Research Article

A Note on Solitary Wave Solutions of the Nonlinear Generalized Camassa-Holm Equation

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We give a simple method for applying ordinary differential equation to solve the nonlinear generalized Camassa-Holm equation $u_t + 2ku_x - u_{xxt} + au^m u_x - 2u_x u_{xx} + uu_{xxx} = 0$. Furthermore we give a new ansatz. In the cases where $m = 1, 2, 3$, the numerical simulations demonstrate the results.

1. Introduction

A new dispersive shallow water equation

$$u_t + \kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \quad (1)$$

known as the Camassa-Holm equation has been derived by Camassa and Holm [1]. They showed that the Camassa-Holm equation has peaked wave solitary solutions which have the first derivative discontinuity at the wave peak called “peakons.” In [2], the integral bifurcation method was used to study a nonlinearly dispersive wave equation of Camassa-Holm equation type. In [3], attractor for a coupled nonhomogeneous Camassa-Holm equation with periodic boundary condition was investigated by means of several inequalities. In [4], sufficient conditions for the modified two-component Camassa-Holm system were established. In [5], a class of nonlinear fourth order analogue of a generalized Camassa-Holm equation was studied by using sine-cosine method. In [6], He et al. studied a generalized Camassa-Holm equation. In [7], Wazwaz established solitary wave solutions to the modified forms of Degasperis-Procesi and Camassa-Holm equations. In [8], a family of Camassa-Holm equations with distinct parameters was investigated. Also many aspects of the problems were studied by researchers [9–16]. Authors in [4, 17] presented periodic wave solutions and traveling wave solutions for some equations. In [18], Khuri investigated the periodic wave and peaked solitary wave solutions of the nonlinear generalized Camassa-Holm equation and gave a

ansatz for demonstrating the existence of a new class of solutions. In [19], Tian and Song derive some new exact peaked solitary wave solutions of the generalized Camassa-Holm equation and two types of new exact traveling wave solutions of the generalized weakly dissipative Camassa-Holm equations. In this paper we give a simple method for applying ordinary differential equation to solve the generalized Camassa-Holm equation and give the improved ansatz. The numerical simulation examples demonstrate that our methods are applicable.

2. Simplification of the Nonlinear Generalized Camassa-Holm Equation

Consider the following nonlinear generalized Camassa-Holm equation:

$$u_t + 2ku_x - u_{xxt} + au^m u_x - 2u_x u_{xx} + uu_{xxx} = 0. \quad (2)$$

With the velocity constant c , we seek the traveling wave solution of the form $u(x, t) = v(\xi)$ of (2) where $\xi = x - ct$. Substituting $u(x, t) = v(\xi)$ into (2), we have

$$-c \frac{dv}{d\xi} + 2k \frac{dv}{d\xi} + c \frac{d^3 v}{d\xi^3} + av^m \frac{dv}{d\xi} - 2 \frac{dv}{d\xi} \frac{d^2 v}{d\xi^2} - v \frac{d^3 v}{d\xi^3} = 0. \quad (3)$$

Integrating both sides of (3), we obtain

$$(v - c) \frac{d^2 v}{d\xi^2} + \frac{1}{2} \left(\frac{dv}{d\xi} \right)^2 = \frac{a}{m+1} v^{m+1} + (2k - c) v + b, \quad (4)$$

where b is an arbitrary constant. Let $dv/d\xi = w$. Then $d^2 v/d\xi^2 = w(dw/dv)$. Substituting this into (4) yields

$$\frac{dw}{dv} + \frac{1}{2(v-c)} w = \frac{(a/(m+1))v^{m+1} + (2k-c)v + b}{v-c} w^{-1}. \quad (5)$$

Solving (4) leads to

$$w^2 = (v - c)^{-1} \left\{ d + \int 2 \left[\frac{a}{m+1} v^{m+1} + (2k - c) v + b \right] dv \right\}, \quad (6)$$

where d is an arbitrary constant. Therefore

$$\begin{aligned} & \left(\frac{dv}{d\xi} \right)^2 \\ &= (v - c)^{-1} \\ & \times \left\{ d + \left[\frac{2a}{(m+1)(m+2)} v^{m+2} + (2k - c) v^2 + bv \right] \right\}. \end{aligned} \quad (7)$$

From $(d\xi)^2 = (d|\xi|)^2$, we have

$$\begin{aligned} & \left(\frac{dv}{d|\xi|} \right)^2 \\ &= (v - c)^{-1} \\ & \times \left\{ d + \left[\frac{2a}{(m+1)(m+2)} v^{m+2} + (2k - c) v^2 + bv \right] \right\}. \end{aligned} \quad (8)$$

Therefore

$$\begin{aligned} & \int \sqrt{\frac{v - c}{d + \left[\frac{2a}{(m+1)(m+2)} v^{m+2} + (2k - c) v^2 + bv \right]}} dv \\ &= \begin{cases} h + \xi, \\ h - \xi, \\ h + |\xi|, \\ h - |\xi|, \end{cases} \end{aligned} \quad (9)$$

where h is an arbitrary constant.

When $m = 1$, (9) is the case of the Camassa-Holm equation. We take $2k \neq c$, $ac - 3c + 2k = 0$, $d = 0$, and $b = 0$, then

$$\sqrt{\frac{3}{a}} \ln \frac{c}{v} = \begin{cases} h + \xi, \\ h - \xi, \\ h + |\xi|, \\ h - |\xi|. \end{cases} \quad (10)$$

So

$$v = \begin{cases} ce^{\sqrt{(a/3)}(h+\xi)}, \\ ce^{\sqrt{(a/3)}(h-\xi)}, \\ ce^{\sqrt{(a/3)}(h+|\xi|)}, \\ ce^{\sqrt{(a/3)}(h-|\xi|)}. \end{cases} \quad (11)$$

It can be checked that $u(x, t) = ce^{h+\sqrt{(a/3)}(x-ct)}$, $u(x, t) = ce^{h-\sqrt{(a/3)}(x-ct)}$, and $u(x, t) = ce^{h+\sqrt{(a/3)}|x-ct|}$ are infinitely great solutions of the Camassa-Holm equation without asymptotic behavior [20]. Only

$$u(x, t) = ce^{h-\sqrt{(a/3)}|x-ct|} \quad (12)$$

are the solitary wave solutions of the Camassa-Holm equation.

When $m = 2$, we take $2k < c$, $ac^2 - 6c + 12k = 0$, $d = 0$, and $b = 0$. In the similar way in the case $m = 1$, we only choose

$$\sqrt{\frac{6}{a}} \int \frac{1}{v\sqrt{v+c}} d\varphi = h - |\xi|, \quad (13)$$

then

$$-\sqrt{\frac{24}{ac}} \operatorname{atanh} \sqrt{\frac{v+c}{c}} = h - |\xi|. \quad (14)$$

From (14), we have

$$v = c \tanh^2 \left[\sqrt{\frac{ac}{24}} (|\xi| - h) \right] - c. \quad (15)$$

Therefore we have the solitary wave solutions of the generalized Camassa-Holm equation as

$$u(x, t) = c \tanh^2 \left[\sqrt{\frac{ac}{24}} (|x - ct| - h) \right] - c. \quad (16)$$

When $m = 3$, we take $2k \neq c$, $ac^3 - 10c + 20k = 0$, $d = 0$, and $b = 0$ then

$$\sqrt{\frac{10}{a}} \int \frac{1}{\sqrt{v^2 + cv + c^2}} dv = h - |\xi|. \quad (17)$$

Then

$$-\sqrt{\frac{10}{ac^2}} \operatorname{atanh} \frac{c(v+2c)}{|c|\sqrt{v^2 + cv + c^2}} = h - |\xi|. \quad (18)$$

Solving (18), we obtain

$$v = \left(-2c \left\{ 1 - \tanh^2 \left[\sqrt{\frac{ac^2}{10}} (|\xi| - h) \right] \right\} \right. \\ \left. \pm 2|c| \left(3 \left\{ 1 - \tanh^2 \left[\sqrt{\frac{ac^2}{10}} (|\xi| - h) \right] \right\} \right. \right. \\ \left. \left. \times \tanh^2 \left[\sqrt{\frac{ac^2}{10}} (|\xi| - h) \right] \right)^{1/2} \right) \\ \times \left(1 - 4\tanh^2 \left[\sqrt{\frac{ac^2}{10}} (|\xi| - h) \right] \right)^{-1}. \quad (19)$$

Therefore

$$u(x, t) = \left(-2c \left[1 - \tanh^2 f(x - ct) \right] \right. \\ \left. + 2|c| \sqrt{3 \left[1 - \tanh^2 f(x - ct) \right] \tanh^2 f(x - ct)} \right) \\ \times \left(1 - 4\tanh^2 f(x - ct) \right)^{-1} \quad (20)$$

or

$$u(x, t) = \left(-2c \left[1 - \tanh^2 f(x - ct) \right] \right. \\ \left. - 2|c| \sqrt{3 \left[1 - \tanh^2 f(x - ct) \right] \tanh^2 f(x - ct)} \right) \\ \times \left(1 - 4\tanh^2 f(x - ct) \right)^{-1}, \quad (21)$$

where $f(x - ct) = \sqrt{(ac^2/10)}(|x - ct| - h)$.

3. Ansatz for the Generalized Camassa-Holm Equation

From [18], we have the following ordinary differential equation in F :

$$2F^3 \frac{d^2 F}{d\xi^2} + 4F^2 \left(\frac{dF}{d\xi} \right)^2 = \frac{a}{m+1} \sum_{k=0}^{m+1} \binom{m+1}{k} F^{2k} c^{m+1-k} \\ + (2k - c) F^2 + 2kc - c^2 + b, \quad (22)$$

where $F = \sqrt{v - c}$.

Let $dF/d\xi = G$. Then $d^2 F/d\xi^2 = G(dG/dF)$. Substituting this into (22) gives the following first order Bernoulli's ordinary differential equation:

$$\frac{dG}{dF} + \frac{2}{F} G = \left(\left(\frac{a}{m+1} \sum_{k=0}^{m+1} \binom{m+1}{k} F^{2k} c^{m+1-k} \right. \right. \\ \left. \left. + (2k - c) F^2 + 2kc - c^2 + b \right) \right. \\ \left. \times (2F^3)^{-1} \right) G^{-1}. \quad (23)$$

Solving (23) leads to

$$G^2 = F^{-4} \left\{ g + \int \left[\frac{a}{m+1} \sum_{k=0}^{m+1} \binom{m+1}{k} F^{2k} c^{m+1-k} \right. \right. \\ \left. \left. + (2k - c) F^2 + 2kc - c^2 + b \right] F dF \right\}. \quad (24)$$

Therefore

$$\left(F \frac{dF}{d\xi} \right)^2 = gF^{-2} + \left[\frac{a}{m+1} \sum_{k=0}^{m+1} \frac{1}{2k+2} \binom{m+1}{k} F^{2k} c^{m+1-k} \right. \\ \left. + \frac{1}{4} (2k - c) F^2 + \frac{1}{2} (2kc - c^2 + b) \right], \quad (25)$$

where g is an arbitrary constant. We observe that $F(dF/d\xi)$ hardly becomes a polynomial in F unless in the particular cases. So this is a new ansatz compared with the ansatz in [18]. Similarly, we have

$$\left(F \frac{dF}{d|\xi|} \right)^2 = gF^{-2} + \left[\frac{a}{m+1} \sum_{k=0}^{m+1} \frac{1}{2k+2} \binom{m+1}{k} F^{2k} c^{m+1-k} \right. \\ \left. + \frac{1}{4} (2k - c) F^2 + \frac{1}{2} (2kc - c^2 + b) \right]. \quad (26)$$

Seemingly, (26) is more difficult than (8). On the contrary, (26) is much easier to be solved than (8). This can be seen in the following.

When $m = 1$, let $g = 0$. For $F^2 = \theta$, we have

$$\int \left(\frac{a}{3} \theta^2 + \frac{1}{4} (2k - c + 4ac) \theta \right. \\ \left. + \frac{1}{2} (2kc - c^2 + 2ac^2 + b) \right)^{-1/2} d\theta = r - |\xi|, \quad (27)$$

where r is an arbitrary constant. So

$$\begin{aligned} & \sqrt{\frac{3}{a}} \ln \left[\left(\theta + \frac{3(2k - c + 4ac)}{8a} \right) \right. \\ & \quad \left. + \sqrt{\theta^2 + \frac{3(2k - c + 4ac)}{4a} \theta + \frac{3(2kc - c^2 + 2ac^2 + b)}{2a}} \right] \\ & = r - |\xi|. \end{aligned} \quad (28)$$

Solving (28), we have that when $3(2kc - c^2 + 2ac^2 + b)/(2a) - 9(2k - c + 4ac)^2/(16a^2) > 0$,

$$\begin{aligned} \theta &= \sqrt{\frac{3(2kc - c^2 + 2ac^2 + b)}{2a} - \frac{9(2k - c + 4ac)^2}{16a^2}} \\ & \quad \times \sinh \left[\sqrt{\frac{3}{a}} (\bar{r} - |\xi|) \right] - \frac{3(2k - c + 4ac)}{8a}, \end{aligned} \quad (29)$$

when $3(2kc - c^2 + 2ac^2 + b)/(2a) - 9(2k - c + 4ac)^2/(16a^2) < 0$,

$$\begin{aligned} \theta &= \sqrt{\frac{3(2kc - c^2 + 2ac^2 + b)}{2a} - \frac{9(2k - c + 4ac)^2}{16a^2}} \\ & \quad \times \cosh \left[\sqrt{\frac{3}{a}} (\bar{r} - |\xi|) \right] - \frac{3(2k - c + 4ac)}{8a}. \end{aligned} \quad (30)$$

Equations (29) and (30) have no asymptotic behavior. When $3(2kc - c^2 + 2ac^2 + b)/(2a) = 9(2k - c + 4ac)^2/(16a^2)$, we obtain

$$\theta = \frac{1}{2} e^{\sqrt{(a/3)}(r - |\xi|)} - \frac{3(2k - c + 4ac)}{16a}. \quad (31)$$

So

$$u(x, t) = \frac{1}{2} e^{\sqrt{(a/3)}(r - |x - ct|)} - \frac{6k - 3c - 4ac}{16a}. \quad (32)$$

These solitary solutions have the first derivative discontinuity at the wave peak.

When $m = 2$, let $g = 0$. Consider the following:

$$\begin{aligned} & \int \left(\frac{a}{6} \theta^3 + \frac{2ac}{3} \theta^2 + \frac{1}{4} (2k - c + 4ac^2) \theta \right. \\ & \quad \left. + \frac{1}{2} \left(2k - c^2 + \frac{4ac^3}{3} + b \right) \right)^{-1/2} d\theta = r - |\xi|. \end{aligned} \quad (33)$$

Take $2k - c + 4ac^2 = 0$ and $2k - c^2 + 4ac^3/3 + b = 0$, then

$$-\sqrt{\frac{6}{ac}} \operatorname{atanh} \sqrt{\frac{\theta + 4c}{4c}} = r - |\xi|. \quad (34)$$

From (34), we have

$$\theta = 4c \tanh^2 \left[\sqrt{\frac{ac}{6}} (|\xi| - r) \right] - 4c. \quad (35)$$

Therefore, we have the solitary wave solutions of the generalized Camassa-Holm equation as

$$u(x, t) = 4c \tanh^2 \left[\sqrt{\frac{ac}{6}} (|x - ct| - r) \right] - 3c. \quad (36)$$

Take $2k - c + 4ac^2 = 8ac^2/3$ and $2k - c^2 + 4ac^3/3 + b = 0$, then

$$\sqrt{\frac{12}{ac}} \operatorname{atan} \sqrt{\frac{\theta}{2c}} = r - |\xi|. \quad (37)$$

Similarly, we have

$$u(x, t) = 2c \tan^2 \left[\sqrt{\frac{ac}{12}} (r - |x - ct|) \right] + c. \quad (38)$$

This result is the same as (28) in [18]. But it is easily to obtain.

When $m = 3$, let $g = 0$. Consider the following:

$$\begin{aligned} & \int \left(\frac{a}{10} \theta^4 + \frac{ac}{2} \theta^3 + ac^2 \theta^2 + \frac{1}{4} (2k - c + 4ac^3) \theta \right. \\ & \quad \left. + \frac{1}{2} (2k - c^2 + ac^4 + b) \right)^{-1/2} d\theta = r - |\xi|. \end{aligned} \quad (39)$$

From (39) and $a, c > 0$, we have seen that $F(dF/d\xi)$ is not a polynomial in F (compared with Equation (35) in [18]). Take $2k - c + 4ac^3 = 0$ and $2k - c^2 + ac^4 + b = 0$, then

$$-\sqrt{\frac{1}{ac^2}} \operatorname{atanh} \frac{\sqrt{10}(c\theta + 4c^2)}{4|c| \sqrt{\theta^2 + 5c\theta + 10c^2}} = r - |\xi|. \quad (40)$$

Solving (40), we obtain

$$\begin{aligned} \theta &= \left(-20c \left[1 - \tanh^2 p(\xi) \right] \right. \\ & \quad \left. \pm 4|c| \sqrt{15 \left[1 - \tanh^2 p(\xi) \right] \tanh^2 p(\xi)} \right) \\ & \quad \times \left(5 - 8 \tanh^2 p(\xi) \right)^{-1}, \end{aligned} \quad (41)$$

where $p(\xi) = \sqrt{a}|c|(|\xi| - r)$. Therefore

$$\begin{aligned} u(x, t) &= \left(\left(-20c \left[1 - \tanh^2 p(x - ct) \right] + 4|c| \right. \right. \\ & \quad \left. \times \sqrt{15 \left[1 - \tanh^2 p(x - ct) \right] \tanh^2 p(x - ct)} \right) \\ & \quad \left. \times \left(5 - 8 \tanh^2 p(x - ct) \right)^{-1} \right) + c, \end{aligned} \quad (42)$$

or

$$\begin{aligned} u(x, t) &= \left(\left(-20c \left[1 - \tanh^2 p(x - ct) \right] - 4|c| \right. \right. \\ & \quad \left. \times \sqrt{15 \left[1 - \tanh^2 p(x - ct) \right] \tanh^2 p(x - ct)} \right) \\ & \quad \left. \times \left(5 - 8 \tanh^2 p(x - ct) \right)^{-1} \right) + c, \end{aligned} \quad (43)$$

where $p(x - ct) = \sqrt{a}|c|(|x - ct| - r)$.

4. Numerical Simulation Examples

Example 1. In (12), take $c = 1$, $h = 0$, and $a = 3$. Then the solitary wave solution is $u(x, t) = e^{-|x-t|}$.

We want to show figures with peakon feature. But to save space, omitting figures we only give MATLAB program: “ $x = -10:0.001:10$; $u = \exp(-\text{abs}(x))$; $\text{plot}(x, u)$ ” for $t = 0$ and “for $n = 1:101$ $x(n) = (n-1)/5 - 10$; $t(n) = x(n)$; end $[x(n), t(n)] = \text{meshgrid}(x(n), t(n))$; for $n = 1:101$ for $m = 1:101$ $u(n, m) = \exp(-\text{abs}(x(n) - t(m)))$; end end $\text{mesh}(u)$ ” for the 3-dimensional case.

Example 2. In (16), take $c = 1$, $h = 0$, and $a = 24$. Then solitary wave solution is $u(x, t) = \tanh^2(|x - t| - 1) - 1$.

The figures with peakon feature will be constructed by “ $x = -10:0.001:10$; $u = (\tanh(\text{abs}(x)-1)).^2 - 1$; $\text{plot}(x, u)$ ” for $t = 0$ and “for $n = 1:101$ $x(n) = (n-1)/5 - 10$; $t(n) = x(n)$; end $[x(n), t(n)] = \text{meshgrid}(x(n), t(n))$; for $n = 1:101$ for $m = 1:101$ $u(n, m) = (\tanh(\text{abs}(x(n) - t(m)) - 1)).^2 - 1$; end end $\text{mesh}(u)$ ” for the 3-dimensional case.

Example 3. In (20), take $c = 1$, $h = 1$, and $a = 10$. Then symmetrically solitary wave solution is

$$u(x, t) = \left\{ -2 \left[1 - \tanh^2(|x - t| - 1) \right] + 2\sqrt{3 \left[1 - \tanh^2(|x - t| - 1) \right] \tanh^2(|x - t| - 1)} \right\} \times (1 - 4\tanh^2(|x - t| - 1))^{-1} \quad (44)$$

For $t = 0$, when $(a = -2, b = 2)$, $(a = 2.31, b = 2.32)$, and $(a = 2.32, b = 5)$, respectively, the figures are with peakon feature. $x = a:0.0001:b$; $y = (-2.* (1 - \tanh(\text{abs}(x)-1)).^2) + 2.*\text{sqrt}(3.* (1 - \tanh(\text{abs}(x)-1)).^2).*\tanh(\text{abs}(x))) ./ (1 - 4.* (1 - \tanh(\text{abs}(x)-1)).^2))$; $\text{plot}(x, y)$. Replace x by $x(n) - t(m)$ and the others are similar to Example 2 program for the 3-dimensional case.

5. Conclusions

In this paper, we make simplification of the nonlinear generalized Camassa-Holm equation and give the improved ansatz for the generalized Camassa-Holm equation. The three numerical simulation examples with MATLAB programs demonstrate solitary wave solutions with peakon feature which show our method applicable. This method may be applied to many other nonlinear equations.

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