*(x) = \frac{Ae'(x)}{1 + Ae(x)}, \quad (51)$$

where A is a known amplitude and $e(x)$ denotes the error function [27]. The fixed-point $f^*(x)$ is then nontrivial, since it differs from the Gaussian one.

6. Asymptotic Solution from QFT

The starting point is the following PDE:

$$\partial_t \varphi = \nu_0 \partial^2 \varphi + V(\varphi, \partial \varphi, \partial \partial \varphi), \quad (52)$$

where the dynamic variable $\varphi(x, t)$ is a scalar field. Here, ν_0 is the diffusion coefficient, and V is a nonlinear function of this field and its first and second derivatives with respect to the space variable x . The above PDE must be completed by the initial condition

$$\varphi(x, 0) = \delta_d(x - x_0). \quad (53)$$

I set $G(x, t | x_0, 0) = \varphi(x, t)$. This function may be written as the double functional integral:

$$G(x, t | x_0, 0) = \int \mathcal{D}\varphi \int \mathcal{D}\tilde{\varphi} \varphi(x, t) e^{-S[\varphi, \tilde{\varphi}] + \tilde{\varphi}(x_0)}, \quad (54)$$

with the effective action

$$S[\varphi, \tilde{\varphi}] = \int d^d x \int dt \tilde{\varphi} [\partial_t \varphi - \nu_0 \partial^2 \varphi + V(\varphi, \partial \varphi, \partial \partial \varphi)]. \quad (55)$$

I shall suppose, thereafter, that the nonlinearity V has the polynomial form

$$V(\varphi, \partial \varphi, \partial \partial \varphi) = \nu_0 \sum_{m,n,p} g_{mnp} \varphi^m (\partial \varphi)^n (\partial \partial \varphi)^p, \quad (56)$$

where the coefficients of this series, g_{mnp} , represent the coupling constants.

The free propagator (with $V = 0$) is

$$G(x, t | x_0, 0) = \int \frac{d^d k}{(2\pi)^d} \int \frac{d\omega}{2\pi} \frac{e^{i(\omega t + k \cdot (x - x_0))}}{i\omega + \nu_0 k^2}. \quad (57)$$

Here, ω is the frequency and k is the d -dimensional wave-vector.

To select the relevant monomials of the above series, I shall make a reasoning using a naive powers counting.

6.1. Powers Counting. First, I introduce time and distance dimensions of an arbitrary X ,

$$[X] = L^{x_F} T^{y_F}, \quad (58)$$

where L and T are time and distance scales. So, I have

$$[\nu_0] = L^2 T^{-1}, \quad [\tilde{\varphi}] = L^0 T^0, \quad [\varphi] = L^{-d} T^0, \quad (59a)$$

$$[V] = L^{-d} T^{-1}, \quad [g_{mnp}] = L^{x_g} T^0, \quad (59b)$$

with the notation

$$x_g = m + 2p - 2 + d(m + n + p - 1). \quad (59c)$$

The dimension x_g suggests that the effective coupling constants are

$$\bar{g}_{mnp} = \frac{g_{mnp}}{L^{x_g}}. \quad (60)$$

Hence, the criterion of relevance is such that

$$\begin{aligned} x_g > 0, & \quad g_{mnp} \text{ is irrelevant,} \\ x_g = 0, & \quad g_{mnp} \text{ is marginal,} \\ x_g < 0, & \quad g_{mnp} \text{ is relevant.} \end{aligned} \quad (61)$$

I note that, for $d = 1$, I recover the relevance criterion presented in the last section.

For a given monomial, that is, at fixed (m, n, p) , the reasoning may be done in terms of space dimensionality d . To this end, I write x_g as follows:

$$x_g = (m + n + p - 1)(d - d_c), \quad (62a)$$

where

$$d_c = \frac{2 - m - 2p}{m + n + p - 1} \quad (62b)$$

accounts for the *critical dimension*. Then, I have an equivalent criterion

$$d < d_c, \quad g_{mnp} \text{ is relevant,} \quad (63a)$$

$$d = d_c, \quad g_{mnp} \text{ is marginal,} \quad (63b)$$

$$d > d_c, \quad g_{mnp} \text{ is irrelevant.} \quad (63c)$$

For example, for nonlinearities $V = \nu_0 g \varphi^m$ and $V = \nu_0 g \varphi \partial \varphi$, the respective critical dimensions are $d_c = 2/(m - 1)$ and $d_c = 1$.

6.2. Applied Renormalization Theory. Without further details concerning the application of the renormalization theory to extract the large-time behavior of solutions of PDEs, I simply draw the strategy as follows.

- (1) Make a perturbative expansion of the propagator G with respect to the coupling constants $\{g\}$, and appear short-distances divergences that present as poles in $\epsilon = d_c - d$.
- (2) Renormalize the theory; that is, $G(\{g\}, \epsilon) = Z(\{g\}, \epsilon) \times G_R(\{g_R\}, \epsilon)$, where G_R is the renormalized propagator, Z is the renormalization factor, and $\{g_R\}$ are the renormalized coupling constants.
- (3) Write, then, the RG-equation satisfied by G_R .
- (4) Its solution using the characteristics method is then the asymptotic behavior.

For the nonlinearity $V = \nu_0 g \varphi^m$, for instance, it has been shown that the solution is exactly self-similar [17]; that is,

$$\varphi(x, t) \sim (\nu_0 t)^{-1/(m-1)} f^*\left(\frac{x}{\sqrt{\nu_0 t}}\right). \quad (64)$$

7. Concluding Remarks

In this paper, I presented a pedagogical review of the application of the renormalization techniques to extract the large-time solutions of PDEs.

I state that the same techniques may be applied to coupled PDEs, that is, with more than one dynamic variable. In this case, the corresponding field theory is constructed with several fields $(\varphi_1, \dots, \varphi_n)$ ($n \geq 1$). This means that one has n kinds of populations (e.g., chemotaxis and morphogenesis in biology) that move and react. The associated PDEs are

$$\partial_t \varphi_i = \nu_i \partial^2 \varphi_i + V_i(\varphi_j, \partial \varphi_j, \partial \partial \varphi_j), \quad i, j = 1, \dots, n. \quad (65)$$

Here, ν_i stands for the diffusion coefficient of species i . I easily show that the associated effective action is

$$S[\varphi_i, \tilde{\varphi}_i] = \sum_{j=1}^n \int d^d x \int dt \tilde{\varphi}_j [\partial_t \varphi_j - \nu_j \partial^2 \varphi_j + V_j(\varphi_i, \partial \varphi_i, \partial \partial \varphi_i)], \quad (66)$$

which generalizes that defined in relationship (55). As before, it is assumed that V_j 's are polynomials in φ_i 's fields and their derivatives $\partial \varphi_i$ and $\partial \partial \varphi_i$.

PDEs with fractional time are also of interest, which describe populations that move and react in random media (or fractals). These equations are as follows:

$$\tau^{\omega-1} \partial_t^\omega \varphi_i = \nu_i \partial^2 \varphi_i + V_i(\varphi_j, \partial \varphi_j, \partial \partial \varphi_j), \quad i, j = 1, \dots, n, \quad (67)$$

with $0 < \omega < 1$ and the (arbitrary) time-scale τ . The quantity ω should not be confused with frequency. Here, the time-fractional derivative is defined by

$$\partial_t^\omega \varphi_i(x, t) = \frac{1}{\Gamma(-\omega)} \int_0^t \frac{\varphi_i(x, s)}{(t-s)^{\omega+1}} ds, \quad (68)$$

where $\Gamma(z)$ denotes the Euler *gamma* function [27]. I show that the associated effective action reads

$$S[\varphi_i, \tilde{\varphi}_i] = \sum_{j=1}^n \int d^d x \int dt \tilde{\varphi}_j [\tau^{\omega-1} \partial_t^\omega \varphi_j - \nu_j \partial^2 \varphi_j + V_j(\varphi_i, \partial \varphi_i, \partial \partial \varphi_i)]. \quad (69)$$

The application of RG to this kind of the problem will be presented elsewhere.

I emphasize that certain diffusion-reaction processes necessitate the introduction of a noise. More exalt, the associated PDEs present as

$$\partial_t \varphi_i = \nu_i \partial^2 \varphi_i + V_i(\varphi_j, \partial \varphi_j, \partial \partial \varphi_j) + \eta_i(x, t), \quad i, j = 1, \dots, n. \quad (70)$$

Here, I suppose that the noise is Gaussian; that is,

$$\begin{aligned} \langle \eta_i(x, t) \rangle &= 0, \\ \langle \eta_i(x, t) \eta_j(x', t') \rangle &= \nu_0 \delta_{ij} \delta(t-t') \delta_d(x-x') \quad (\nu_i = \nu_0). \end{aligned} \quad (71)$$

The above analysis may extend without difficulty to these kinds of situations.

Finally, I note that RG machineries presented in this review article may be extended to some class of elliptic partial differential equations [15–17].

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