Research Article

Some Inclusion Relations Associated with Generalized Fractional Differintegral Operator

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A generalized fractional differintegral operator is used to define some new subclasses of analytic functions in the open unit disk $\mathbb{U}$. For each of these new function classes, several inclusion relationships are established.

1. Introduction and Definitions

Let $\mathbb{A}$ be the class of normalized functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in D, |z| < 1\}$. If $f \in \mathbb{A}$ is given by (1) and $g \in \mathbb{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in $z \in \mathbb{U}$, then the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \quad (2)$$

Let $P_k(\alpha)$ denote the class of functions $h(z)$ analytic in the unit disk $\mathbb{U}$, satisfying the properties $h(0) = 1$ and

$$\int_0^{2\pi} \left| \Re \left(\frac{h(z) - \alpha}{1 - \alpha} \right) \right| d\theta \leq k\pi \quad (z = re^{i\theta}; 0 \leq \alpha < 1; k \geq 2). \quad (3)$$

This class $P_k(\alpha)$ has been introduced in [1]. Note that, for $\alpha = 0$, we obtain the class P_k defined and studied in [2], and for $k = 2$, we have the class $P(\alpha)$ of functions with positive real part greater than α . In particular, $P(0)$ is the class P of functions

with positive real part. From (3), we can easily deduce that $h \in P_k(\alpha)$ if and only if

$$h(z) = \left(\frac{k}{4} + \frac{1}{2} \right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2} \right) h_2(z), \quad h_1, h_2 \in P(\alpha). \quad (4)$$

Following the recent investigations [3] (see also [4, 5]), we have the following subclasses:

$$R_k(\alpha) = \left\{ f \in A : \frac{zf'(z)}{f(z)} \in P_k(\alpha), z \in \mathbb{U} \right\},$$

$$V_k(\alpha) = \left\{ f \in A : \frac{(zf'(z))'}{f'(z)} \in P_k(\alpha), z \in \mathbb{U} \right\},$$

$$P'_k(\alpha) = \{ f \in A : f'(z) \in P_k(\alpha), z \in \mathbb{U} \},$$

$$T_k(\beta, \alpha) = \left\{ f \in A : g \in R_2(\alpha), \frac{zf'(z)}{g(z)} \in P_k(\beta), z \in \mathbb{U} \right\},$$

$$T_k^*(\beta, \alpha)$$

$$= \left\{ f \in A : g \in V_2(\alpha), \frac{(zf'(z))'}{g'(z)} \in P_k(\beta), z \in \mathbb{U} \right\}. \quad (5)$$

We note that the class $R_2(\alpha) = S^*(\alpha)$ and $V_2(\alpha) = \mathcal{K}(\alpha)$ are, respectively, the subclasses of $\mathbb{A}$, consisting of functions which are starlike of order α and convex of order α in $\mathbb{U}$. The class $T_2^*(\beta, \alpha) = C^*(\beta, \alpha)$ was considered by Noor [6], and $T_2^*(0, 0) = C^*$ is the class of quasiconvex univalent functions which was first introduced and studied in [7]. It can be easily seen from the above definition that

$$f(z) \in V_k(\alpha) \iff zf'(z) \in R_k(\alpha), \quad (6)$$

$$f(z) \in T_k^*(\beta, \alpha) \iff zf'(z) \in T_k(\beta, \alpha). \quad (7)$$

We recall here the following family of generalized fractional integral operators due to Srivastava et al. [8] and generalized fractional derivative operators due to Raina and Nahar [9] (see also [10, 11]).

Definition 1. Let $\lambda > 0$ and $\mu, \eta \in \mathbb{R}$. Then the fractional integral operator $\mathcal{I}_{0,z}^{\lambda, \mu, \eta}$ is defined by

$$\begin{aligned} \mathcal{I}_{0,z}^{\lambda, \mu, \eta} f(z) &= \frac{z^{-\lambda-\mu}}{\Gamma(\lambda)} \int_0^z (z-\zeta)^{\lambda-1} {}_2F_1\left(\lambda+\mu, -\eta; \lambda; 1-\frac{\zeta}{z}\right) f(\zeta) d\zeta, \end{aligned} \quad (8)$$

where the function $f(z)$ is analytic in a simply connected region of the complex z -plane containing the origin, with the order

$$f(z) = O(|z|^\varepsilon) \quad (z \rightarrow 0; \varepsilon > \max\{0, \mu - \eta\} - 1), \quad (9)$$

and the multiplicity $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 2. For $m-1 \leq \lambda < m$ and $\mu, \eta \in \mathbb{R}$ and $m \in \mathbb{N}$, the fractional derivative operator $\mathcal{J}_{0,z}^{\lambda, \mu, \eta}$ is defined by

$$\begin{aligned} \mathcal{J}_{0,z}^{\lambda, \mu, \eta} f(z) &= \frac{d^m}{dz^m} \left(\frac{z^{\lambda-\mu}}{\Gamma(m-\lambda)} \right. \\ &\quad \times \int_0^z (z-\zeta)^{m-\lambda-1} {}_2F_1\left(\mu-\lambda, m-\eta; m-\lambda; \right. \\ &\quad \left. \left. 1-\frac{\zeta}{z}\right) f(\zeta) d\zeta \right), \end{aligned} \quad (10)$$

where the multiplicity $(z-\zeta)^{m-\lambda}$ is removed as in the above definition.

The operators $\mathcal{I}_{0,z}^{\lambda, \mu, \eta}$ and $\mathcal{J}_{0,z}^{\lambda, \mu, \eta}$ include the Riemann-Liouville and Erdelyi-Kober operators of fractional calculus (see, e.g., [10]). Using the hypotheses of Definitions 1 and 2 the generalized fractional differintegral operator

$$\mathcal{S}_{0,z}^{\lambda, \mu, \eta} : \mathbb{A} \longrightarrow \mathbb{A} \quad (11)$$

is defined by (see also [12–15])

$$\mathcal{S}_{0,z}^{\lambda, \mu, \eta} f(z) = \begin{cases} \frac{\Gamma(2-\mu)\Gamma(2+\eta-\lambda)}{\Gamma(2+\eta-\mu)} z^\mu \mathcal{I}_{0,z}^{\lambda, \mu, \eta} f(z), & (0 \leq \lambda < \eta+2), \\ \frac{\Gamma(2-\mu)\Gamma(2+\eta-\lambda)}{\Gamma(2+\eta-\mu)} z^\mu \mathcal{J}_{0,z}^{-\lambda, \mu, \eta} f(z), & (\lambda < 0), \end{cases} \quad (12)$$

$(z \in \mathbb{U}, \mu, \eta \in \mathbb{R}, \min\{-\mu, \eta-\lambda, \eta-\mu\} > -2).$

It is easily seen from (12) that for a function f of the form (1), we have

$$\begin{aligned} \mathcal{S}_{0,z}^{\lambda, \mu, \eta} f(z) &= z + \frac{\Gamma(2-\mu)\Gamma(2+\eta-\lambda)}{\Gamma(2+\eta-\mu)} \\ &\quad \times \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(n+\eta-\mu+1)}{\Gamma(n-\mu+1)\Gamma(n+\eta-\lambda+1)} a_n z^n \\ &= {}_3F_2(1, 2, 2+\eta-\mu; 2-\mu, 2+\eta-\lambda; z) * f(z). \end{aligned} \quad (13)$$

We note that the operator $\mathcal{S}_{0,z}^{\lambda, \mu, \eta}$ is a generalization of many other operators, for example,

$$\begin{aligned} \text{(i)} \quad \mathcal{S}_{0,z}^{0,0,0} f(z) &= f(z), \\ \text{(ii)} \quad \mathcal{S}_{0,z}^{1,1,1} f(z) &= \mathcal{S}_{0,z}^{1,0,0} f(z) = zf'(z), \\ \text{(iii)} \quad \mathcal{S}_{0,z}^{2,1,1} f(z) &= zf'(z) + z^2 f''(z), \\ \text{(iv)} \quad \mathcal{S}_{0,z}^{\lambda, \lambda, \eta} f(z) &= \mathcal{S}_{0,z}^{\lambda, \mu, 0} f(z), \\ \text{(v)} \quad \mathcal{S}_{0,z}^{-\lambda, 0, \alpha-1} f(z) &= \Omega_\alpha^\lambda f(z) \\ &= \left(\frac{\lambda+\alpha}{\alpha} \right) \frac{\lambda}{z^\alpha} \\ &\quad \times \int_0^z \zeta^{\alpha-1} \left(1 - \frac{\zeta}{z} \right)^{\lambda-1} f(\zeta) d\zeta, \end{aligned} \quad (14)$$

which is a multiplier transformation operator Ω_α^λ studied by Jung et al. [16],

$$\begin{aligned} \text{(vi)} \quad \mathcal{S}_{0,z}^{-1,0,\alpha-1} f(z) &= \mathcal{J}_\alpha f(z) \\ &= \frac{\alpha+1}{z^\alpha} \int_0^z \zeta^{\alpha-1} f(\zeta) d\zeta \quad (\alpha > -1) \end{aligned} \quad (15)$$

which is the generalized Bernardi-Libra-Livingston integral operator. It can easily be verified from (12) that

$$\begin{aligned} z \left(\mathcal{S}_{0,z}^{\lambda, \mu, \eta} f(z) \right)' &= (\eta - \lambda + 1) \mathcal{S}_{0,z}^{\lambda+1, \mu, \eta} f(z) - (\eta - \lambda) \mathcal{S}_{0,z}^{\lambda, \mu, \eta} f(z). \end{aligned} \quad (16)$$

For the generalized fractional differintegral operator $\mathcal{S}_{0,z}^{\lambda,\mu,\eta}$, we now define the following subclasses of $\mathbb{A}$.

Definition 3. Let $f(z) \in \mathbb{A}$. Then $f(z) \in R^{\lambda,\mu,\eta}(k, \alpha)$ if and only if $\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z) \in R_k(\alpha)$, for $z \in \mathbb{U}$.

Definition 4. Let $f(z) \in \mathbb{A}$. Then $f(z) \in V^{\lambda,\mu,\eta}(k, \alpha)$ if and only if $\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z) \in V_k(\alpha)$, for $z \in \mathbb{U}$.

Definition 5. Let $f(z) \in \mathbb{A}$. Then $f(z) \in T^{\lambda,\mu,\eta}(k, \beta, \alpha)$ if and only if $\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z) \in T_k(\beta, \alpha)$, for $z \in \mathbb{U}$.

Definition 6. Let $f(z) \in \mathbb{A}$. Then $f(z) \in T_*^{\lambda,\mu,\eta}(k, \beta, \alpha)$ if and only if $\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z) \in T_k^*(\beta, \alpha)$, for $z \in \mathbb{U}$.

Definition 7. Let $f(z) \in \mathbb{A}$. Then $f(z) \in P'_k(\lambda, \mu, \eta, \alpha)$ if and only if $\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z) \in P'_k(\alpha)$, for $z \in \mathbb{U}$.

In this paper we establish some inclusion relationships and some other interesting properties for these subclasses.

2. Main Inclusion Relationships

We recall first the following necessary lemma.

Lemma 8 (see [17]). Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$, and let $\phi(u, v)$ be a complex-valued function satisfying the conditions:

- (i) $\phi(u, v)$ is continuous in $D \subset \mathbb{C}^2$,
- (ii) $(1, 0) \in D$ and $\Re \phi(1, 0) > 0$,
- (iii) $\Re \phi(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq -(1/2)(1 + u_2^2)$.

If $h(z) = 1 + \sum_{n=2}^{\infty} c_n z^n$ is a function analytic in $\mathbb{U}$ such that $(h(z), zh'(z)) \in D$ and $\Re e(\phi(h(z), zh'(z))) > 0$, for $z \in \mathbb{U}$, then $\Re e(h(z)) > 0$ for $z \in \mathbb{U}$.

Our first main inclusion relationship is given by the theorem below.

Theorem 9. Let $f \in \mathbb{A}$, $0 \leq \alpha < 1$, $\eta - \lambda > 1/8$, and $\min\{-\mu, \eta - \mu\} > -2$; then

$$R^{\lambda+1,\mu,\eta}(k, \alpha) \subset R^{\lambda,\mu,\eta}(k, \alpha_1), \quad (17)$$

where

$$\begin{aligned} \alpha_1 &= 2(2\alpha\eta - 2\alpha\lambda + 1) \\ &\times \left((2(\eta - \lambda - \alpha) + 1) \right. \\ &\quad \left. + \sqrt{4(\eta - \lambda + \alpha)^2 + 4(\eta - \lambda - \alpha) + 9} \right)^{-1} \end{aligned} \quad (18)$$

and $0 \leq \alpha < \alpha_1 < 1$.

Proof. Let $f \in R^{\lambda+1,\mu,\eta}(k, \alpha)$. Then upon setting

$$\begin{aligned} \frac{z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z))'}{\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)} &= p(z) \\ &= \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \end{aligned} \quad (19)$$

$(z \in \mathbb{U}),$

we see that the function $p(z)$ is analytic in $\mathbb{U}$, with $p(0) = 1$ in $z \in \mathbb{U}$. Using the identity (16) in (19) and differentiating with respect to z , we get

$$\begin{aligned} \frac{z(\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} f(z))'}{\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} f(z)} &= \left(p(z) + \frac{zp'(z)}{p(z) + \eta - \lambda} \right) \in P_k(\alpha) \quad (z \in \mathbb{U}). \end{aligned} \quad (20)$$

Let

$$\phi(z) = \sum_{j=1}^{\infty} \frac{\eta - \lambda + j}{\eta - \lambda + 1} z^j. \quad (21)$$

Then, by convolution technique, (see [18]), we have

$$\begin{aligned} p(z) * \frac{\phi(z)}{z} &= p(z) + \frac{zp'(z)}{p(z) + \eta - \lambda} \\ &= \left(\frac{k}{4} + \frac{1}{2}\right) \left(p_1(z) * \frac{\phi(z)}{z} \right) \\ &\quad - \left(\frac{k}{4} - \frac{1}{2}\right) \left(p_2(z) * \frac{\phi(z)}{z} \right) \end{aligned} \quad (22)$$

and this implies that

$$\left(p_i(z) + \frac{zp'_i(z)}{p_i(z) + \eta - \lambda} \right) \in P(\alpha) \quad (z \in \mathbb{U}, i = 1, 2). \quad (23)$$

We want to show that $p_i(z) \in P(\alpha_1)$, where α_1 is given by (18), and this will show that $p(z) \in P_k(\alpha_1)$ for $z \in \mathbb{U}$. Let

$$p_i(z) = (1 - \alpha_1) h_i(z) + \alpha_1 \quad (z \in \mathbb{U}, i = 1, 2). \quad (24)$$

Then in view of (23) and (24), we obtain, for $z \in \mathbb{U}$, $i = 1, 2$

$$\begin{aligned} \Re \left((1 - \alpha_1) h_i(z) + \alpha_1 - \alpha + \frac{(1 - \alpha_1) zh'_i(z)}{(1 - \alpha_1) h_i(z) + \alpha_1 + \eta - \lambda} \right) \\ > 0. \end{aligned} \quad (25)$$

We now form a function $\phi(u, v)$ by choosing $u = h_i(z)$ and $v = zh'_i(z)$ in (25). Thus

$$\phi(u, v) = (1 - \alpha_1) u + \alpha_1 - \alpha + \frac{(1 - \alpha_1) v}{(1 - \alpha_1) u + \alpha_1 + \eta - \lambda}. \quad (26)$$

We can easily see that the first two conditions of Lemma 8 are easily satisfied as $\phi(u, v)$ is continuous in $D = \mathbb{C} - (-\alpha_1 + \eta - \lambda)/(1 - \alpha_1) \times \mathbb{C}$, $(1, 0) \in D$, and $\Re\{\phi(1, 0)\} > 0$. Now for $v_1 \leq -(1/2)(1 + u_2^2)$ we obtain

$$\begin{aligned} & \Re\{\phi(iu_2, v_1)\} \\ &= \Re\left((1 - \alpha_1)iu_2 + \alpha_1 - \alpha + \frac{(1 - \alpha_1)v_1}{(1 - \alpha_1)iu_2 + \alpha_1 + \eta - \lambda}\right) \\ &= \alpha_1 - \alpha + \frac{(1 - \alpha_1)v_1\{\alpha_1 + \eta - \lambda\}}{(\alpha_1 + \eta - \lambda)^2 + (1 - \alpha_1)^2 u_2^2} \\ &\leq \alpha_1 - \alpha - \frac{1}{2} \frac{(1 - \alpha_1)(\alpha_1 + \eta - \lambda)(1 + u_2^2)}{(\alpha_1 + \eta - \lambda)^2 + (1 - \alpha_1)^2 u_2^2} \\ &= \frac{A + Bu_2^2}{2C}, \end{aligned} \quad (27)$$

where $A = (\alpha_1 + \eta - \lambda)\{2(\alpha_1 - \alpha)(\alpha_1 + \eta - \lambda) - (1 - \alpha_1)\}$, $B = (1 - \alpha_1)\{2(\alpha_1 - \alpha)(1 - \alpha_1) - (\alpha_1 + \eta - \lambda)\}$, and $C = (\alpha_1 + \eta - \lambda)^2 + (1 - \alpha_1)^2 u_2^2 > 0$.

We note that $\Re\{\phi(iu_2, v_1)\} \leq 0$ if and only if $A \leq 0$ and $B \leq 0$. From $A \leq 0$, we obtain α_1 as given by (18), and $B \leq 0$ gives us $0 \leq \alpha_1 < 1$. This completes the proof. $\square$

Theorem 10. Let $f \in \mathbb{A}$, $0 \leq \alpha < 1$, $\eta - \lambda > 1/8$, and $\min\{-\mu, \eta - \mu\} > -2$; then

$$V^{\lambda+1, \mu, \eta}(k, \alpha) \subset V^{\lambda, \mu, \eta}(k, \alpha_1), \quad (28)$$

where α_1 is given by (18).

Proof. To prove the inclusion relationship, we observe (in view of Theorem 9) that

$$\begin{aligned} f(z) \in V^{\lambda+1, \mu, \eta}(k, \alpha) &\iff zf'(z) \in R^{\lambda+1, \mu, \eta}(k, \alpha) \\ &\implies zf'(z) \in R^{\lambda, \mu, \eta}(k, \alpha_1) \\ &\iff f(z) \in V^{\lambda, \mu, \eta}(k, \alpha_1), \end{aligned} \quad (29)$$

which establishes Theorem 10. $\square$

Theorem 11. Let $f \in \mathbb{A}$, $0 \leq \alpha, \beta < 1$, $\eta - \lambda > 1/8$, and $\min\{-\mu, \eta - \mu\} > -2$; then

$$T^{\lambda+1, \mu, \eta}(k, \beta, \alpha) \subset T^{\lambda, \mu, \eta}(k, \beta_1, \alpha_1), \quad (30)$$

where α_1 is given by (18) and $\beta < \beta_1 < 1$ is defined in the proof.

Proof. Let $f(z) \in T^{\lambda+1, \mu, \eta}(k, \beta, \alpha)$. Then there exists $g(z) \in R^{\lambda+1, \mu, \eta}(2, \alpha)$ such that

$$\frac{z(\mathcal{S}_{0,z}^{\lambda+1, \mu, \eta} f(z))'}{\mathcal{S}_{0,z}^{\lambda+1, \mu, \eta} g(z)} \in P_k(\beta) \quad (z \in \mathbb{U}, 0 \leq \beta < 1). \quad (31)$$

Since $R^{\lambda+1, \mu, \eta}(2, \alpha) \subset R^{\lambda, \mu, \eta}(2, \alpha_1)$, therefore we have

$$\begin{aligned} \frac{z(\mathcal{S}_{0,z}^{\lambda, \mu, \eta} g(z))'}{\mathcal{S}_{0,z}^{\lambda, \mu, \eta} g(z)} &= (1 - \alpha_1)q(z) + \alpha_1 \\ &= (1 - \alpha_1)(q_1 + iq_2) + \alpha_1, \end{aligned} \quad (32)$$

where $q(z) \in P$. Next consider,

$$\frac{z(\mathcal{S}_{0,z}^{\lambda, \mu, \eta} f(z))'}{\mathcal{S}_{0,z}^{\lambda, \mu, \eta} g(z)} = (1 - \beta_1)h(z) + \beta_1 \quad (33)$$

and proceed on the same technique of Theorem 9 (see also [5]); we obtain $f(z) \in T^{\lambda, \mu, \eta}(k, \beta_1, \alpha_1)$, where

$$\begin{aligned} \beta_1 &= (2\beta [((1 - \alpha_1)q_1 + \alpha_1 + \eta - \lambda)^2 + (1 - \alpha_1)^2 q_2^2] \\ &\quad + [(1 - \alpha_1)q_1 + \alpha_1 + \eta - \lambda]) \\ &\quad \times (2 [((1 - \alpha_1)q_1 + \alpha_1 + \eta - \lambda)^2 + (1 - \alpha_1)^2 q_2^2] \\ &\quad + [(1 - \alpha_1)q_1 + \alpha_1 + \eta - \lambda])^{-1}. \end{aligned} \quad (34)$$

$\square$

In view of relation (7), an application of Theorem 11 is obtained as follows.

Theorem 12. Let $f \in \mathbb{A}$, $0 \leq \alpha, \beta < 1$, $\eta - \lambda > 1/8$, and $\min\{-\mu, \eta - \mu\} > -2$; then

$$T_*^{\lambda+1, \mu, \eta}(k, \beta, \alpha) \subset T_*^{\lambda, \mu, \eta}(k, \beta_1, \alpha_1), \quad (35)$$

where β_1 and α_1 are as in Theorem 11.

Lemma 13 (see [19]). Let $p(z)$ be analytic in $\mathbb{U}$ with $p(0) = 1$ and $\Re p(z) > 0$, $z \in \mathbb{U}$. Then, for $s > 0$ and $\eta_1 \neq -1$ (complex),

$$\Re \left(p(z) + \frac{szp'(z)}{p(z) + \eta_1} \right) > 0, \quad \text{for } |z| < r_0, \quad (36)$$

where r_0 is given by $r_0 = |1 + \eta_1|/\sqrt{m + (m^2 - |\eta_1^2 - 1|)^{1/2}}$ and $m = 2(s+1)^2 + |\eta_1|^2 - 1$, and this result is the best possible.

Theorem 14. Let $z \in \mathbb{U}$, $f \in R^{\lambda, \mu, \eta}(k, 0)$. Then $f \in R^{\lambda+1, \mu, \eta}(k, 0)$ for $|z| < r_0$, where

$$r_0 = \frac{\eta - \lambda + 1}{\sqrt{m + \sqrt{m^2 - |(\eta - \lambda)^2 - 1|}}}, \quad m = 7 + (\eta - \lambda)^2. \quad (37)$$

This radius is exact.

Proof. Let

$$\frac{z(\mathcal{S}_{0,z}^{\lambda, \mu, \eta} f(z))'}{\mathcal{S}_{0,z}^{\lambda, \mu, \eta} f(z)} = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right)p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)p_2(z), \quad (38)$$

where $p \in P_k$ and $p_1, p_2 \in P$ in $\mathbb{U}$. Using a similar argument as in Theorem 9, we obtain

$$\begin{aligned} \frac{z(\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} f(z))'}{\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} f(z)} &= p(z) + \frac{zp'(z)}{\eta - \lambda + p(z)} \\ &= \left(\frac{k}{4} + \frac{1}{2}\right) \left(p_1(z) + \frac{zp_1'(z)}{p_1(z) + \eta - \lambda}\right) \\ &\quad - \left(\frac{k}{4} - \frac{1}{2}\right) \left(p_2(z) + \frac{zp_2'(z)}{p_2(z) + \eta - \lambda}\right). \end{aligned} \quad (39)$$

Applying Lemma 13, we get

$$\Re \left(p_i(z) + \frac{zp_i'(z)}{p_i(z) + \eta - \lambda} \right) > 0 \quad \text{for } |z| < r_0, \quad (40)$$

where r_0 is given by (37). This completes our proof. $\square$

Lemma 15 (see [20]). Let p be analytic in $\mathbb{E}$ with $p(0) = 1$, and δ is a complex number satisfying $\Re \delta \geq 0$, ($\delta \neq 0$); then $\Re[p(z) + \delta zp'(z)] > \beta$, ($0 \leq \beta < 1$) implies $\Re p(z) > \beta + (1 - \beta)(2\gamma - 1)$, where γ is given by

$$\gamma = \int_0^1 (1 + t^{\Re \delta})^{-1} dt, \quad (41)$$

which is an increasing function of $\Re \delta$ and $1/2 \leq \gamma \leq 1$. The estimate is sharp in the sense that bound cannot be improved.

Theorem 16. Let $\min\{\eta - \lambda - 1, -\mu, \eta - \mu\} > -2$. Then

$$P'_k(\lambda + 1, \mu, \eta, \alpha) \subset P'_k(\lambda, \mu, \eta, \alpha + (1 - \alpha)(2\gamma - 1)), \quad (42)$$

where

$$\gamma = \int_0^1 (1 + t^{1/(\eta - \lambda + 1)})^{-1} dt, \quad (43)$$

which is an increasing function of $1/(\eta - \lambda + 1)$ and $1/2 \leq \gamma < 1$.

Proof. Let $f(z) \in P'_k(\lambda + 1, \mu, \eta, \alpha)$

$$(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z))' = H(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z), \quad (44)$$

where $H(z)$ is analytic and $H(0) = 1$ in $\mathbb{U}$. Identity (16) gives us

$$\begin{aligned} (\mathcal{S}_{0,z}^{\lambda+1,\mu,\eta} f(z))' &= H(z) + \frac{zH'(z)}{\eta - \lambda + 1} \\ &= \left(\frac{k}{4} + \frac{1}{2}\right) \left(h_1(z) + \frac{zh_1'(z)}{\eta - \lambda + 1}\right) \\ &\quad - \left(\frac{k}{4} - \frac{1}{2}\right) \left(h_2(z) + \frac{zh_2'(z)}{\eta - \lambda + 1}\right). \end{aligned} \quad (45)$$

This implies that

$$\Re \left(h_i(z) + \frac{zh_i'(z)}{\eta - \lambda + 1} \right) > \alpha, \quad i = 1, 2. \quad (46)$$

Now using Lemma 15, we get the desired result. $\square$

Lemma 17 (see [21]). Let ϕ be convex and g starlike in $\mathbb{U}$. Then, for F analytic in $\mathbb{U}$ with $F(0) = 1$, $(\Phi * Fg)/(\Phi * g)$ is contained in the convex hull of $F(\mathbb{U})$. By convex hull of a set X , one means the intersection of all convex sets that contain X .

Theorem 18. Let ϕ be a convex function and $f \in R^{\lambda,\mu,\eta}(2, \alpha)$. Then $\phi * f \in R^{\lambda,\mu,\eta}(2, \alpha)$.

Proof. Let $G = \phi * f$, and let

$$\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad (47)$$

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Then

$$\begin{aligned} \mathcal{S}_{0,z}^{\lambda,\mu,\eta} G(z) &= \mathcal{S}_{0,z}^{\lambda,\mu,\eta} \left[z + \sum_{n=2}^{\infty} a_n b_n z^n \right] \\ &= z + \frac{\Gamma(2 - \mu) \Gamma(2 + \eta - \lambda)}{\Gamma(2 - \mu + \eta)} \\ &\quad \times \sum_{n=2}^{\infty} \frac{\Gamma(n + 1) \Gamma(n - \mu + \eta + 1)}{\Gamma(n - \mu + 1) \Gamma(n + \eta - \lambda + 1)} a_n b_n z^n \\ &= (\phi * \mathcal{S}_{0,z}^{\lambda,\mu,\eta} f)(z). \end{aligned} \quad (48)$$

Also, $f(z) \in R^{\lambda,\mu,\eta}(2, \alpha)$. Therefore, $\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z) \in R_2(\alpha) = S^*(\alpha)$. By logarithmic differentiation of (48), we have

$$\frac{z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} G(z))'}{\mathcal{S}_{0,z}^{\lambda,\mu,\eta} G(z)} = \frac{\phi(z) * F \mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)}{\phi(z) * \mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)}, \quad (49)$$

where

$$F(z) = \frac{z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z))'}{\mathcal{S}_{0,z}^{\lambda,\mu,\eta} f(z)} \quad (50)$$

is analytic in $\mathbb{U}$, and $F(0) = 1$. From Lemma 17, we see that $z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} G(z))'/\mathcal{S}_{0,z}^{\lambda,\mu,\eta} G(z)$ is contained in the convex hull of $F(\mathbb{U})$. Since $z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} G(z))'/\mathcal{S}_{0,z}^{\lambda,\mu,\eta} G(z)$ is analytic in $\mathbb{U}$ and $F(\mathbb{U}) \subset \Omega = \{W : (z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} W(z))')/(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} W(z)) \in P_2(\alpha)\}$, then $z(\mathcal{S}_{0,z}^{\lambda,\mu,\eta} G(z))'/\mathcal{S}_{0,z}^{\lambda,\mu,\eta} G(z)$ lies in Ω . This implies that $G = \phi * f \in R^{\lambda,\mu,\eta}(2, \alpha)$. $\square$

This is application of Theorem 18.

Theorem 19. The class $R^{\lambda, \mu, \eta}(2, \alpha)$ is invariant under the following differintegral operators. That is if $f \in R^{\lambda, \mu, \eta}(2, \alpha)$, then so does f_i , where f_i are given to us:

- (i) $f_1(z) = \int_0^z (f(t)/t)dt,$
- (ii) $f_2(z) = (2/z) \int_0^z f(t)dt,$
- (iii) $f_3(z) = \int_0^z ((f(t) - f(xt))/(t - xt))dt, |x| \leq 1, x \neq 1,$
- (iv) $f_4(z) = ((1+c)/z^c) \int_0^z t^{c-1} f(t)dt, \operatorname{Re}(c) > 0.$

The proof immediately follows from Theorem 18. Since we can write $f_i = f * \phi_i$ with

$$\begin{aligned}\phi_1(z) &= -\log(1-z), \\ \phi_2(z) &= -2 \left[\frac{z + \log(1-z)}{z} \right], \\ \phi_3(z) &= \frac{1}{1-x} \log \left(\frac{1-xz}{1-z} \right), \\ \phi_4(z) &= \sum_{m=1}^{\infty} \frac{1+c}{m+c} z^m \quad \operatorname{Re} c > 0,\end{aligned}\tag{51}$$

each ϕ_i is convex for $i = 1, 2, 3$, and 4.

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