Research Article

An Upper Bound for the Symmetric Tensor Rank of a Low Degree Polynomial in a Large Number of Variables

E. Ballico

Department of Mathematics, University of Trento, 38123 Povo, Italy

Correspondence should be addressed to E. Ballico; ballico@science.unitn.it

Received 22 December 2012; Accepted 15 February 2013

Academic Editor: Michel Planat

Copyright © 2013 E. Ballico. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Fix integers $m \geq 5$ and $d \geq 3$. Let f be a degree d homogeneous polynomial in $m + 1$ variables. Here, we prove that f is the sum of at most $d \cdot \lceil \binom{m+d}{m} / (m+1) \rceil$ d -powers of linear forms (of course, this inequality is nontrivial only if $m \gg d$).

1. An Upper Bound for the Symmetric Tensor Rank

Fix positive integers m, d , and an algebraically closed field $\mathbb{K}$ such that either $\text{char}(\mathbb{K}) = 0$ or $\text{char}(\mathbb{K}) > d$. Let $\mathbb{K}[x_0, \dots, x_m]_d$ be the $\mathbb{K}$ -vector space of all degree d homogeneous polynomials in $m + 1$ variables. For any $f \in \mathbb{K}[x_0, \dots, x_m]_d$, the symmetric rank (or the symmetric tensor rank or just the rank) $r_{m,d}(f)$ of f is the minimal integer t such that $f = \sum_{i=1}^t \ell_i^d$ for some $\ell_i \in \mathbb{K}[x_0, \dots, x_m]_1$ [1–5]. Let $\rho(m, d)$ be the maximum of all integers $r_{m,d}(f)$, $f \in \mathbb{K}[x_0, \dots, x_m]_d$. It is known that $\rho(m, d) \leq \binom{m+d}{m} - m$ [5, Proposition 5.1], but this is a general upper bound for the X -rank with respect to any projective variety $X \subset \mathbb{P}^r$ (although it is sharp in the case $m = 1$, since $\rho(1, d) = d$ for all d by a theorem of Sylvester ([6], [5, Theorem 5.1], and [2, Theorem 23]). In this paper, we prove the following result.

Theorem 1. Fix integers m, d such that $m \geq 5$ and $d \geq 3$. Then,

$$\rho(m, d) \leq d \cdot \left\lceil \frac{\binom{m+d}{m}}{(m+1)} \right\rceil. \quad (1)$$

Of course, by [5, Proposition 5.1], Theorem 1 is nontrivial only if $m \gg d$. Since either $\text{char}(\mathbb{K}) = 0$ or $\text{char}(\mathbb{K}) > d$, Theorem 1 is equivalent to the following result.

Corollary 2. A symmetric d -tensor in $m + 1$ variables has symmetric tensor rank at most $d \cdot \lceil \binom{m+d}{m} / (m+1) \rceil$.

2. The proof

For any subset $S \subset \mathbb{P}^r$, let $\langle S \rangle$ denote its linear span, that is, the intersection of all hyperplanes containing S , with the convention $\langle S \rangle = \mathbb{P}^r$ if there is no such a hyperplane. For any integral variety $Y \subset \mathbb{P}^r$ and any integer $t > 0$, the t -secant variety $\sigma_t(Y)$ of Y is the closure of the union of all linear spaces $\langle S \rangle$, where $S \subset Y$ is the union of any t linearly independent points. Let $\mu_d : \mathbb{P}^m \rightarrow \mathbb{P}^N$, $N := \binom{m+d}{m} - 1$, denote the Veronese embedding of $\mathbb{P}^m$, that is, the embedding of $\mathbb{P}^m$ induced by the complete linear system $|\mathcal{O}_{\mathbb{P}^m}(d)|$. Set $X_{m,d} := \mu_d(\mathbb{P}^m)$. For any $Q \in \mathbb{P}^m$, let $2Q$ denote the first infinitesimal neighborhood of Q in $\mathbb{P}^m$, that is, the closed subscheme of $\mathbb{P}^m$ with $(\mathcal{I}_Q)^2$ as its ideal sheaf. For any finite set $B \subset \mathbb{P}^m$ set $2B := \cup_{Q \in B} 2Q$. For any $O \in X_{m,d}$, let $T_O X_{m,d} \subset \mathbb{P}^N$ denote the Zariski tangent space of $X_{m,d}$ at O . Notice that $T_{v_d(Q)} X_{m,d} = \langle v_d(2Q) \rangle$; that is, $T_{v_d(Q)} X_{m,d}$ is the minimal linear subspace of $\mathbb{P}^N$ containing the zero-dimensional scheme $v_d(2Q)$ of $X_{m,d}$. Hence, $\langle \cup_{O \in v_d(B)} T_O X_{m,d} \rangle = \langle v_d(2B) \rangle$ for any finite $B \subset \mathbb{P}^m$. Notice that $\langle \cup_{O \in v_d(B)} T_O X_{m,d} \rangle = \mathbb{P}^N$ if and only if $h^0(\mathbb{P}^m, \mathcal{I}_{2B}(d)) = 0$.

Proof of Theorem 1. Set $\alpha := \lceil \binom{m+d}{m} / (m+1) \rceil$. Since $m \geq 5$ and $d \geq 3$, a theorem of Alexander and Hirschowitz says that α is the first positive integer t such that $\sigma_t(X_{m,d}) = \mathbb{P}^N$ [7–10]. In characteristic zero, this is equivalent to say that α

is the first positive integer t such that $\langle \cup_{Q \in B} T_{v_d(Q)} X_{m,d} \rangle = \mathbb{P}^N$ for a general $B \subset \mathbb{P}^m$ with cardinality t , that is, the first integer t such that $h^0(\mathbb{P}^m, \mathcal{F}_{2B}(d)) = 0$ for the union $B \subset \mathbb{P}^m$ of t general points of $\mathbb{P}^m$ [11, Terracini's lemma, part 2 of Corollary 1.11]. In a positive characteristic, only one implication holds; that is, if $h^0(\mathbb{P}^m, \mathcal{F}_{2B}(d)) = 0$ for a general $B \subset \mathbb{P}^m$ with $\#(B) = t$, then $\sigma_t(X_{m,d}) = \mathbb{P}^N$ [11, part (1) of Corollary 1.11]. This is exactly the version of Alexander-Hirschowitz theorem proved in arbitrary characteristic by Chandler [9, Theorem 1]. Hence, for a general $A \subset X_{m,d}$ such that $\#(A) = \alpha$, we have $\langle \cup_{O \in A} T_O X_{m,d} \rangle = \mathbb{P}^N$. Fix a general $A \subset X_{m,d}$ with $\#(A) = \alpha$. Fix any $f \in \mathbb{K}[x_0, \dots, x_m] \setminus \{0\}$, and let $P \in \mathbb{P}^N$ be the point associated to any λf , $\lambda \in \mathbb{K} \setminus \{0\}$. By the definition of the Veronese embedding, the symmetric tensor rank $r_{m,d}(f)$ is the minimal cardinality of a set $S \subset X_{m,d}$ such that $P \in \langle S \rangle$. Since $\langle \cup_{O \in A} T_O X_{m,d} \rangle = \mathbb{P}^N$, for each $O \in A$, there is $P_O \in T_O X_{m,d}$ such that $P \in \langle \cup_{O \in A} P_O \rangle$. If $P_O \in X_{m,d}$, then it has a symmetric rank 1, while if $P_O \notin X_{m,d}$, then it has symmetric rank d [2, Theorem 32]. Hence, for each $O \in A$, there is $S_O \subset X_{m,d}$ such that $P_O \in \langle S_O \rangle$ and $\#(S_O) \leq d$. Set $S := \cup_{O \in A} S_O$. Since $\#(S) \leq d\alpha$ and $P \in \langle S \rangle$, we have $r_{m,d}(f) \leq d\alpha$. $\square$

Remark 3. Fix any integer $b \geq \lceil \binom{m+d}{m} / (m+1) \rceil$ and any $A \subset X_{m,d}$ such that $\langle \cup_{O \in A} T_O X_{m,d} \rangle = \mathbb{P}^N$ and $\#(A) = b$. We saw that we may take A as any general subset of $X_{m,d}$ with cardinality b . Fix any $f \in \mathbb{K}[x_0, \dots, x_m] \setminus \{0\}$, and let $P \in \mathbb{P}^N$ be the point associated to f . Our choice of the set A implies $P \in \langle \cup_{O \in A} T_O X_{m,d} \rangle$. If (a very big IF) it is possible algorithmically to find $P_O \in T_O X_{m,d}$, $O \in A$, such that $P \in \langle \cup_{O \in A} P_O \rangle$, then it is possible to find explicitly and algorithmically a set $S \subset X_{m,d}$ such that $\#(S) \leq bd$ and $P \in \langle S \rangle$. Indeed, there is an algorithm to find $S_O \subset X_{m,d}$ such that $P_O \in \langle S_O \rangle$ and $\#(S_O) = d$ [2, Algorithm 3]. Take $S := \cup_{O \in A} S_O$.

Acknowledgment

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

- [1] L. H. Lim and V. de Silva, "Tensor rank and the ILL-posedness of the best low-rank approximation problem," *SIAM Journal on Matrix Analysis and Applications*, vol. 30, no. 3, pp. 1084–1127, 2008.
- [2] A. Bernardi, A. Gimigliano, and M. Idà, "On the stratification of secant varieties of Veronese varieties via symmetric rank," *Journal of Symbolic Computation*, vol. 46, pp. 34–55, 2011.
- [3] J. Buczyński and J. M. Landsberg, "Ranks of tensors and a generalization of secant varieties," *Linear Algebra and Its Applications*, vol. 438, no. 2, pp. 668–689, 2013.
- [4] J. M. Landsberg, *Tensors: Geometry and Applications*, vol. 128 of *Graduate Studies in Mathematics*, The American Mathematical Society, 2012.
- [5] J. M. Landsberg and Z. Teitler, "On the ranks and border ranks of symmetric tensors," *Foundations of Computational Mathematics*, vol. 10, no. 3, pp. 339–366, 2010.
- [6] G. Comas and M. Seiguer, "On the rank of a binary form," *Foundations of Computational Mathematics*, vol. 11, no. 1, pp. 65–78, 2011.
- [7] J. Alexander and A. Hirschowitz, "La méthode d'Horace éclatée: application à l'interpolation en degré quatre," *Inventiones Mathematicae*, vol. 107, no. 1, pp. 585–602, 1992.
- [8] J. Alexander and A. Hirschowitz, "Polynomial interpolation in several variables," *The Journal of Algebraic Geometry*, vol. 4, no. 2, pp. 201–222, 1995.
- [9] K. A. Chandler, "A brief proof of a maximal rank theorem for generic double points in projective space," *Transactions of the American Mathematical Society*, vol. 353, no. 5, pp. 1907–1920, 2000.
- [10] M. C. Brambilla and G. Ottaviani, "On the Alexander-Hirschowitz theorem," *Journal of Pure and Applied Algebra*, vol. 212, no. 5, pp. 1229–1251, 2008.
- [11] B. Ådlandsvik, "Joins and higher secant varieties," *Mathematica Scandinavica*, vol. 61, pp. 213–222, 1987.