Hindawi Publishing Corporation
ISRN Applied Mathematics
Volume 2013, Article ID 710643, 11 pages
http://dx.doi.org/10.1155/2013/710643

Research Article

Global Stability of an SEIS Epidemic Model with General Saturation Incidence

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Received 18 February 2013; Accepted 11 March 2013

Academic Editors: J. R. Fernandez, C. Lu, E. Skubalska-Rafajlowicz, and F. Tadeo

We present an SEIS epidemic model with infective force in both latent period and infected period, which has different general saturation incidence rates. It is shown that the global dynamics are completely determined by the basic reproductive number R_0. If $R_0 \leq 1$, the disease-free equilibrium is globally asymptotically stable in T by LaSalle's Invariance Principle, and the disease dies out. Moreover, using the method of autonomous convergence theorem, we obtain that the unique epidemic equilibrium is globally asymptotically stable in T^0, and the disease spreads to be endemic.

1. Introduction

Epidemiology is the study of hot spots of the spread of infectious disease, with the objective to trace factors that contribute to their occurrence. Mathematical epidemiology models describing the population dynamics of infectious diseases have been playing an important role in better understanding of epidemiological patterns and disease control for a long time. Epidemiological models are now widely used as more epidemiologists realize the role that modeling can play in basic understanding and policy development. In recent years, many epidemiological models of ordinary differential equations have been studies by a number of authors [1–4].

The most general form of an epidemiological model is an SEIRS model consisting of four population subclasses: S-susceptible, E—exposed, I—infected, and R—recovered. All other models are limiting cases of the SEIRS model under various parameter restrictions. If there is no immunity and hence no R class, the SEIS model is obtained, which can be regarded when the average period of immunity tends to zero.

Many epidemic models with the infectious force in the latent period have been performed. Guihua and Zhen [5, 6] studied global stability of an SEI model with general incidence or standard incidence. Mukhopadhyay and Bhattacharyya [7] discussed global stability of an SEIS model with standard incidence. Global dynamics of an SEI model with acute and chronic stages were given by Yuan and Yang [8].

Incidence rate plays a very important role in the research of epidemiological models. Comparing with bilinear and standard incidence rate, saturating incidence rate may be more suitable for our real word, which should generally be written as $\beta C(N)SI/N$, where N is the total population size. Michaelis and Menten combined the two previous approaches by assuming that if the number of available partners N is low, the number of actual per capita partners $C(N)$ is proportional to N whereas if the number of available partners is large, there is a saturation effect which makes the number of actual partners constant. Specifically, it has the form (Michaelis-Menten contact rate):

$$C(N) = \frac{aN}{1 + bN}. \tag{1}$$

Obviously, incidence with above form suggests that the number of new cases per unit time is saturated with the total population. Using a mechanistic argument, Heesterbeek and Metz [9] derived the expression for the saturating contact rate of individual contacts in a population that mixes randomly; that is,

$$C(N) = \frac{bN}{1 + bN + \sqrt{1 + 2bN}}. \tag{2}$$

Furthermore, $C(N)$ is nondecreasing and $C(N)/N$ is nonincreasing.

The above discussion reveals the importance of incidence functions in epidemic models. Different nonlinear forms of incidence can exhibit very dynamics and hence are able to unearth some otherwise unknown features of disease dynamics. Though the aspect of nonlinearity in incidence has found a significant importance in the existing literature, the fact that population subclasses with different infection statuses should have different incidence rates has received little attention among mathematical epidemiologists. Thus in an SEIS epidemic model, since there is a difference in relative measure of infectiousness between the exposed and the infected populations, the incidence rate between the susceptible fraction S and the infected fraction I should be different from that between S with the exposed fraction E.

The present analysis aims to explore the impact of this distinct incidence for exposed and infected populations under the influence of spatial heterogeneity. As a model system, We have divided the population in researched area into three classes: S—susceptible, E—exposed with the infectious force, and I—infected.

In the next section, we establish the model discussed in this paper and determine the basic reproductive number. In Section 3, we analyze the global stability of the disease-free equilibrium. In Section 4, we resolve the unique existence and global stability of the epidemic equilibrium. In Section 5, we present some numerical simulation of examples which validate these theoretical results. The paper ends with a brief discussion in Section 6.

2. The Model and the Basic Reproductive Number

The model, we consider, has the following population subclasses: (i) S—the susceptible, (ii) E—the exposed, and (iii) I—the infected. The total population size, denoted by N, is $N(t) = S(t) + E(t) + I(s)$. The transfer mechanism from the class S to the class E is guided by the function

$$f(t) = \frac{\beta_1 C_1(N)}{N} SE + \frac{\beta_2 C_2(N)}{N} SI, \tag{3}$$

where β_1 and β_2 are average numbers of adequate contacts of an exposed individual and an infectious individual per unit time, respectively, and $C_i(N)$ $(i = 1, 2)$ are relevant saturation contact rate, which satisfy the following assumptions, for $N > 0$,

(i) $C_i(N) > 0$;

(ii) $C_i'(N) \geq 0$;

(iii) $[C_i(N)/N]' \leq 0$.

The assumptions (i) and (ii) are biologically motivated. As the total population N increases, the probability of a contact with a susceptible individual decreases, and thus the force of the exposed or the infected is expected to be a decreasing function of N. And the assumption (iii) implies that the contact rate $C_i(N)$ is saturated.

The population transfer among compartments is schematically depicted in the transfer diagram in Figure 1.

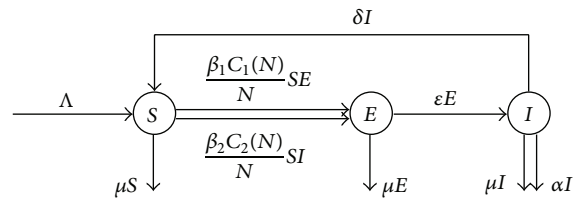

Figure 1: The transfer diagram for model (4).

The transfer diagram leads to the following SEIS epidemic model of ordinary differential equations:

$$\begin{aligned} S' &= \Lambda - \mu S - \frac{\beta_1 C_1(N)}{N} SE - \frac{\beta_2 C_2(N)}{N} SI + \delta I, \\ E' &= \frac{\beta_1 C_1(N)}{N} SE + \frac{\beta_2 C_2(N)}{N} SI - (\mu + \varepsilon) E, \\ I' &= \varepsilon E - (\mu + \alpha + \delta) I, \end{aligned} \tag{4}$$

where Λ is the recruitment rate of the population, μ is the natural death rate, and α is the death rate for the infected. E individuals move to the class I at the rate ε and I individuals recover at the rate δ, which are assumed to join the susceptible class. The above parameters are positive.

Summing up the three equations in system (4), then the time derivative of $N(t)$ along a solution of system (4) is

$$N' = \Lambda - \mu(S + E + I) - \alpha I. \tag{5}$$

Therefore, $N' \leq \Lambda - \mu N$, equivalently, $N' + \mu N \leq \Lambda$. Applying a theorem on differential inequalities [10], we get $0 \leq N \leq \Lambda/\mu$ for $t \to +\infty$. Thus, the three-dimensional simplex

$$T := \left\{ (S, E, I) \in \mathbf{R}_+^3 \mid 0 \leq S + E + I \leq \frac{\Lambda}{\mu} \right\} \tag{6}$$

is positively invariant with respect to system (4), where $\mathbf{R}_+^3$ denotes the nonnegative cone of $\mathbf{R}^3$ including its lower dimensional faces.

By using $S = N - E - I$ and (5), we get the following system:

$$\begin{aligned} E' &= \frac{\beta_1 C_1(N) E + \beta_2 C_2(N) I}{N} (N - E - I) - (\mu + \varepsilon) E, \\ I' &= \varepsilon E - (\mu + \alpha + \delta) I, \\ N' &= \Lambda - \mu N - \alpha I. \end{aligned} \tag{7}$$

The dynamical behavior of system (4) in T is equivalent to that of system (7). Thus, in the rest of the paper, we will study the system (7) in the feasible region

$$G := \left\{ (E, I, N) \in \mathbf{R}_+^3 \mid 0 \leq E + I \leq N \leq \frac{\Lambda}{\mu} \right\}, \tag{8}$$

which can be shown to be a positive invariant set for system (7).

Now, we derive the basic reproductive number of system (4) by the method of next-generation matrix formulated in [11].

Let $x = (E, I, S)^T$, then system (4) can be written as

$$x' = \mathscr{F}(x) - \mathscr{V}(x), \tag{9}$$

where

$$\mathscr{F}(x) = \begin{pmatrix} [\beta_1 C_1(N) E + \beta_2 C_2(N) I] \dfrac{S}{N} \\ 0 \\ 0 \end{pmatrix},$$

$$\mathscr{V}(x) = \begin{pmatrix} (\mu+\varepsilon) E \\ -\varepsilon E + (\mu+\alpha+\delta) I \\ -\Lambda + \mu S + [\beta_1 C_1(N) E + \beta_2 C_2(N) I] \dfrac{S}{N} - \delta I \end{pmatrix}. \tag{10}$$

Then, $x_0 = (0, 0, \Lambda/\mu)^T$ is the unique disease-free equilibrium of system (9), and the Jacobian matrices of $\mathscr{F}(x)$ and $\mathscr{V}(x)$ at equilibrium x_0 are, respectively,

$$D\mathscr{F}(x_0) = \begin{pmatrix} F & O_{2\times1} \\ O_{1\times2} & 0 \end{pmatrix}, \qquad D\mathscr{V}(x_0) = \begin{pmatrix} V & O_{2\times1} \\ J_1 & \mu \end{pmatrix}, \tag{11}$$

where

$$F = \begin{pmatrix} \beta_1 C_1\left(\dfrac{\Lambda}{\mu}\right) & \beta_2 C_2\left(\dfrac{\Lambda}{\mu}\right) \\ 0 & 0 \end{pmatrix},$$
$$V = \begin{pmatrix} \mu+\varepsilon & 0 \\ -\varepsilon & \mu+\alpha+\delta \end{pmatrix}, \tag{12}$$
$$J_1 = \left(\beta_1 C_1\left(\frac{\Lambda}{\mu}\right), \beta_2 C_2\left(\frac{\Lambda}{\mu}\right) - \delta\right).$$

Obviously, all eigenvalues of $-D\mathscr{V}(x_0)$ have negative real parts.

We call

$$FV^{-1} = \frac{1}{(\mu+\varepsilon)(\mu+\alpha+\delta)} \begin{pmatrix} \beta_1 C_1\left(\dfrac{\Lambda}{\mu}\right)(\mu+\alpha+\delta) + \varepsilon\beta_2 C_2\left(\dfrac{\Lambda}{\mu}\right) & \beta_2 C_2\left(\dfrac{\Lambda}{\mu}\right)(\mu+\varepsilon) \\ 0 & 0 \end{pmatrix} \tag{13}$$

the next generation matrix for system (9). According to [11, Theorem 2], the basic reproductive number of system (4), which is the number of secondary infectious cases produced by an exposed individual and an infectious individual during their effective infectious period when introduced in a population of susceptible, is

$$R_0 = \rho\left(FV^{-1}\right) = \frac{\beta_1 C_1(\Lambda/\mu)(\mu+\alpha+\delta) + \varepsilon\beta_2 C_2(\Lambda/\mu)}{(\mu+\varepsilon)(\mu+\alpha+\delta)}, \tag{14}$$

where $\rho(A)$ denotes the spectral radius of matrix A.

3. Stability Analysis of the Disease-Free Equilibrium

In this section, we discuss the global stability of the disease-free equilibrium. It is obvious that system (7) always has the unique disease-free equilibrium $P_0 = (0, 0, \Lambda/\mu)$ in G. About P_0, we have the following main results.

Theorem 1. *The disease-free equilibrium P_0 is globally asymptotically stable in G if $R_0 \le 1$ and it is unstable if $R_0 > 1$.*

Proof. The Jacobian matrix of system (7) at $P_0 = (0, 0, \Lambda/\mu)$ goes as follows:

$$J(P_0) = \begin{pmatrix} \beta_1 C_1\left(\dfrac{\Lambda}{\mu}\right) - (\mu+\varepsilon) & \beta_2 C_2\left(\dfrac{\Lambda}{\mu}\right) & 0 \\ \varepsilon & -\omega & 0 \\ 0 & -\alpha & -\mu \end{pmatrix}, \tag{15}$$

which has a eigenvalue $\lambda_1 = -\mu < 0$, obviously. The other two eigenvalues λ_2 and λ_3 are determined by the following equation:

$$\lambda^2 - \left[\beta_1 C_1\left(\frac{\Lambda}{\mu}\right) - (\mu+\omega+\varepsilon)\right]\lambda - \omega\left[\beta_1 C_1\left(\frac{\Lambda}{\mu}\right) - (\mu+\varepsilon)\right] - \varepsilon\beta_2 C_2\left(\frac{\Lambda}{\mu}\right) = 0. \tag{16}$$

If $R_0 > 1$, we can have easily

$$\begin{aligned} \lambda_2\lambda_3 &= -\omega\left[\beta_1 C_1\left(\frac{\Lambda}{\mu}\right) - (\mu+\varepsilon)\right] - \varepsilon\beta_2 C_2\left(\frac{\Lambda}{\mu}\right) \\ &= \omega(\mu+\varepsilon)(1-R_0) < 0. \end{aligned} \tag{17}$$

Therefore, λ_2 and λ_3 are two opposite-sign real roots. Thus, P_0 is unstable.

Since $R_0 < 1$ implies $\beta_1 C_1(\Lambda/\mu) < (\mu+\varepsilon)-(\varepsilon/\omega)\beta_2 C_2(\Lambda/\mu)$, then we get

$$\begin{aligned}\lambda_2+\lambda_3 &= \beta_1 C_1\left(\frac{\Lambda}{\mu}\right)-(\mu+\omega+\varepsilon)\\ &< -\omega-\frac{\varepsilon}{\omega}\beta_2 C_2\left(\frac{\Lambda}{\mu}\right)<0,\\ \lambda_2\lambda_3 &= \omega(\mu+\varepsilon)(1-R_0)>0.\end{aligned} \tag{18}$$

Therefore, λ_2 and λ_3 have negative real parts. Hence, P_0 is locally asymptotically stable.

When $R_0 = 1$, it implies that $\lambda_2\lambda_3 = 0$, $\beta_1 C_1(\Lambda/\mu)-(\mu+\varepsilon) = -(\varepsilon/\omega)\beta_2 C_2(\Lambda/\mu)$. We may as well assume that $\lambda_2 = 0$; then $\lambda_3 = -\omega-(\varepsilon/\omega)\beta_2 C_2(\Lambda/\mu)$. The characteristic matrix of $J(P_0)$ has three invariable factors: 1, 1, and $\lambda(\lambda+\mu)(\lambda+\omega+(\varepsilon/\omega)\beta_2 C_2(\Lambda/\mu))$. Because the elementary factor with respect to $\lambda_2 = 0$ is λ, which is single, P_0 is stable.

Constructing a suitable Lyapunov function

$$V = R_0 E+\frac{\beta_2}{\omega}C_2\left(\frac{\Lambda}{\mu}\right)I, \tag{19}$$

then the time derivative of V along a solution of system (7) gives

$$\begin{aligned}\dot V &= R_0\frac{\beta_1 C_1(N)E+\beta_2 C_2(N)I}{N}(N-E-I)\\ &\quad -R_0(\mu+\varepsilon)E+\frac{\varepsilon\beta_2}{\omega}C_2\left(\frac{\Lambda}{\mu}\right)E-\beta_2 C_2\left(\frac{\Lambda}{\mu}\right)I\\ &= R_0\frac{\beta_1 C_1(N)E+\beta_2 C_2(N)I}{N}(N-E-I)\\ &\quad -\beta_1 C_1\left(\frac{\Lambda}{\mu}\right)E-\beta_2 C_2\left(\frac{\Lambda}{\mu}\right)I\\ &\le R_0\frac{\beta_1 C_1(N)E+\beta_2 C_2(N)I}{N}(N-E-I)\\ &\quad -\beta_1 C_1(N)E-\beta_2 C_2(N)I\\ &= \frac{\beta_1 C_1(N)E+\beta_2 C_2(N)I}{N}\left[(R_0-1)N-R_0E-R_0I\right].\end{aligned} \tag{20}$$

Hence, $\dot V \le 0$ holds if $R_0 \le 1$. Furthermore, $\dot V = 0$, if and only if $E = I = 0$. Let $F = \{(E,I,N)\in G \mid \dot V = 0\} = \{(0,0,N)\}$, then the largest compact invariant set in F for system (7) is the set $\{(0,0,N)\}$. Thus, the solution of system (7) satisfies $E \to 0, I \to 0$ as $t \to +\infty$ by LaSalle's Invariance Principle [12]. Therefore, the limit system of system (7) is

$$\begin{aligned}E' &= 0,\\ I' &= 0,\\ N' &= \Lambda-\mu N.\end{aligned} \tag{21}$$

It is obviously known that the equilibrium $(0,0,\Lambda/\mu)$ of system (21) is globally asymptotically stable; thus, the disease-free equilibrium P_0 of system (7) is globally attractive in G. On the basis of local stability, P_0 is globally asymptotically stable in G if $R_0 \le 1$. This completes the proof. □

About system (4), we also obtain.

Theorem 2. *The unique disease-free equilibrium $\overline{P}_0 = (\Lambda/\mu, 0, 0)$ of system (4) is globally asymptotically stable in T if $R_0 \le 1$ and it is unstable if $R_0 > 1$.*

4. Existence and Stability of the Endemic Equilibrium

In this section, we first discuss the existence and uniqueness of the endemic equilibrium P^* of system (7) when $R_0 > 1$. Whereafter, we focus on investigating the local stability of P^*. We have to prove that the Jacobian matrix $J(P^*)$ is stable; namely, all its eigenvalues have negative real parts. This is routinely done by verifying the Routh-Hurwitz conditions. Finally, we study the global stability of the endemic equilibrium $\overline{P}^*$ of system (4) with the method of autonomous convergence theorem of Li and Muldowney in [13].

The coordinates of the endemic equilibrium (positive equilibrium) of system (7) are the positive solutions of equations

$$\begin{aligned}\frac{\beta_1 C_1(N)E+\beta_2 C_2(N)I}{N}(N-E-I)-(\mu+\varepsilon)E &= 0,\\ \varepsilon E-(\mu+\alpha+\delta)I &= 0,\\ \Lambda-\mu N-\alpha I &= 0\end{aligned} \tag{22}$$

in G^o.

Let $\omega = \mu+\alpha+\delta$, by the direct calculation, we can get the following equation of N easily as

$$\begin{aligned}\varphi(N) := &\frac{\alpha\varepsilon+\mu\omega+\mu\varepsilon}{\alpha\varepsilon}\left[\beta_1 C_1(N)+\frac{\varepsilon}{\omega}\beta_2 C_2(N)\right]\\ &-\frac{\Lambda(\omega+\varepsilon)}{\alpha\varepsilon}\left[\frac{\beta_1 C_1(N)}{N}+\frac{\varepsilon}{\omega}\frac{\beta_2 C_2(N)}{N}\right]\\ &-(\mu+\varepsilon)=0.\end{aligned} \tag{23}$$

Because $C_i(N)$ $(i = 1,2)$ satisfy conditions (i), (ii), and (iii), thus $\varphi(N)$ is an increasing continuous function, and $\varphi(\Lambda/\mu) = (\mu+\varepsilon)(R_0-1)$. When N is sufficiently small, $\varphi(N) < 0$. If $R_0 > 1$, then $\varphi(\Lambda/\mu) > 0$. According to the zero-point theorem, $\varphi(N)$ has the unique positive solution N^* in the open interval $(0, \Lambda/\mu)$. Then, $I^* = (\Lambda-\mu N^*)/\alpha$, $E^* = (\omega/\varepsilon)I^*$. Otherwise, if $R_0 \le 1$, N^* does not exist in $(0, \Lambda/\mu)$. Therefore, we have the following theorem.

Theorem 3. *When $R_0 > 1$, system (7) has the unique endemic equilibrium $P^* = (E^*, I^*, N^*)$ besides the disease-free equilibrium P_0 in G.*

Theorem 4. *When $R_0 > 1$, the unique endemic equilibrium P^* is locally asymptotically stable in G^o.*

Proof. The Jacobian matrix of system (7) at $P^* = (E^*, I^*, N^*)$ is

$$J(P^*) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \varepsilon & -\omega & 0 \\ 0 & -\alpha & -\mu \end{pmatrix}, \tag{24}$$

where

$$\begin{aligned}
a_{11} &= -\frac{\varepsilon\beta_2 C_2(N^*)(N^* - E^* - I^*)}{\omega N^*} - W^* < 0, \\
a_{12} &= \frac{\beta_2 C_2(N^*)(N^* - E^* - I^*)}{N^*} - W^*, \\
a_{13} &= \left[\beta_1\left(\frac{C_1(N^*)}{N^*}\right)' E^* + \beta_2\left(\frac{C_2(N^*)}{N^*}\right)' I^*\right] \\
&\quad \times (N^* - E^* - I^*) + W^* \\
&= \beta_1 E^*\left[C_1'(N^*) - (E^* + I^*)\left(\frac{C_1(N^*)}{N^*}\right)'\right] \\
&\quad + \beta_2 I^*\left[C_2'(N^*) - (E^* + I^*)\left(\frac{C_2(N^*)}{N^*}\right)'\right] > 0,
\end{aligned} \tag{25}$$

thereinto $W^* = (\beta_1 C_1(N^*)E^* + \beta_2 C_2(N^*)I^*)/N^*$.

Therefore, the characteristic equation of $J(P^*)$ is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{26}$$

where

$$\begin{aligned}
a_1 &= \mu + \omega - a_{11} > 0, \\
a_2 &= (\mu - a_{11})\omega - \mu a_{11} - \varepsilon a_{12} \\
&= (\omega - a_{11})\mu + (\omega + \varepsilon)W^* > 0, \\
a_3 &= -\omega\mu a_{11} - \mu\varepsilon a_{12} + \alpha\varepsilon a_{13} \\
&= \mu(\omega + \varepsilon)W^* + \alpha\varepsilon a_{13} > 0.
\end{aligned} \tag{27}$$

By calculation, we have

$$\begin{aligned}
H_1 &= a_1 > 0, \\
H_2 &= a_1 a_2 - a_3 \\
&= (\mu + \omega - a_{11})\left[(\omega - a_{11})\mu + (\omega + \varepsilon)W^*\right] \\
&\quad - \mu(\omega + \varepsilon)W^* \\
&\quad - \alpha\varepsilon\left\{\left[\beta_1\left(\frac{C_1(N^*)}{N^*}\right)' E^* + \beta_2\left(\frac{C_2(N^*)}{N^*}\right)' I^*\right]\right. \\
&\qquad \left. \times (N^* - E^* - I^*) + W^*\right\} \\
&= (\omega - a_{11})\mu^2 + (\omega - a_{11})^2\mu \\
&\quad + \left[(\mu + \omega - a_{11})(\omega + \varepsilon) - \mu(\omega + \varepsilon) - \alpha\varepsilon\right]W^* \\
&\quad - \alpha\varepsilon\left[\beta_1\left(\frac{C_1(N^*)}{N^*}\right)' E^* + \beta_2\left(\frac{C_2(N^*)}{N^*}\right)' I^*\right] \\
&\quad \times (N^* - E^* - I^*) \\
&= (\omega - a_{11})\mu^2 + (\omega - a_{11})^2\mu \\
&\quad + \left[(\mu + \alpha + \delta - a_{11})(\omega + \varepsilon) - \alpha\varepsilon\right]W^* \\
&\quad - \alpha\varepsilon\left[\beta_1\left(\frac{C_1(N^*)}{N^*}\right)' E^* + \beta_2\left(\frac{C_2(N^*)}{N^*}\right)' I^*\right] \\
&\quad \times (N^* - E^* - I^*) > 0, \\
H_3 &= a_3 H_2 > 0.
\end{aligned} \tag{28}$$

By Routh-Hurwitz stability theorem [10], all the three eigenvalues of $J(P^*)$ have negative real parts. Thus, the endemic equilibrium P^* is locally asymptotically stable in G^o, when $R_0 > 1$. □

Denote the boundary and the interior of T by ∂T and T^o, we also obtain for system (4).

Theorem 5. *When $R_0 > 1$, system (4) has a unique endemic equilibrium $\overline{P}^* = (S^*, E^*, I^*)$, and it is locally asymptotically stable in T^o, thereinto $S^* = N^* - E^* - I^*$.*

Now, we briefly outline the autonomous convergence theorem in [13] for proving global stability of the endemic equilibrium $\overline{P}^*$.

Let $D \subset \mathbf{R}^n$ be an open set, and let $x \to f(x) \in \mathbf{R}^n$ be a C^1 function defined in D. We consider the autonomous system in $\mathbf{R}^n$:

$$\dot{x} = f(x). \tag{29}$$

Let $\overline{x}$ be an equilibrium of (29); that is, $f(\overline{x}) = 0$. We recall that $\overline{x}$ is said to be globally stable in D if it is locally stable and all trajectories in D converge to $\overline{x}$.

Assume that the following hypothesis hold:

(H1) D is simply connected;

(H2) there exists a compact absorbing set $\Gamma \subset D$;

(H3) $\overline{x}$ is the only equilibrium of (29) and is locally stable in D.

The basic job is to find conditions under which the global stability of $\overline{x}$ with respect to D is implied by its local stability. The difficulty associated with this problem is largely due to the lack of practical tools. A new approach to the global stability problem has emerged from a series of papers on higher-dimensional generalizations of the criteria of Bendixson and Dulac for planar systems and on so-called autonomous convergence theorems. First, we now introduce a definition, which will appear in the following context.

Definition 6 (see [13]). Suppose system (29) has a periodic solution $x = p(t)$ with least period $\omega > 0$ and orbit $\gamma = \{p(t) : 0 \le t \le \omega\}$. This orbit is *orbitally stable* if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that any solution $x(t)$, for which the distance of $x(0)$ from γ is less than δ, remains at a distance less than ε from γ for all $t \ge 0$. It is *asymptotically orbitally stable* if the distance of $x(t)$ from γ also tends to zero as $t \to \infty$. This orbit γ is *asymptotically orbitally stable with asymptotic phase* if it is asymptotically orbitally stable and there is a $b > 0$ such that any solution $x(t)$, for which the distance of $x(0)$ from γ is less than b, satisfies $|x(t) - p(t-\tau)| \to 0$ as $t \to \infty$ for some τ which may depend on $x(0)$.

Theorem 7 (see [14]). *A sufficient condition for a period orbit $\gamma = \{p(t) : 0 \le t \le \omega\}$ of* (29) *is asymptotically orbitally stable with asymptotic phase such that the linear system*

$$z'(t) = \left(\frac{\partial f^{[2]}}{\partial x}(p(t))\right) z(t) \tag{30}$$

is asymptotically stable.

Remark 8. Equation (30) is called the second compound equation of (29) and $\partial f^{[2]}/\partial x$ is the second compound matrix of the Jacobian matrix $\partial f/\partial x$ of f.

It is also demonstrated that Theorem 7 generalizes a class of Poincare for the orbital stability of periodic solutions to planar autonomous systems.

Theorem 9 (see [13]). *Under assumptions (H1), (H2), and (H3), $\overline{x}$ is globally asymptotically stable in D provided that*

(H4) *the system* (29) *satisfies a Poincare-Bendixson criterion;*

(H5) *a periodic orbit of the system* (29) *is asymptotically orbitally stable.*

As a matter of fact, the condition (H2) is true, if and only if the system (4) *is uniformly persistent in T^o.*

Definition 10 (see [15, 16]). System (4) is said to be uniformly persistent if there exists a constant $\eta \in (0,1)$ such that any solution $(S(t), E(t), I(t))$ with initial point $(S(0), E(0), I(0)) \in T^o$ satisfies

$$\min\left\{\lim_{t\to\infty}\inf S(t), \lim_{t\to\infty}\inf E(t), \lim_{t\to\infty}\inf I(t)\right\} \ge \eta. \tag{31}$$

Lemma 11. *When $R_0 > 1$, system* (4) *is uniformly persistent in T^o.*

Proof. Any solution of system (4) which begins from $\{(S,0,0), 0 \le S \le \Lambda/\mu\}$ always, in fact, converges at the point $\overline{P}_0 = (\Lambda/\mu, 0, 0)$ along the S-axis. Except the S-axis, the solution of system (4) which begin from ∂T will converge in the region T^o. Thus, $\overline{P}_0$ is the unique ω-limit point in ∂T of system (4).

Let

$$U = R_0 E + \frac{\beta_2}{\omega} C_2\left(\frac{\Lambda}{\mu}\right) I, \tag{32}$$

then the time derivative of U along a solution of system (4) gives

$$\begin{aligned}\dot{U} &= \frac{\beta_1 C_1(N) E + \beta_2 C_2(N) I}{N} S R_0 \\ &\quad - \beta_1 C_1\left(\frac{\Lambda}{\mu}\right) E - \beta_2 C_2\left(\frac{\Lambda}{\mu}\right) I \\ &\ge \left[\beta_1 C_1\left(\frac{\Lambda}{\mu}\right) E + \beta_2 C_2\left(\frac{\Lambda}{\mu}\right) I\right]\left(\frac{\mu}{\Lambda} R_0 S - 1\right).\end{aligned} \tag{33}$$

When $R_0 > 1$, if the trajectories (S, E, I) in T^o sufficiently converge to P_0, it implies that $\dot{U} > 0$. That is to say, there exists a neighborhood $\mathbf{U}(\overline{P}_0)$ of $\overline{P}_0$, such that when the trajectories of system (4) begin from $T^o \cap \mathbf{U}(\overline{P}_0)$, it will come out of $\mathbf{U}(\overline{P}_0)$. Therefore, $\overline{P}_0$ is not a ω-limit point of any trajectory in T^o. Thus, $M = \{(S,0,0) \mid 0 \le S \le \Lambda/\mu\}$ is the largest invariant set in ∂T of system (4). When $R_0 > 1$, M is isolated. Also the invariant set $W^s(M) \subseteq \partial T$, where $W^s(M) := \{x \in D : f^n(x) \to M \text{ as } n \to +\infty\}$ [15] is the stable set of M. According to [15, Theorem 4.1], system (4) is uniformly persistent in T^o when $R_0 > 1$. Thus, there exists a compact absorbing subset in T^o for system (4). □

Lemma 12. *When $R_0 > 1$, system* (4) *satisfies the Poincare-Bendixson criterion in T^o.*

Proof. Because the system (4) is not quasimonotone, we cannot verify that the system (4) is competitive by examining its Jacobian matrix. Thus, we can replace the system (4) by

$$\begin{aligned} x_1' &= \Lambda - \mu x_1 - \frac{\beta_1 C_1(N)}{N} S(N - x_1 - x_3) \\ &\quad - \frac{\beta_2 C_2(N)}{N} S(N - x_1 - x_2) + \delta x_3, \\ x_2' &= \frac{\beta_1 C_1(N)}{N} S x_2 + \frac{\beta_2 C_2(N)}{N} S x_3 - (\mu + \varepsilon) x_2, \\ x_3' &= \varepsilon x_2 - (\mu + \alpha + \delta) x_3. \end{aligned} \tag{34}$$

Then, system (34) has a solution $u(t) = (S(t), E(t), I(t))$.

Let $\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbf{R}^3$, we have

$$\mathbf{x}' = (B - \mu I)\mathbf{x} + C(S, E, I), \tag{35}$$

where I denotes the 3×3 unit matrix, $C(S, E, I)$ is a function that need not concern us, and

$$B = \begin{pmatrix} \frac{\beta_1 C_1(N) + \beta_2 C_2(N)}{N} S & \frac{\beta_2 C_2(N)}{N} S & \frac{\beta_1 C_1(N)}{N} S + \delta \\ 0 & \frac{\beta_1 C_1(N)}{N} S - \varepsilon & \frac{\beta_2 C_2(N)}{N} S \\ 0 & \varepsilon & -(\alpha + \delta) \end{pmatrix}. \tag{36}$$

The off-diagonal entries in this matrix are nonnegative; thus, the system (34) as a whole is quasimonotone [17]. Then, we can verify that the system (34) is competitive [18] with respect to the partial ordering defined by the orthant $K = \{(S, E, I) \in \mathbf{R}^3_+\}$. Since T^o is convex, system (4) satisfies the Poincare-Bendixson criterion [10, 19] in T^o when $R_0 > 1$. □

Lemma 13. *When $R_0 > 1$, the trajectory of any nonconstant periodic solution $p(t) = (S(t), E(t), I(t))$ to system (4), if it exists, is asymptotically orbitally stable with asymptotically phase.*

Proof. Suppose that the period solution $p(t)$ is periodic of least period $\tau > 0$ such that $(S(0), E(0), I(0)) \in T^o$. The period orbit is $p = \{p(t) : 0 \le t \le \tau\}$. The Jacobian matrix of system (4) at (S, E, I) is given by

$$\mathbf{J}(p(t)) = \begin{pmatrix} -\mu - \mathbf{M} - \mathbf{N} & -\frac{\beta_1 C_1(N)}{N} S - \mathbf{N} & \delta - \frac{\beta_2 C_2(N)}{N} S - \mathbf{N} \\ \mathbf{M} + \mathbf{N} & \frac{\beta_1 C_1(N)}{N} S + \mathbf{N} - (\mu + \varepsilon) & \frac{\beta_2 C_2(N)}{N} S + \mathbf{N} \\ 0 & \varepsilon & -(\mu + \alpha + \delta) \end{pmatrix}, \tag{37}$$

where

$$\mathbf{M} = \frac{\beta_1 C_1(N)}{N} E + \frac{\beta_2 C_2(N)}{N} I \ge 0,$$
$$\mathbf{N} = \left(\frac{\beta_1 C_1(N)}{N}\right)' SE + \left(\frac{\beta_2 C_2(N)}{N}\right)' SI \le 0. \tag{38}$$

Then, the second compound matrix of $\mathbf{J}(p(t))$ is

$$\mathbf{J}^{[2]}(p(t)) = \begin{pmatrix} -(2\mu + \varepsilon) + \frac{\beta_1 C_1(N)}{N} S - \mathbf{M} & \frac{\beta_2 C_2(N)}{N} S + \mathbf{N} & -\delta + \frac{\beta_2 C_2(N)}{N} S + \mathbf{N} \\ \varepsilon & -(\mu + \omega) - \mathbf{M} - \mathbf{N} & -\frac{\beta_1 C_1(N)}{N} S - \mathbf{N} \\ 0 & \mathbf{M} + \mathbf{N} & -(\mu + \varepsilon + \omega) + \frac{\beta_1 C_1(N)}{N} S + \mathbf{N} \end{pmatrix}, \tag{39}$$

whose definition can be found in the appendix.

Furthermore, the second compound system of (4) is the following periodic linear system:

$$\begin{aligned} X' &= -\left[\mathbf{M} - \frac{\beta_1 C_1(N)}{N} S + 2\mu + \varepsilon\right] X + \left[\frac{\beta_2 C_2(N)}{N} S + \mathbf{N}\right] Y \\ &\quad + \left[\frac{\beta_2 C_2(N)}{N} S + \mathbf{N} - \delta\right] Z, \\ Y' &= \varepsilon X - (\mathbf{M} + \mathbf{N} + \mu + \omega) Y - \left[\frac{\beta_1 C_1(N)}{N} S + \mathbf{N}\right] Z, \\ Z' &= (\mathbf{M} + \mathbf{N}) Y + \left[\frac{\beta_1 C_1(N)}{N} S + \mathbf{N} - \mu - \varepsilon - \omega\right] Z. \end{aligned} \tag{40}$$

Let (x, y, z) be a vector in $\mathbf{R}^3$. We choose a vector norm in $\mathbf{R}^3$ as

$$\|(x(t), y(t), z(t))\| = \sup\{|x(t)|, |y(t)| + |z(t)|\}. \tag{41}$$

Let

$$L(t) = \sup\left\{|X(t)|, \frac{E(t)}{I(t)}(|Y(t)| + |Z(t)|)\right\}. \tag{42}$$

When $R_0 > 1$, system (4) is uniformly persistent in T^o. Then, there exists constant $k > 0$ such that

$$L(t) \ge k \sup\{|X(t)|, |Y(t)| + |Z(t)|\} \tag{43}$$

for all $(X(t), Y(t), Z(t)) \in \mathbf{R}^3$.

By direct calculations, we can obtain the following differential inequalities:

$$\begin{aligned} D_+|X(t)| &\le -\left[\mathbf{M} - \frac{\beta_1 C_1(N)}{N} S + 2\mu + \varepsilon\right] |X(t)| \\ &\quad + \left[\frac{\beta_2 C_2(N)}{N} S + \mathbf{N}\right] |Y(t)| \\ &\quad + \left[\frac{\beta_2 C_2(N)}{N} S + \mathbf{N} - \delta\right] |Z(t)|, \\ &\le -\left[\mathbf{M} - \frac{\beta_1 C_1(N)}{N} S + 2\mu + \varepsilon\right] |X(t)| \\ &\quad + \frac{I}{E}\left[\frac{\beta_2 C_2(N)}{N} S + \mathbf{N}\right] \frac{E}{I}(|Y| + |Z|), \end{aligned} \tag{44}$$

$$D_+|Y(t)| \le \varepsilon|X(t)| - (\mathbf{M}+\mathbf{N}+\mu+\omega)|Y(t)| - \left[\frac{\beta_1 C_1(N)}{N}S + \mathbf{N}\right]|Z(t)|, \tag{45}$$

$$D_+|Z(t)| \le (\mathbf{M}+\mathbf{N})|Y(t)| + \left[\frac{\beta_1 C_1(N)}{N}S + \mathbf{N} - \mu - \omega\right]|Z(t)|. \tag{46}$$

Using (45) and (46), we have

$$\begin{aligned} D_+\frac{E}{I}(|Y(t)|+|Z(t)|) &= \left(\frac{E'}{E}-\frac{I'}{I}\right)\frac{E}{I}(|Y(t)|+|Z(t)|) \\ &\quad + \frac{E}{I}\left(D_+|Y(t)|+D_+|Z(t)|\right) \\ &\le \frac{\varepsilon E}{I}|X(t)| + \left(\frac{E'}{E}-\frac{I'}{I}-\mu-\omega\right)\frac{E}{I} \\ &\quad \times (|Y(t)|+|Z(t)|). \end{aligned} \tag{47}$$

Therefore, we obtain from (44) and (47),

$$D_+L(t) \le \sup\{g_1(t), g_2(t)\}L(t), \tag{48}$$

where

$$g_1(t) = -\left[\mathbf{M} - \frac{\beta_1 C_1(N)}{N}S + 2\mu + \varepsilon\right] + \frac{I}{E}\left[\frac{\beta_2 C_2(N)}{N}S + \mathbf{N}\right], \tag{49}$$

$$g_2(t) = \frac{\varepsilon E}{I} + \frac{E'}{E} - \frac{I'}{I} - \mu - \omega. \tag{50}$$

The system (4) implies

$$\frac{\beta_2 C_2(N)SI}{NE} = \frac{E'}{E} - \frac{\beta_1 C_1(N)S}{N} + \mu + \varepsilon, \tag{51}$$

$$\frac{\varepsilon E}{I} = \frac{I'}{I} + \omega. \tag{52}$$

Substituting (51) into (49) and (52) into (50), we have

$$\begin{aligned} g_1(t) &= \frac{E'}{E} - \mu - \mathbf{M} + \frac{I}{E}\mathbf{N} \le \frac{E'}{E} - \mu, \\ g_2(t) &= \frac{E'}{E} - \mu. \end{aligned} \tag{53}$$

Thus,

$$\begin{aligned} \sup\{g_1(t), g_2(t)\} &\le \frac{E'}{E} - \mu, \\ D_+L(t) &\le \left(\frac{E'}{E} - \mu\right)L(t), \\ \int_0^\tau \sup\{g_1(t), g_2(t)\}\,dt &\le \int_0^\tau \left(\frac{E'}{E} - \mu\right)dt \\ &= \ln E(t)|_0^\tau - \mu\tau = -\mu\tau < 0, \end{aligned} \tag{54}$$

which implies that $L(t) \to 0$ as $t \to \infty$, and in turn that $(X(t), Y(t), Z(t)) \to 0$ as $t \to \infty$. Aa a result, the second compound system (40) is asymptotically stable. Thus, the periodic solution $(S(t), E(t), I(t))$ is asymptotically orbitally stable with asymptotically phase. □

By Lemmas 11–13, we know that system (4) is satisfied with every condition of Theorem 9; thus we can obtain the following.

Theorem 14. *If $R_0 > 1$, the unique endemic equilibrium $\overline{P}^* = (S^*, E^*, I^*)$ of system (4) is globally asymptotically stable in T^o.*

Theorem 15. *If $R_0 > 1$, the unique endemic equilibrium $P^* = (E^*, I^*, N^*)$ of system (7) is globally asymptotically stable in G^o.*

5. Example and Numerical Simulation

In this paper, we considered an SEIS model with saturation incidence. Now, we give the number simulations for system (4) (see Figures 2 and 3).

Choose $C_1(N) = (10N/3)/(1 + 10N/3 + \sqrt{1+20N/3})$ and $C_2(N) = (N/3)/(1 + N/3 + \sqrt{1+2N/3})$. Assume that $\Lambda = 0.6$, $\mu = 0.05$, $\varepsilon = 0.15$, $\delta = 0.15$, $\beta_2 = 0.36$, and $\alpha = 0.1$. We choose randomly six initial values: (1, 2.2, 5.7), (5.1, 2.2, 1.3), (3.3, 1.8, 2.7), (4.4, 2.1, 0.6), (0.8, 5.4, 2.2), and (5.4, 1.3, 1.6) in $T^o = \{(S, E, I) \in \mathbf{R}_+^3 \mid 0 < S + E + I < 12\}$.

If $\beta_1 = 0.1$, $R_0 = 0.85$. We give the trajectory plot and its tridimensional figure by Matlab software.

If $\beta_1 = 0.36$, $R_0 = 1.4$. We give the trajectory plot and its tridimensional figure by Matlab software.

6. Discussion

In this paper, we present a complete mathematical analysis for the global stability problem at the equilibria of an SEIS epidemic model with saturation incidence. The basic reproductive number R_0 is obtained as a sharp threshold parameter, which represents the average number of secondary infections from a single exposed host and infectious host. If $R_0 \le 1$, the disease-free equilibrium $\overline{P}_0$ is globally asymptotically stable in the feasible region T by Lyapunov function, and thus the disease always dies out. If $R_0 > 1$, the unique disease equilibrium $\overline{P}^*$ is globally asymptotically stable in T^o, so that the disease, if initially present, will persist at the unique endemic

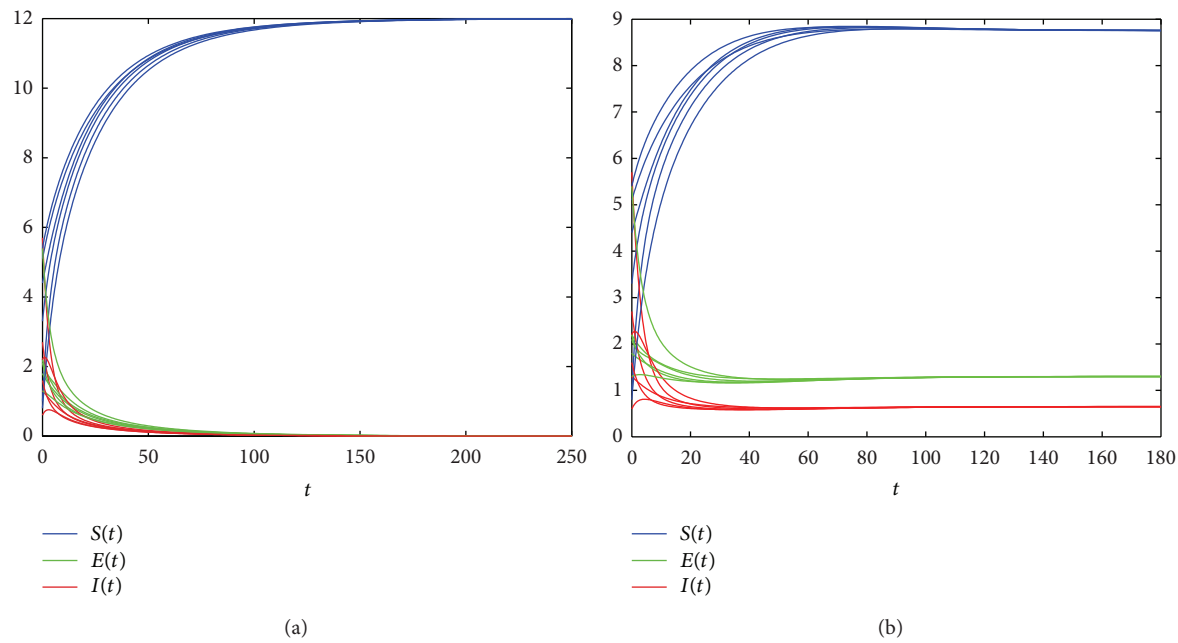

FIGURE 2: Movement paths of S, E, and I as functions of time t. For (a), we have $R_0 = 0.85$ and $\overline{P}_0$ is globally stable. The disease is extinct. For (b), we have $R_0 = 1.4$ and $\overline{P}^*$ is globally stable. The disease spreads to be endemic.

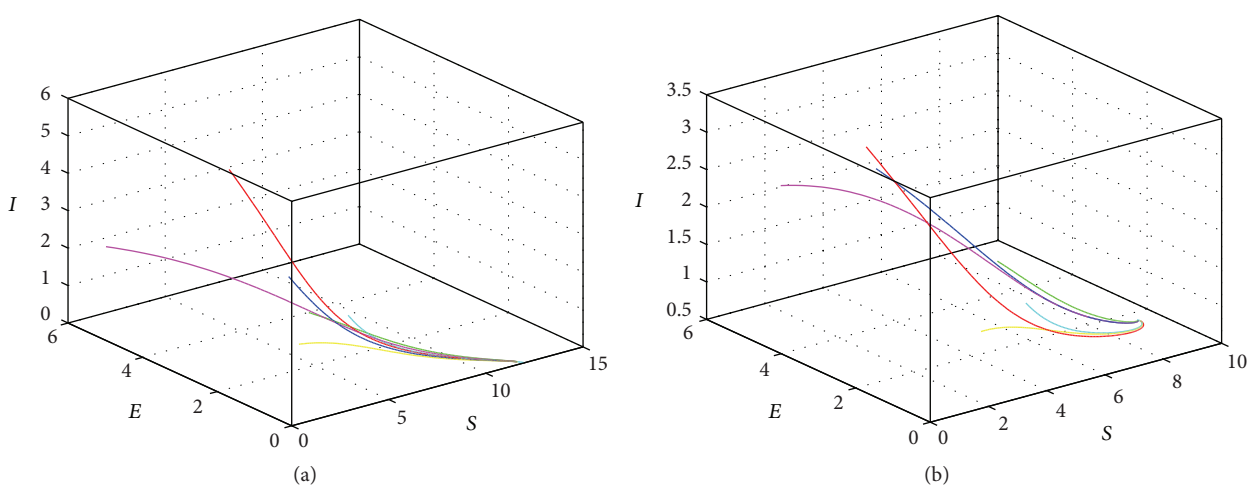

FIGURE 3: The graph of the trajectory in (S, E, I)-space. (a) and (b) correspond with Figures 2(a) and 2(b), respectively.

equilibrium level. The global stability of $\overline{P}^*$ in model is proved using a geometrical approach in [13]. We expect that these approaches can be applied to solve global stability problems in many other epidemic models.

Appendix

Compound Matrices

Let A be a linear operator on $\mathbf{R}^n$ and also denote its matrix representation with respect to the standard basis of $\mathbf{R}^n$. Let $\wedge^2\mathbf{R}^n$ denote the exterior product of $\mathbf{R}^n$. A induces canonically a linear operator $A^{[2]}$ on $\wedge^2\mathbf{R}^n$; for $u_1, u_2 \in \mathbf{R}^n$, define

$$A^{[2]}(u_1 \wedge u_2) := A(u_1) \wedge u_2 + u_1 \wedge A(u_2) \tag{A.1}$$

and extend the definition over $\wedge^2\mathbf{R}^n$ by linearity. The matrix representation of $A^{[2]}$ with respect to the canonical basis in $\wedge^2\mathbf{R}^n$ is called the second additive compound matrix of A. This is an $\binom{n}{2}\times\binom{n}{2}$ matrix and satisfies the property $(A+B)^{[2]} = A^{[2]}+B^{[2]}$. The entries in $A^{[2]}$ are linear relations of those in A. Let $A = (a_{ij})$. For any integer $i = 1, 2 \ldots, \binom{n}{2}$, let $(i) = (i_1, i_2)$ be the ith member in the lexicographic ordering of integer

pairs such that $1 \le i_1 < i_2 \le m$. Then, the entry in the jth column of $Z = A^{[2]}$ is

$$z_{ij} = \begin{cases} a_{i_1 i_1} + a_{i_2 i_2} & \text{if } (i) = (j), \\ (-1)^{r+s} a_{i_s j_r} & \text{if exactly one entry } i_s \text{ of} \\ & (i) \text{ does not occur in } (j) \\ & \text{and } j_r \text{ does not occur in } (i), \\ 0 & \text{if } (i) \text{ differs from } (j) \text{ in two} \\ & \text{or more entries.} \end{cases} \tag{A.2}$$

For any integer $1 \le k \le n$, the kth additive compound matrix A^k of A is defined canonically. For detailed discussions of compound matrices and their properties, we refer the reader to [20]. A comprehensive survey on compound matrices and their relations to differential equations is given in [20]. For $n = 2, 3$, and 4, the second additive compound matrix $A^{[2]}$ of an $n \times n$ matrix $A = (a_{ij})$ is, respectively,

$$\begin{aligned} n = 2: &\quad a_{11} + a_{22}, \\ n = 3: &\quad \begin{pmatrix} a_{11} + a_{22} & a_{23} & -a_{13} \\ a_{32} & a_{11} + a_{33} & a_{12} \\ -a_{31} & a_{21} & a_{22} + a_{33} \end{pmatrix}, \\ n = 4: &\quad \begin{pmatrix} a_{11} + a_{22} & a_{23} & -a_{24} & -a_{13} & -a_{14} & 0 \\ a_{32} & a_{11} + a_{33} & a_{34} & a_{12} & 0 & -a_{14} \\ a_{42} & a_{43} & a_{11} + a_{44} & 0 & a_{12} & a_{13} \\ -a_{31} & a_{21} & 0 & a_{22} + a_{33} & a_{34} & -a_{24} \\ -a_{41} & 0 & a_{21} & a_{43} & a_{22} + a_{44} & a_{23} \\ 0 & -a_{41} & a_{31} & -a_{42} & a_{32} & a_{33} + a_{44} \end{pmatrix}. \end{aligned} \tag{A.3}$$

Acknowledgments

This work is supported by the National Natural Science Foundation of China (10531030). There is no financial conflict of interests between the authors and the commercial identity. Also it partially contains the results obtained in [21] and develops the ideas formulated in [21].

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