Research Article

On Twisted Products Finsler Manifolds

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Received 16 May 2013; Accepted 10 June 2013

Academic Editors: I. Biswas and A. Borowiec

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On the product of two Finsler manifolds $M_1 \times M_2$, we consider the twisted metric F which is constructed by using Finsler metrics F_1 and F_2 on the manifolds M_1 and M_2 , respectively. We introduce horizontal and vertical distributions on twisted product Finsler manifold and study C-reducible and semi-C-reducible properties of this manifold. Then we obtain the Riemannian curvature and some of non-Riemannian curvatures of the twisted product Finsler manifold such as Berwald curvature, mean Berwald curvature, and we find the relations between these objects and their corresponding objects on M_1 and M_2 . Finally, we study locally dually flat twisted product Finsler manifold.

1. Introduction

Twisted and warped product structures are widely used in geometry to construct new examples of semi-Riemannian manifolds with interesting curvature properties (see [1–3]). Twisted product metric tensors, as a generalization of warped product metric tensors, have also been useful in the study of several aspects of submanifold theory, namely, in hypersurfaces of complex space forms [4], in Lagrangian submanifolds [5], in decomposition of curvature netted hypersurfaces [6], and so forth.

The notion of twisted product of Riemannian manifolds was mentioned first by Chen in [7] and was generalized for the pseudo-Riemannian case by Ponge and Reckziegel [8]. Chen extended the study of twisted product for CR-submanifolds in Kähler manifolds [9].

On the other hand, Finsler geometry is a natural extension of Riemannian geometry without the quadratic restriction. Therefore, it is natural to extend the construction of twisted product manifolds for Finsler geometry. In [10], Kozma-Peter-Shimada extended the construction of twisted product for the Finsler geometry.

Let (M_1, F_1) and (M_2, F_2) be two Finsler manifolds with Finsler metrics F_1 and F_2 , respectively, and let $f : M_1 \times M_2 \rightarrow \mathbb{R}^+$ be a smooth function. On the product manifold $M_1 \times M_2$, we consider the metric

$$F(v_1, v_2) = \sqrt{F_1^2(v_1) + f^2(x, y) F_2^2(v_2)} \quad (1)$$

for all $(x, y) \in M_1 \times M_2$ and $(v_1, v_2) \in TM_1^\circ \times TM_2^\circ$, where TM_1° is the slit tangent manifold $TM_1^\circ = TM_1 \setminus \{o\}$. The manifold $M_1 \times M_2$ endowed with this metric, we call the twisted product of the manifolds M_1 and M_2 and denote it by $M_1 \times_f M_2$. The function f will be called the twisted function. In particular, if f is constant on M_2 , then $M_1 \times_f M_2$ is called warped product manifold.

Let (M, F) be a Finsler manifold. The second and third order derivatives of $(1/2)F_x^2$ at $y \in T_x M_0$ are the symmetric trilinear forms $\mathbf{g}_y$ and $\mathbf{C}_y$ on $T_x M$, which called the fundamental tensor and Cartan torsion, respectively. A Finsler metric is called semi-C-reducible if its Cartan tensor is given by

$$C_{ijk} = \frac{p}{1+n} \{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} + \frac{q}{C^2} I_i I_j I_k, \quad (2)$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on TM , h_{ij} is the angular metric, and $C^2 = I^i I_i$ [11]. If $q = 0$, then F is called C-reducible Finsler metric, and if $p = 0$, then F is called C2-like metric.

The geodesic curves of a Finsler metric F on a smooth manifold M are determined by the system of second-order differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients. F is called a Berwald metric, if G^i are quadratic in $y \in T_x M$ for any $x \in M$. Taking a trace of Berwald curvature yields mean Berwald curvature E . Then F is said to be isotropic mean

Berwald metric if $\mathbf{E} = ((n+1)/2)cF^{-1}\mathbf{h}$, where $\mathbf{h} = h_{ij}dx^i \otimes dx^j$ is the angular metric and $c = c(x)$ is a scalar function on M [12].

The second variation of geodesics gives rise to a family of linear maps $\mathbf{R}_y = R^i_k dx^k \otimes (\partial/\partial x^i)|_x : T_x M \rightarrow T_x M$ at any point $y \in T_x M$. R_y is called the Riemann curvature in the direction y . A Finsler metric F is said to be of scalar flag curvature, if for some scalar function $\mathbf{K}$ on TM_0 the Riemann curvature is in the form $R^i_k = \mathbf{K}F^2 h^i_j$. If $\mathbf{K} = \text{constant}$, then F is said to be of constant flag curvature.

In this paper, we introduce the horizontal and vertical distributions on tangent bundle of a doubly warped product Finsler manifold and construct the Finsler connection on this manifold. Then, we study some geometric properties of this product manifold such as C-reducible and semi-C-reducible. Then, we introduce the Riemannian curvature of twisted product Finsler manifold $(M_1 \times_f M_2, F)$ and find the relation between it and Riemannian curvatures of its components (M_1, F_1) and (M_2, F_2) . In the cases that $(M_1 \times_f M_2, F)$ is flat or it has the scalar flag curvature, we obtain some results on its components. Then, we study twisted product Finsler metrics with vanishing Berwald curvature and isotropic mean Berwald curvature, respectively. Finally, we study locally dually flat twisted product Finsler manifold. We prove that there is not exist any locally dually flat proper twisted product Finsler manifold.

2. Preliminary

Let M be an n -dimensional C^∞ manifold. Denote by $T_x M$ the tangent space at $x \in M$, by $TM = \cup_{x \in M} T_x M$ the tangent bundle of M , and by $TM^\circ = TM \setminus \{0\}$ the slit tangent bundle on M [13]. A Finsler metric on M is a function $F : TM \rightarrow [0, \infty)$ which has the following properties:

- (i) F is C^∞ on TM° ;
- (ii) F is positively 1-homogeneous on the fibers of tangent bundle TM ;
- (iii) for each $y \in T_x M$, the following quadratic form $\mathbf{g}_y$ on $T_x M$ is positive definite:

$$\mathbf{g}_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(y + su + tv)]|_{s,t=0}, \quad u, v \in T_x M. \quad (3)$$

Let $x \in M$ and $F_x := F|_{T_x M}$. To measure the non-Euclidean feature of F_x , define $\mathbf{C}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2} \frac{d}{dt} [\mathbf{g}_{y+tw}(u, v)]|_{t=0}, \quad u, v, w \in T_x M. \quad (4)$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y \in TM^\circ}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian [14].

For $y \in T_x M^\circ$, define mean Cartan torsion $\mathbf{I}_y$ by $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$, $C_{ijk} = (1/2)(\partial g_{ij}/\partial y^k)$ and $u = u^i(\partial/\partial x^i)|_x$. By Deicke's theorem, F is Riemannian if and only if $\mathbf{I}_y = 0$.

Let (M, F) be a Finsler manifold. For $y \in T_x M^\circ$, define the Matsumoto torsion $\mathbf{M}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R}$ by $\mathbf{M}_y(u, v, w) := M_{ijk}(y)u^i v^j w^k$, where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1} \{I_i h_{jk} + I_j h_{ik} + I_k h_{ij}\}, \quad (5)$$

where $h_{ij} := FF_{y^i y^j}$ is the angular metric. In [15], it is proved that a Finsler metric F on a manifold M of dimension $n \geq 3$ is a Randers metric if and only if $\mathbf{M}_y = 0$, for all $y \in TM_0$. A Randers metric $F = \alpha + \beta$ on a manifold M is just a Riemannian metric $\alpha = \sqrt{a_{ij}y^i y^j}$ perturbed by a one form $\beta = b_i(x)y^i$ on M such that $\|\beta\|_\alpha < 1$.

A Finsler metric is called semi-C-reducible if its Cartan tensor is given by

$$C_{ijk} = \frac{p}{1+n} \{h_{ij}I_k + h_{jk}I_i + h_{ki}I_j\} + \frac{q}{C^2} I_i I_j I_k, \quad (6)$$

where $p = p(x, y)$ and $q = q(x, y)$ are scalar function on TM and $C^2 = I^i I_i$ with $p + q = 1$. In [11], Matsumoto-Shibata proved that every (α, β) metric on a manifold M of dimension $n \geq 3$ is semi-C-reducible.

Given a Finsler manifold (M, F) , then a global vector field $\mathbf{G}$ is induced by F on TM° , which in a standard coordinate (x^i, y^i) for TM° is given by $\mathbf{G} = y^i(\partial/\partial x^i) - 2G^i(x, y)(\partial/\partial y^i)$, where

$$G^i := \frac{1}{4} g^{il} \left\{ \frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l} \right\}, \quad y \in T_x M. \quad (7)$$

$\mathbf{G}$ is called the spray associated to (M, F) . In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$ [16].

A Finsler metric $F = F(x, y)$ on a manifold M is said to be locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients are in the following form:

$$G^i = -\frac{1}{2} g^{ij} H_{y^j}, \quad (8)$$

where $H = H(x, y)$ is a C^∞ scalar function on TM° satisfying $H(x, \lambda y) = \lambda^3 H(x, y)$ for all $\lambda > 0$. Such a coordinate system is called an adapted coordinate system. In [17], Shen proved that the Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is dually flat if and only if it satisfies $(F^2)_{x^k y^l} y^k = 2(F^2)_{x^l}$.

For a tangent vector $y \in T_x M^\circ$, define $\mathbf{B}_y : T_x M \otimes T_x M \otimes T_x M \rightarrow \mathbb{R}$ by $\mathbf{B}_y(u, v, w) := B^i_{jkl}(y)u^j v^k w^l (\partial/\partial x^i)|_x$ and $\mathbf{E}_y(u, v) := E_{jk}(y)u^j v^k$, where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2} B^m_{jkm}. \quad (9)$$

$\mathbf{B}$ and $\mathbf{E}$ are called the Berwald curvature and mean Berwald curvature, respectively. Then F is called a Berwald metric and weakly Berwald metric if $\mathbf{B} = 0$ and $\mathbf{E} = 0$, respectively [14]. It is proved that on a Berwald space, the parallel translation along any geodesic preserves the Minkowski functionals [18].

A Finsler metric F is said to be isotropic Berwald metric and isotropic mean Berwald metric if its Berwald curvature and mean Berwald curvature are in the following form, respectively:

$$B_{jkl}^i = c \{F_{y^j y^k} \delta_l^i + F_{y^k y^l} \delta_j^i + F_{y^l y^j} \delta_k^i + F_{y^j y^k y^l} y^i\}, \quad (10)$$

$$E_{ij} = \frac{1}{2} (n+1) c F^{-1} h_{ij},$$

where $c = c(x)$ is a scalar function on M [19].

The Riemann curvature $\mathbf{R}_y = R^i_k dx^k \otimes (\partial/\partial x^i)|_x : T_x M \rightarrow T_x M$ is a family of linear maps on tangent spaces defined by

$$R^i_k = 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2 G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}. \quad (11)$$

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry was first introduced by L. Berwald [20]. For a flag $P = \text{span}\{y, u\} \subset T_x M$ with flagpole y , the flag curvature $\mathbf{K} = \mathbf{K}(P, y)$ is defined by

$$\mathbf{K}(P, y) := \frac{\mathbf{g}_y(u, \mathbf{R}_y(u))}{\mathbf{g}_y(y, y) \mathbf{g}_y(u, u) - \mathbf{g}_y(y, u)^2}. \quad (12)$$

We say that a Finsler metric F is of scalar curvature if for any $y \in T_x M$, the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function on the slit tangent bundle TM° . If $\mathbf{K} = \text{constant}$, then F is said to be of constant flag curvature.

3. Nonlinear Connection

Let (M_1, F_1) and (M_2, F_2) be two Finsler manifolds. Then the functions

$$(i) \quad g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F_1^2(x, y)}{\partial y^i \partial y^j}, \quad (13)$$

$$(ii) \quad g_{\alpha\beta}(u, v) = \frac{1}{2} \frac{\partial^2 F_2^2(u, v)}{\partial v^\alpha \partial v^\beta}$$

define a Finsler tensor field of type $(0, 2)$ on TM_1° and TM_2° , respectively. Now let $(M_1 \times_f M_2, F)$ be a doubly warped Finsler manifold, $\mathbf{x} = (x, u) \in M$, $\mathbf{y} = (y, v) \in T_x M$, $M = M_1 \times M_2$, and $T_x M = T_x M_1 \oplus T_u M_2$. Then by using (13), we conclude that

$$(\mathbf{g}_{ab}(x, u, y, v)) = \left(\frac{1}{2} \frac{\partial^2 F^2(x, u, y, v)}{\partial y^a \partial y^b} \right) = \begin{bmatrix} g_{ij} & 0 \\ 0 & f^2 g_{\alpha\beta} \end{bmatrix}, \quad (14)$$

where $\mathbf{y}^a = (y^i, v^\alpha)$, $\mathbf{g}_{ij} = g_{ij}$, $\mathbf{g}_{\alpha\beta} = f^2 g_{\alpha\beta}$, $\mathbf{g}_{i\beta} = \mathbf{g}_{\alpha j} = 0$, $i, j, \dots \in \{1, \dots, n_1\}$, $\alpha, \beta, \dots \in \{1, \dots, n_2\}$, and $a, b, \dots \in \{1, \dots, n_1 + n_2\}$.

Now we consider spray coefficients of F_1 , F_2 , and F as

$$G^i(x, y) = \frac{1}{4} g^{ih} \left(\frac{\partial^2 F_1^2}{\partial y^h \partial x^j} y^j - \frac{\partial F_1^2}{\partial x^h} \right)(x, y), \quad (15)$$

$$G^\alpha(u, v) = \frac{1}{4} g^{\alpha\gamma} \left(\frac{\partial^2 F_2^2}{\partial v^\gamma \partial u^\beta} v^\beta - \frac{\partial F_2^2}{\partial u^\gamma} \right)(u, v), \quad (16)$$

$$\mathbf{G}^a(\mathbf{x}, \mathbf{y}) = \frac{1}{4} \mathbf{g}^{ab} \left(\frac{\partial^2 F^2}{\partial y^b \partial x^c} y^c - \frac{\partial F^2}{\partial x^b} \right)(\mathbf{x}, \mathbf{y}). \quad (17)$$

Taking into account the homogeneity of both F_1^2 and F_2^2 and using (15) and (16), we can conclude that G^i and G^α are positively homogeneous of degree two with respect to (y^i) and (v^α) , respectively. Hence from Euler theorem for homogeneous functions, we infer that

$$\frac{\partial G^i}{\partial y^j} y^j = 2G^i, \quad \frac{\partial G^\alpha}{\partial v^\beta} v^\beta = 2G^\alpha. \quad (18)$$

By setting $a = i$ in (17), we have

$$\mathbf{G}^i(x, u, y, v) = \frac{1}{4} \mathbf{g}^{ih} \left(\frac{\partial^2 F^2}{\partial y^h \partial x^j} y^j + \frac{\partial^2 F^2}{\partial y^h \partial u^\alpha} v^\alpha - \frac{\partial F^2}{\partial x^h} \right). \quad (19)$$

Direct calculations give us

$$\begin{aligned} \frac{\partial F^2}{\partial x^h} &= \frac{\partial F_1^2}{\partial x^h} + \frac{\partial f^2}{\partial x^h} F_2^2, \\ \frac{\partial^2 F^2}{\partial y^h \partial x^j} &= \frac{\partial^2 F_1^2}{\partial y^h \partial x^j}, \\ \frac{\partial^2 F^2}{\partial y^h \partial u^\alpha} &= 0. \end{aligned} \quad (20)$$

Putting these equations together $\mathbf{g}^{ih} = g^{ih}$ in the previous equation and using (15) imply that

$$\mathbf{G}^i(x, u, y, v) = G^i(x, y) - \frac{1}{2} f f^i F_2^2. \quad (21)$$

Similarly, by setting $a = \alpha$ in (17) and using (16), we obtain

$$\begin{aligned} \mathbf{G}^\alpha(x, u, y, v) &= G^\alpha(u, v) \\ &+ f^{-1} \left(f_j v^\alpha y^j + f_\lambda v^\alpha v^\lambda - \frac{1}{2} f_\gamma g^{\alpha\gamma} F_2^2 \right), \end{aligned} \quad (22)$$

where $f_i = \partial f / \partial x^i$, $f_\gamma = \partial f / \partial u^\gamma$, $f^i = g^{ih} f_h$, and $f^\gamma = g^{\lambda\gamma} f_\lambda$. Therefore we have $\mathbf{G}^a = (\mathbf{G}^i, \mathbf{G}^\alpha)$, where $\mathbf{G}^a$, $\mathbf{G}^i$, and $\mathbf{G}^\alpha$ are given by (17), (21), and (22), respectively.

Now, we put

$$\begin{aligned} (i) \quad \mathbf{G}_b^a &:= \frac{\partial \mathbf{G}^a}{\partial y^b}, \\ (ii) \quad G_j^i &:= \frac{\partial G^i}{\partial y^j}, \\ (iii) \quad G_\beta^\alpha &:= \frac{\partial G^\alpha}{\partial v^\beta}. \end{aligned} \quad (23)$$

Then we have the following.

Lemma 1. The coefficients $\mathbf{G}_b^a$ defined by (23) satisfy in the following:

$$(\mathbf{G}_b^a(x, u, y, v)) = \begin{bmatrix} \mathbf{G}_j^i(x, u, y, v) & \mathbf{G}_j^\alpha(x, u, y, v) \\ \mathbf{G}_\beta^i(x, u, y, v) & \mathbf{G}_\beta^\alpha(x, u, y, v) \end{bmatrix}, \quad (24)$$

where

$$\mathbf{G}_j^i(x, u, y, v) := \frac{\partial \mathbf{G}^i}{\partial y^j} = G_j^i + C_j^{ih} f f_h F_2^2, \quad (25)$$

$$\mathbf{G}_\beta^i(x, u, y, v) := \frac{\partial \mathbf{G}^i}{\partial v^\beta} = -f f^i v_\beta, \quad (26)$$

$$\mathbf{G}_j^\alpha(x, u, y, v) := \frac{\partial \mathbf{G}^\alpha}{\partial y^j} = f^{-1} f_j v^\alpha, \quad (27)$$

$$\begin{aligned} \mathbf{G}_\beta^\alpha(x, u, y, v) &:= \frac{\partial \mathbf{G}^\alpha}{\partial v^\beta} \\ &= G_\beta^\alpha + f^{-1} \left(C_\beta^{\alpha\gamma} f_\gamma F_2^2 + f_j y^j \delta_\beta^\alpha \right. \\ &\quad \left. - f^\alpha v_\beta + f_\beta v^\alpha + f_\gamma v^\gamma \delta_\beta^\alpha \right). \end{aligned} \quad (28)$$

Next, VTM° kernel of the differential of the projection map

$$\pi := (\pi_1, \pi_2) : TM_1^\circ \oplus TM_2^\circ \longrightarrow M_1 \times M_2, \quad (29)$$

which is a well-defined subbundle of TTM° , is considered. Locally, $\Gamma(VTM^\circ)$ is spanned by the natural vector fields $\{\partial/\partial y^1, \dots, \partial/\partial y^{n_1}, \partial/\partial v^1, \dots, \partial/\partial v^{n_2}\}$, and it is called the *twisted vertical distribution* on TM° . Then, using the functions given by (25)–(28), the nonholonomic vector fields are defined as follows:

$$\frac{\delta^t}{\delta^t x^i} := \frac{\partial}{\partial x^i} - \mathbf{G}_i^j \frac{\partial}{\partial y^j} - \mathbf{G}_i^\beta \frac{\partial}{\partial v^\beta}, \quad (30)$$

$$\frac{\delta^t}{\delta^t u^\alpha} := \frac{\partial}{\partial u^\alpha} - \mathbf{G}_\alpha^j \frac{\partial}{\partial y^j} - \mathbf{G}_\alpha^\beta \frac{\partial}{\partial v^\beta}, \quad (31)$$

which make it possible to construct a complementary vector subbundle HTM° to VTM° in TTM° as follows:

$$HTM^\circ := \text{span} \left\{ \frac{\delta^t}{\delta^t x^1}, \dots, \frac{\delta^t}{\delta^t x^{n_1}}, \frac{\delta^t}{\delta^t u^1}, \dots, \frac{\delta^t}{\delta^t u^{n_2}} \right\}. \quad (32)$$

HTM° is called the *twisted horizontal distribution* on TM° . Thus the tangent bundle of TM° admits the decomposition

$$TTM^\circ = HTM^\circ \oplus VTM^\circ. \quad (33)$$

It is shown that $\mathbf{G} := (\mathbf{G}_b^a)$ is a nonlinear connection on $TM = TM_1 \oplus TM_2$. In the following, we compute the nonlinear connection of a twisted product Finsler manifold.

Proposition 2. If $(M_1 \times_f M_2, F)$ is a twisted product Finsler manifold, then $\mathbf{G} = (\mathbf{G}_b^a)$ is the nonlinear connection on TM . Further, one has

$$\begin{aligned} \frac{\partial \mathbf{G}_j^i}{\partial y^k} y^k + \frac{\partial \mathbf{G}_j^i}{\partial v^\gamma} v^\gamma &= \mathbf{G}_j^i, \\ \frac{\partial \mathbf{G}_\beta^i}{\partial y^k} y^k + \frac{\partial \mathbf{G}_\beta^i}{\partial v^\gamma} v^\gamma &= \mathbf{G}_\beta^i, \\ \frac{\partial \mathbf{G}_j^\alpha}{\partial y^k} y^k + \frac{\partial \mathbf{G}_j^\alpha}{\partial v^\gamma} v^\gamma &= \mathbf{G}_j^\alpha, \\ \frac{\partial \mathbf{G}_\beta^\alpha}{\partial y^k} y^k + \frac{\partial \mathbf{G}_\beta^\alpha}{\partial v^\gamma} v^\gamma &= \mathbf{G}_\beta^\alpha. \end{aligned} \quad (34)$$

Definition 3. Using decomposition (33), the twisted vertical morphism $v^t : TTM^\circ \rightarrow VTM^\circ$ is defined by

$$v^t := \frac{\partial}{\partial y^i} \otimes \delta^t y^i + \frac{\partial}{\partial v^\alpha} \otimes \delta^t v^\alpha, \quad (35)$$

where

$$\begin{aligned} \delta^t y^i &:= dy^i + \mathbf{G}_j^i dx^j + \mathbf{G}_\beta^i du^\beta, \\ \delta^t v^\alpha &:= dv^\alpha + \mathbf{G}_j^\alpha dx^j + \mathbf{G}_\beta^\alpha du^\beta. \end{aligned} \quad (36)$$

For this projective morphism, the following hold:

$$\begin{aligned} v^t \left(\frac{\partial}{\partial y^i} \right) &= \frac{\partial}{\partial y^i}, & v^t \left(\frac{\partial}{\partial v^\alpha} \right) &= \frac{\partial}{\partial v^\alpha}, \\ v^t \left(\frac{\delta^t}{\delta^t x^i} \right) &= 0, & v^t \left(\frac{\delta^t}{\delta^t u^i} \right) &= 0. \end{aligned} \quad (37)$$

From the previous equations, we conclude that

$$(v^t)^2 = v^t, \quad \ker(v^t) = HTM^\circ. \quad (38)$$

This mapping is called the *twisted vertical projective*.

Definition 4. Using decomposition (33), the doubly warped horizontal projective $h^t : TTM^\circ \rightarrow HTM^\circ$ is defined by

$$h^t = id - v^t \quad (39)$$

or

$$h^t = \frac{\delta^t}{\delta^t x^i} \otimes dx^i + \frac{\delta^t}{\delta^t u^\alpha} \otimes du^\alpha. \quad (40)$$

For this projective morphism, the following hold:

$$\begin{aligned} h^t \left(\frac{\delta^t}{\delta^t x^i} \right) &= \frac{\delta^t}{\delta^t x^i}, & h^t \left(\frac{\delta^t}{\delta^t u^\alpha} \right) &= \frac{\delta^t}{\delta^t u^\alpha}, \\ h^t \left(\frac{\partial}{\partial y^i} \right) &= 0, & h^t \left(\frac{\partial}{\partial v^\alpha} \right) &= 0. \end{aligned} \quad (41)$$

Thus we result that

$$(h^t)^2 = h^t, \quad \ker(h^t) = VTM^\circ. \quad (42)$$

Definition 5. Using decomposition (33), the twisted almost tangent structure $J^t : HTM^\circ \rightarrow VTM^\circ$ is defined by

$$J^t : \frac{\partial}{\partial y^i} \otimes dx^i + \frac{\partial}{\partial v^\alpha} \otimes du^\alpha, \quad (43)$$

or

$$\begin{aligned} J^t \left(\frac{\delta^t}{\delta^t x^i} \right) &= \frac{\partial}{\partial y^i}, & J^t \left(\frac{\delta^t}{\delta^t u^\alpha} \right) &= \frac{\partial}{\partial v^\alpha}, \\ J^t \left(\frac{\partial}{\partial y^i} \right) &= J^t \left(\frac{\partial}{\partial v^\alpha} \right) = 0. \end{aligned} \quad (44)$$

Thus we result that

$$(J^t)^2 = 0, \quad \ker J^t = \text{Im} J^t = VTM^\circ. \quad (45)$$

Here, we introduce some geometrical objects of twisted product Finsler manifold. In order to simplify the equations, we rewritten the basis of HTM° and VTM° as follows:

$$\begin{aligned} \frac{\delta^t}{\delta^t x^a} &= \frac{\delta^t}{\delta^t x^i} \delta_a^i + \frac{\delta^t}{\delta^t u^\alpha} \delta_a^\alpha, \\ \frac{\partial}{\partial y^a} &= \frac{\partial}{\partial y^i} \delta_a^i + \frac{\partial}{\partial v^\alpha} \delta_a^\alpha. \end{aligned} \quad (46)$$

Thus

$$TTM^\circ = \text{span} \left\{ \frac{\delta^t}{\delta^t x^a}, \frac{\partial}{\partial y^a} \right\}. \quad (47)$$

The Lie brackets of this basis is given by

$$\begin{aligned} \left[\frac{\delta^t}{\delta^t x^a}, \frac{\delta^t}{\delta^t x^b} \right] &= \mathbf{R}^c_{ab} \frac{\partial}{\partial y^c}, \\ \left[\frac{\delta^t}{\delta^t x^a}, \frac{\partial}{\partial y^b} \right] &= \mathbf{G}^c_{ab} \frac{\partial}{\partial y^c}, \\ \left[\frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b} \right] &= 0, \end{aligned} \quad (48)$$

where

$$(i) \quad \mathbf{R}^c_{ab} = \frac{\delta^t \mathbf{G}^c_a}{\delta^t x^b} - \frac{\delta^t \mathbf{G}^c_b}{\delta^t x^a}, \quad (49)$$

$$(ii) \quad \mathbf{G}^c_{ab} = \frac{\partial \mathbf{G}^c_a}{\partial y^b}. \quad (50)$$

Therefore, we have the following.

Corollary 6. Let $(M_1 \times_f M_2, F)$ be a twisted product Finsler manifold. Then

$$\mathbf{R}^c_{ab} = \left(\mathbf{R}^k_{ij}, \mathbf{R}^k_{i\beta}, \mathbf{R}^k_{\alpha j}, \mathbf{R}^k_{\alpha\beta}, \mathbf{R}^\gamma_{ij}, \mathbf{R}^\gamma_{i\beta}, \mathbf{R}^\gamma_{\alpha j}, \mathbf{R}^\gamma_{\alpha\beta} \right), \quad (51)$$

where

$$\begin{aligned} \mathbf{R}^k_{ij} &:= \frac{\delta^t \mathbf{G}^k_i}{\delta^t x^j} - \frac{\delta^t \mathbf{G}^k_j}{\delta^t x^i}, & \mathbf{R}^k_{i\beta} &:= \frac{\delta^t \mathbf{G}^k_i}{\delta^t u^\beta} - \frac{\delta^t \mathbf{G}^k_\beta}{\delta^t x^i}, \\ \mathbf{R}^k_{\alpha j} &:= \frac{\delta^t \mathbf{G}^k_\alpha}{\delta^t x^j} - \frac{\delta^t \mathbf{G}^k_j}{\delta^t u^\alpha}, & \mathbf{R}^k_{\alpha\beta} &:= \frac{\delta^t \mathbf{G}^k_\alpha}{\delta^t u^\beta} - \frac{\delta^t \mathbf{G}^k_\beta}{\delta^t u^\alpha}, \\ \mathbf{R}^\gamma_{ij} &:= \frac{\delta^t \mathbf{G}^\gamma_i}{\delta^t x^j} - \frac{\delta^t \mathbf{G}^\gamma_j}{\delta^t x^i}, & \mathbf{R}^\gamma_{i\beta} &:= \frac{\delta^t \mathbf{G}^\gamma_i}{\delta^t u^\beta} - \frac{\delta^t \mathbf{G}^\gamma_\beta}{\delta^t x^i}, \\ \mathbf{R}^\gamma_{\alpha j} &:= \frac{\delta^t \mathbf{G}^\gamma_\alpha}{\delta^t x^j} - \frac{\delta^t \mathbf{G}^\gamma_j}{\delta^t u^\alpha}, & \mathbf{R}^\gamma_{\alpha\beta} &:= \frac{\delta^t \mathbf{G}^\gamma_\alpha}{\delta^t u^\beta} - \frac{\delta^t \mathbf{G}^\gamma_\beta}{\delta^t u^\alpha}. \end{aligned} \quad (52)$$

With a simple calculation, we have the following.

Corollary 7. Let $(M_1 \times_f M_2, F)$ be a twisted product Finsler manifold. Then

$$\mathbf{G}^c_{ab} = \left(\mathbf{G}^k_{ij}, \mathbf{G}^k_{i\beta}, \mathbf{G}^k_{\alpha j}, \mathbf{G}^k_{\alpha\beta}, \mathbf{G}^\gamma_{ij}, \mathbf{G}^\gamma_{i\beta}, \mathbf{G}^\gamma_{\alpha j}, \mathbf{G}^\gamma_{\alpha\beta} \right), \quad (53)$$

where

$$\begin{aligned} \mathbf{G}^\gamma_{\alpha\beta} &= \frac{\partial \mathbf{G}^\gamma_\alpha}{\partial v^\beta} \\ &= \mathbf{G}^\gamma_{\alpha\beta} + f^{-1} \left(C^{\gamma\lambda}_{\alpha\beta} f_\lambda F_2^2 + 2C^{\gamma\lambda}_\alpha f_\lambda v_\beta + 2C^{\gamma\lambda}_\beta f_\lambda v_\alpha \right. \\ &\quad \left. - f^\gamma g_{\alpha\beta} + f_\beta \delta^\gamma_\alpha + f_\alpha \delta^\gamma_\beta \right) = \mathbf{G}^\gamma_{\beta\alpha}, \\ \mathbf{G}^k_{ij} &= \frac{\partial \mathbf{G}^k_i}{\partial y^j} = \mathbf{G}^k_{ij} + C^{kh}_{i;j} f f_h F_2^2 = \mathbf{G}^k_{ji}, \\ \mathbf{G}^k_{i\beta} &= \frac{\partial \mathbf{G}^k_i}{\partial v^\beta} = 2C^{kh}_i f f_h v_\beta = \mathbf{G}^k_{\beta i}, \\ \mathbf{G}^k_{\alpha\beta} &= \frac{\partial \mathbf{G}^k_\alpha}{\partial v^\beta} = -f f^\beta g_{\alpha\beta} = \mathbf{G}^k_{\beta\alpha}, \\ \mathbf{G}^\gamma_{i\beta} &= \frac{\partial \mathbf{G}^\gamma_i}{\partial v^\beta} = f^{-1} f_i \delta^\gamma_\beta = \mathbf{G}^\gamma_{\beta i}, \\ \mathbf{G}^\gamma_{ij} &= \frac{\partial \mathbf{G}^\gamma_i}{\partial y^j} = \mathbf{G}^\gamma_{ji} = 0, \end{aligned} \quad (54)$$

where $C^{kh}_{i;j} = \partial C^{kh}_i / \partial y^j$. Apart from $\mathbf{G}^c_{ab}$, the functions $\mathbf{F}^c_{ab}$ are given by

$$\mathbf{F}^c_{ab} = \frac{1}{2} \mathbf{g}^{ce} \left(\frac{\delta^t \mathbf{g}_{ea}}{\delta^t x^b} + \frac{\delta^t \mathbf{g}_{eb}}{\delta^t x^a} - \frac{\delta^t \mathbf{g}_{ab}}{\delta^t x^e} \right). \quad (55)$$

Corollary 8. Let $(M_1 \times_f M_2, F)$ be a twisted product Finsler manifold. Then

$$\mathbf{F}^c_{ab} = \left(\mathbf{F}^k_{ij}, \mathbf{F}^k_{i\beta}, \mathbf{F}^k_{\alpha j}, \mathbf{F}^k_{\alpha\beta}, \mathbf{F}^\gamma_{ij}, \mathbf{F}^\gamma_{i\beta}, \mathbf{F}^\gamma_{\alpha j}, \mathbf{F}^\gamma_{\alpha\beta} \right), \quad (56)$$

where

$$\mathbf{F}_{ij}^k = F_{ij}^k - (M_j^r C_{ir}^k + M_i^r C_{jr}^k - M_h^r C_{ijr}^k g^{kh}), \quad (57)$$

$$\mathbf{F}_{i\beta}^k = -\mathbf{G}_\beta^r C_{ir}^k = \mathbf{F}_{\beta i}^k, \quad (58)$$

$$\mathbf{F}_{\alpha\beta}^k = -f f^k g_{\alpha\beta} + f^2 g^{kh} \mathbf{G}_h^\lambda C_{\alpha\beta\lambda}^k, \quad (59)$$

$$\mathbf{F}_{ij}^\gamma = f^{-2} g^{\gamma\lambda} \mathbf{G}_\lambda^r C_{ijr}^\gamma, \quad (60)$$

$$\mathbf{F}_{i\beta}^\gamma = f^{-1} f_i \delta_\beta^\gamma - \mathbf{G}_i^\alpha C_{\alpha\beta}^\gamma = \mathbf{F}_{\beta i}^\gamma, \quad (61)$$

$$\mathbf{F}_{\alpha\beta}^\gamma = F_{\alpha\beta}^\gamma + N_{\alpha\beta}^\gamma - (M_\beta^\mu C_{\alpha\mu}^\gamma + M_\alpha^\mu C_{\beta\mu}^\gamma - M_\lambda^\mu C_{\alpha\beta\mu}^\gamma g^{\gamma\lambda}), \quad (62)$$

$$F_{ij}^k = \frac{1}{2} g^{kh} \left(\frac{\delta g_{hi}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h} \right),$$

$$F_{\alpha\beta}^\gamma = \frac{1}{2} g^{\gamma\lambda} \left(\frac{\delta g_{\lambda\alpha}}{\delta u^\beta} + \frac{\delta g_{\lambda\beta}}{\delta u^\alpha} - \frac{\delta g_{\alpha\beta}}{\delta u^\lambda} \right),$$

$$M_i^r = C_i^{rh} f f_h F_2^2,$$

$$M_\alpha^\mu = f^{-1} (C_\alpha^{\mu\gamma} f_\gamma F_2^2 + f_r \gamma^r \delta_\alpha^\mu + f_\gamma \gamma^\gamma \delta_\alpha^\mu - g^{\mu\gamma} f_\gamma \nu_\alpha + f_\alpha \nu^\mu),$$

$$N_{\alpha\beta}^\gamma = f^{-1} (f_\beta \delta_\alpha^\gamma + f_\alpha \delta_\beta^\gamma - f_\lambda g^{\gamma\lambda} g_{\alpha\beta}). \quad (63)$$

Proof. By using (55), we have

$$\mathbf{F}_{ij}^k = \frac{1}{2} g^{kh} \left(\frac{\delta^t g_{hi}}{\delta^t x^j} + \frac{\delta^t g_{hj}}{\delta^t x^i} - \frac{\delta^t g_{ij}}{\delta^t x^h} \right). \quad (64)$$

Since g_{ij} is a function with respect to (x, y) , then by (25) and (30) we obtain

$$\frac{\delta^t g_{hi}}{\delta^t x^j} = \frac{\delta g_{hi}}{\delta x^j} - 2M_j^r C_{hir}^r. \quad (65)$$

Interchanging i, j , and h in the previous equation gives us

$$\begin{aligned} \frac{\delta^t g_{hj}}{\delta^t x^i} &= \frac{\delta g_{hj}}{\delta x^i} - 2M_i^r C_{hjr}^r, \\ \frac{\delta^t g_{ij}}{\delta^t x^h} &= \frac{\delta g_{ij}}{\delta x^h} - 2M_h^r C_{ijr}^r. \end{aligned} \quad (66)$$

Putting these equation in (64) give us (57). In the similar way, we can prove the another relation. $\square$

By using (i) of (23) and (57)–(62), we can conclude the following.

Lemma 9. Let $(M_1 \times_f M_2, F)$ be a twisted product Finsler manifold. Then $\mathbf{y}^c \mathbf{F}_{bc}^a = \mathbf{G}_b^a$, where $\mathbf{F}_{bc}^a$ and $\mathbf{G}_b^a$ are defined by (55) and (i) of (23), respectively.

The Cartan torsion is one of the most important non-Riemannian quantity in Finsler geometry, and it is first

introduced by Finsler and emphasized by Cartan which measures a departure from a Riemannian manifold. More precisely, a Finsler metric reduces to a Riemannian metric if and only if it has vanishing Cartan torsion. The local components of Cartan tensor field of the twisted Finsler manifold $(M_1 \times_f M_2, F)$ is defined by

$$\mathbf{C}_{bc}^a = \frac{1}{2} \mathbf{g}^{ae} \frac{\partial \mathbf{g}_{be}}{\partial \mathbf{y}^c}. \quad (67)$$

From this definition, we conclude the following.

Lemma 10. Let $\mathbf{C}_{ij}^k$ and $\mathbf{C}_{\alpha\beta}^\gamma$ be the local components of Cartan tensor field on M_1 and M_2 , respectively. Then one has

$$\mathbf{C}_{ab}^c = (\mathbf{C}_{ij}^k, \mathbf{C}_{i\beta}^k, \mathbf{C}_{\alpha j}^k, \mathbf{C}_{\alpha\beta}^k, \mathbf{C}_{ij}^\gamma, \mathbf{C}_{i\beta}^\gamma, \mathbf{C}_{\alpha j}^\gamma, \mathbf{C}_{\alpha\beta}^\gamma), \quad (68)$$

where

$$\mathbf{C}_{ij}^k = \frac{1}{2} g^{kh} \frac{\partial g_{ij}}{\partial y^h} = C_{ij}^k, \quad (69)$$

$$\mathbf{C}_{\alpha\beta}^\gamma = \frac{1}{2} g^{\gamma\lambda} \frac{\partial g_{\alpha\beta}}{\partial v^\lambda} = C_{\alpha\beta}^\gamma,$$

$$\text{and } \mathbf{C}_{i\beta}^k = \mathbf{C}_{\alpha j}^k = \mathbf{C}_{\alpha\beta}^k = \mathbf{C}_{ij}^\gamma = \mathbf{C}_{i\beta}^\gamma = \mathbf{C}_{\alpha j}^\gamma = 0.$$

By using the Lemma 10, we can get the following.

Corollary 11. Let $(M_1 \times_f M_2, F)$ be a twisted product Finsler manifold. Then $(M_1 \times_f M_2, F)$ is a Riemannian manifold if and only if (M_1, F_1) and (M_2, F_2) are Riemannian manifold.

Various interesting special forms of Cartan tensors have been obtained by some Finslerians [11]. The Finsler spaces having such special forms have been called C-reducible, C2-like, semi-C-reducible, and so forth. In [21], Matsumoto introduced the notion of C-reducible Finsler metrics and proved that any Randers metric is C-reducible. Later on, Matsumoto-Hōjō proves that the converse is true too [15].

Here, we define the Matsumoto twisted tensor $\mathbf{M}_{abc}$ for a twisted product Finsler manifold $(M_1 \times_f M_2, F)$ as follows:

$$\mathbf{M}_{abc} = \mathbf{C}_{abc} - \frac{1}{n+1} \{ \mathbf{I}_a \mathbf{h}_{bc} + \mathbf{I}_b \mathbf{h}_{ac} + \mathbf{I}_c \mathbf{h}_{ab} \}, \quad (70)$$

where $\mathbf{I}_a = \mathbf{g}^{bc} \mathbf{C}_{abc}$, $\mathbf{C}_{abc} = \mathbf{g}_{cd} \mathbf{C}_{ab}^d$, and $\mathbf{h}_{ab} = \mathbf{g}_{ab} - (1/F^2) \mathbf{y}_a \mathbf{y}_b$. By attention to the previous equation and relations

$$\mathbf{C}_{ijk} = C_{ijk}, \quad \mathbf{C}_{\alpha\beta\gamma} = f^2 C_{\alpha\beta\gamma}, \quad (71)$$

we obtain

$$\mathbf{M}_{\alpha j k} = -\frac{1}{n+1} \left\{ I_\alpha \left(g_{jk} - \frac{1}{F^2} y_j y_k \right) - \frac{f^2}{F^2} v_\alpha (I_j y_k + I_k y_j) \right\}. \quad (72)$$

Contracting the previous equation in $y^j y^k$ gives us

$$y^j y^k \mathbf{M}_{\alpha j k} = -\frac{f^2 F_1^2 F_2^2}{(n+1) F^2} I_\alpha. \quad (73)$$

Similarly, we obtain

$$\nu^\lambda \nu^\beta \mathbf{M}_{i\beta\lambda} = -\frac{f^2 F_1^2 F_2^2}{(n+1) F^2} I_i. \quad (74)$$

Therefore if $\mathbf{M}_{i\beta\lambda} = \mathbf{M}_{\alpha jk} = 0$, then we get $I_i = I_\alpha = 0$; that is, (M_1, F_1) and (M_2, F_2) are Riemannian manifolds. Thus we have the following.

Theorem 12. *There is not exist any C-reducible twisted product Finsler manifold.*

Now, we are going to consider semi-C-reducible twisted product Finsler manifold $(M_1 \times_f M_2, F)$. Let $(M_1 \times_f M_2, F)$ be a semi-C-reducible twisted product Finsler manifold. Then we have

$$\mathbf{C}_{abc} = \frac{p}{n+1} \{ \mathbf{I}_a \mathbf{h}_{bc} + \mathbf{I}_b \mathbf{h}_{ac} + \mathbf{I}_c \mathbf{h}_{ab} \} + \frac{q}{C^2} \mathbf{I}_a \mathbf{I}_b \mathbf{I}_c, \quad (75)$$

where $C^2 = \mathbf{I}^a \mathbf{I}_a$ and p and q are scalar function on $M_1 \times_f M_2$ with $p + q = 1$. This equation gives us

$$\begin{aligned} 0 &= \mathbf{C}_{\alpha jk} \\ &= \frac{p}{n+1} \left\{ I_\alpha \left(g_{jk} - \frac{1}{F^2} y_j y_k \right) - \frac{f^2}{F^2} \nu_\alpha (I_j y_k + I_k y_j) \right\} \\ &\quad + \frac{q}{C^2} I_\alpha I_j I_k. \end{aligned} \quad (76)$$

Contracting the previous equation with $y^j y^k$ implies that

$$p f^2 F_1^2 F_2^2 I_\alpha = 0. \quad (77)$$

Therefore we have $p = 0$ or $I_\alpha = 0$. If $p = 0$, then F is C_2 -like metric. But if $p \neq 0$, then $I_\alpha = 0$; that is, F_2 is Riemannian metric. In this case, with similar way, we conclude that F_1 is Riemannian metric. But, definition F cannot be a Riemannian metric. Therefore we have the following.

Theorem 13. *Every semi-C-reducible twisted product Finsler manifold $(M_1 \times_f M_2, F)$ is a C_2 -like manifold.*

4. Riemannian Curvature

The Riemannian curvature of twisted product Finsler manifold $(M_1 \times_f M_2, F)$ with respect to Berwald connection is given by

$$\mathbf{R}_{cd}^a = \frac{\delta^t \mathbf{F}_{bc}^a}{\delta^t \mathbf{x}^d} - \frac{\delta^t \mathbf{F}_{bd}^a}{\delta^t \mathbf{x}^c} + \mathbf{F}_{de}^a \mathbf{F}_{bc}^e - \mathbf{F}_{ce}^a \mathbf{F}_{bd}^e. \quad (78)$$

Lemma 14. *Let $(M_1 \times_f M_2, F)$ be a twisted product Finsler manifold. Then one has*

$$\mathbf{R}_{cd}^a = \mathbf{y}^b \mathbf{R}_{bcd}^a, \quad (79)$$

where $\mathbf{R}_{cd}^a$ and $\mathbf{y}^b \mathbf{R}_{bcd}^a$ are given by (50) and (78).

Proof. By using (78), we have

$$\mathbf{y}^b \mathbf{R}_{bkl}^i = \mathbf{y}^b \frac{\delta^t \mathbf{F}_{bk}^i}{\delta^t \mathbf{x}^l} - \mathbf{y}^b \frac{\delta^t \mathbf{F}_{bl}^i}{\delta^t \mathbf{x}^k} + \mathbf{y}^b \mathbf{F}_{le}^i \mathbf{F}_{bk}^e - \mathbf{y}^b \mathbf{F}_{ke}^i \mathbf{F}_{bl}^e. \quad (80)$$

By using Corollary 8 and Lemma 9, we obtain

$$\begin{aligned} \mathbf{y}^b \frac{\delta^t \mathbf{F}_{bk}^i}{\delta^t \mathbf{x}^l} &= \frac{\delta^t \mathbf{G}_k^i}{\delta^t \mathbf{x}^l} + \mathbf{F}_{jk}^i \mathbf{G}_l^j + \mathbf{F}_{\beta k}^i \mathbf{G}_l^\beta, \\ \mathbf{y}^b \mathbf{F}_{le}^i \mathbf{F}_{bk}^e &= \mathbf{F}_{lh}^i \mathbf{G}_k^h + \mathbf{F}_{ly}^i \mathbf{G}_k^\gamma. \end{aligned} \quad (81)$$

Interchanging i and j in the previous equation implies that

$$\begin{aligned} \mathbf{y}^b \frac{\delta^t \mathbf{F}_{bl}^i}{\delta^t \mathbf{x}^k} &= \frac{\delta^t \mathbf{G}_l^i}{\delta^t \mathbf{x}^k} + \mathbf{F}_{jl}^i \mathbf{G}_k^j + \mathbf{F}_{\beta l}^i \mathbf{G}_k^\beta, \\ \mathbf{y}^b \mathbf{F}_{ke}^i \mathbf{F}_{bl}^e &= \mathbf{F}_{kh}^i \mathbf{G}_l^h + \mathbf{F}_{ky}^i \mathbf{G}_l^\gamma. \end{aligned} \quad (82)$$

Setting (81) and (82) in (80) gives us $\mathbf{y}^b \mathbf{R}_{bkl}^i = \mathbf{R}_{kl}^i$. In the similar way, we can obtain this relation for another indices. $\square$

Using (78), we can compute the Riemannian curvature of a twisted product Finsler manifold.

Lemma 15. *Let $(M_1 \times_f M_2, F)$ be a twisted product Finsler manifold. Then the coefficients of Riemannian curvature are as follows:*

$$\begin{aligned} \mathbf{R}_{jkl}^i &= \mathbf{R}_{jkl}^i \\ &- \left\{ \left\{ M_l^r \frac{\partial F_{jk}^i}{\partial y^r} + \frac{\delta^t M_{jk}^i}{\delta^t \mathbf{x}^l} + \mathbf{F}_{lh}^i M_{jk}^h + M_{lh}^i F_{jk}^h - M_{lh}^i M_{jk}^h \right. \right. \\ &\quad \left. \left. + f^{-2} g^{\alpha\gamma} \mathbf{G}_\alpha^r \mathbf{G}_\gamma^m C_{lr}^i C_{jkm} \right\} - \mathfrak{C}_l^k \right\}, \end{aligned} \quad (83)$$

$$\begin{aligned} \mathbf{R}_{\alpha kl}^i &= \left\{ -\frac{\delta^t}{\delta^t \mathbf{x}^l} (\mathbf{G}_\alpha^r C_{kr}^i) - (F_{rl}^i - M_{rl}^i) \mathbf{G}_\alpha^m C_{km}^r \right. \\ &\quad \left. - f^{-1} \mathbf{G}_\beta^r C_{lr}^i f_k \delta_\alpha^\beta + \mathbf{G}_\beta^r \mathbf{G}_k^\mu C_{lr}^i C_{\alpha\mu}^\beta \right\} - \mathfrak{C}_l^k, \end{aligned} \quad (84)$$

$$\begin{aligned} \mathbf{R}_{j\beta\lambda}^i &= \left\{ -\frac{\delta^t}{\delta^t \mathbf{u}^\lambda} (\mathbf{G}_\beta^r C_{jr}^i) + \mathbf{G}_\lambda^m \mathbf{G}_\beta^l C_{rm}^i C_{jl}^r \right. \\ &\quad \left. - (f^i g_{\alpha\lambda} - f \mathbf{G}_h^\mu g^{ih} C_{\alpha\lambda\mu}) (f_j \delta_\beta^\alpha - f \mathbf{G}_j^\gamma C_{\beta\gamma}^\alpha) \right\} \\ &- \mathfrak{C}_\lambda^\beta, \end{aligned} \quad (85)$$

$$\begin{aligned}
\mathbf{R}_{\alpha\beta l}^i &= \frac{\delta^t}{\delta^t u^\beta} (\mathbf{G}_\alpha^r C_{lr}^i) - \frac{\delta^t}{\delta^t x^l} f (f^i g_{\alpha\beta} - f \mathbf{G}_h^\lambda g^{ih} C_{\alpha\beta\lambda}) \\
&\quad - \mathbf{G}_\beta^m \mathbf{G}_\alpha^s C_{rm}^i C_{ls}^r + (f^i g_{\mu\beta} - f \mathbf{G}_h^\lambda g^{ih} C_{\mu\beta\lambda}) \\
&\quad \times (f_l \delta_\alpha^\mu - f \mathbf{G}_l^\gamma C_{\alpha\nu}^\mu) - f g^{rh} (F_{rl}^i - M_{rl}^i) \\
&\quad \times (f_h g_{\alpha\beta} - f \mathbf{G}_h^\lambda C_{\alpha\beta\lambda}) - \mathbf{G}_\mu^r C_{lr}^i (F_{\alpha\beta}^\mu + N_{\alpha\beta}^\mu - M_{\alpha\beta}^\mu), \quad (86)
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{j\beta l}^i &= -\frac{\delta^t}{\delta^t x^l} (\mathbf{G}_\beta^r C_{jr}^i) - \frac{\delta^t}{\delta^t u^\beta} (F_{jl}^i - M_{jl}^i) \\
&\quad - (F_{lr}^i - M_{lr}^i) \mathbf{G}_\beta^s C_{js}^r - f^{-1} \mathbf{G}_\alpha^r C_{lr}^i \\
&\quad \times (f_j \delta_\beta^\alpha - f \mathbf{G}_j^\mu C_{\beta\mu}^\alpha) + \mathbf{G}_\beta^s C_{rs}^i (F_{jl}^r - M_{jl}^r) \\
&\quad + f^{-1} \mathbf{G}_\mu^r C_{jlr} (f^i \delta_\beta^\mu - f \mathbf{G}_h^\lambda g^{ih} C_{\beta\lambda}^\mu), \quad (87)
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{\alpha\beta\lambda}^i &= \left\{ -\frac{\delta^t}{\delta^t u^\lambda} (f f^i g_{\alpha\beta} - f^2 g^{ih} \mathbf{G}_h^\mu C_{\alpha\beta\mu}) + f \mathbf{G}_\lambda^s C_{rs}^i \right. \\
&\quad \times (f^r g_{\alpha\beta} - f \mathbf{G}_l^\mu C_{\alpha\beta\mu} g^{rl}) - f (f^i g_{\lambda\mu} - f g^{ih} \mathbf{G}_h^\kappa C_{\lambda\mu\kappa}) \\
&\quad \times (F_{\alpha\beta}^\mu + N_{\alpha\beta}^\mu - M_{\alpha\beta}^\mu) \left. \right\} - \mathfrak{E}_\lambda^\beta, \quad (88)
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{jkl}^\gamma &= \left\{ \frac{\delta^t}{\delta^t x^l} (f^{-2} g^{\gamma\lambda} \mathbf{G}_\lambda^r C_{jkr}) + f^{-2} g^{\gamma\lambda} \mathbf{G}_\lambda^s C_{lrs} \right. \\
&\quad \times (F_{jk}^r - M_{jk}^r) + f^{-3} \mathbf{G}_\mu^r C_{jkr} (f_l g^{\gamma\mu} - f \mathbf{G}_l^\alpha C_{\alpha}^{\gamma\mu}) \left. \right\} \\
&\quad - \mathfrak{E}_l^k, \quad (89)
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{j\beta l}^\gamma &= \frac{\delta^t}{\delta^t x^l} (f^{-1} f_j \delta_\beta^\gamma - f \mathbf{G}_j^\alpha C_{\alpha\beta}^\gamma) \\
&\quad - \frac{\delta^t}{\delta^t u^\beta} (f^{-2} g^{\gamma\lambda} \mathbf{G}_\lambda^r C_{jlr}) - f^{-2} g^{\gamma\lambda} \mathbf{G}_\lambda^s \mathbf{G}_\beta^m C_{ls}^h C_{hjm} \\
&\quad + f^{-2} (f_l \delta_\mu^\gamma - f \mathbf{G}_l^\alpha C_{\mu\alpha}^\gamma) (f_j \delta_\beta^\mu - f \mathbf{G}_j^\gamma C_{\beta\nu}^\mu) \\
&\quad - f^{-1} (f_r \delta_\beta^\gamma - f \mathbf{G}_r^\alpha C_{\beta\alpha}^\gamma) (F_{jl}^r - M_{jl}^r) \\
&\quad - f^{-2} g^{\mu\lambda} \mathbf{G}_\lambda^r C_{jlr} (F_{\beta\mu}^\gamma + N_{\beta\mu}^\gamma - M_{\beta\mu}^\gamma), \quad (90)
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{\alpha\beta l}^\gamma &= \frac{\delta^t}{\delta^t x^l} (F_{\alpha\beta}^\gamma + N_{\alpha\beta}^\gamma - M_{\alpha\beta}^\gamma) \\
&\quad - \frac{\delta^t}{\delta^t u^\beta} (f^{-1} f_l \delta_\alpha^\gamma - \mathbf{G}_l^\mu C_{\alpha\mu}^\gamma) \\
&\quad - f^{-1} g^{\gamma\lambda} \mathbf{G}_\lambda^s C_{ls}^h (f_h g_{\alpha\beta} - f \mathbf{G}_h^\mu C_{\alpha\beta\mu}) \\
&\quad + f^{-1} (f_l \delta_\mu^\gamma - f \mathbf{G}_l^\kappa C_{\mu\kappa}^\gamma) (F_{\alpha\beta}^\mu + N_{\alpha\beta}^\mu - M_{\alpha\beta}^\mu)
\end{aligned}$$

$$\begin{aligned}
&+ f^{-1} \mathbf{G}_\alpha^s C_{ls}^r (f_r \delta_\beta^\gamma - f \mathbf{G}_r^\kappa C_{\beta\kappa}^\gamma) \\
&- f^{-1} (f_l \delta_\alpha^\mu - f \mathbf{G}_l^\kappa C_{\alpha\kappa}^\mu) (F_{\beta\mu}^\gamma + N_{\beta\mu}^\gamma - M_{\beta\mu}^\gamma), \quad (91)
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{j\beta\lambda}^\gamma &= \left\{ \frac{\delta^t}{\delta^t u^\lambda} (f^{-1} f_j \delta_\beta^\gamma - \mathbf{G}_j^\alpha C_{\alpha\beta}^\gamma) \right. \\
&\quad + f^{-1} (F_{\alpha\lambda}^\gamma + N_{\alpha\lambda}^\gamma - M_{\alpha\lambda}^\gamma) (f_j \delta_\beta^\alpha - f \mathbf{G}_j^\gamma C_{\beta\nu}^\alpha) \\
&\quad \left. - f^{-1} \mathbf{G}_\beta^m C_{jm}^r (f_r \delta_\lambda^\gamma - f \mathbf{G}_r^\alpha C_{\lambda\alpha}^\gamma) \right\} - \mathfrak{E}_\lambda^\beta, \quad (92)
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{\alpha kl}^\gamma &= \left\{ \frac{\delta^t}{\delta^t x^l} (f^{-1} f_k \delta_\alpha^\gamma - \mathbf{G}_k^\mu C_{\alpha\mu}^\gamma) \right. \\
&\quad + f^{-2} (f_l \delta_\beta^\gamma - f \mathbf{G}_l^\kappa C_{\beta\kappa}^\gamma) (f_k \delta_\alpha^\beta - f \mathbf{G}_k^\gamma C_{\alpha\nu}^\beta) \\
&\quad \left. - f^{-2} \mathbf{G}_\mu^s \mathbf{G}_\alpha^m g^{\gamma\mu} C_{ls}^h C_{hkm} \right\} - \mathfrak{E}_l^k, \quad (93)
\end{aligned}$$

$$\begin{aligned}
\mathbf{R}_{\alpha\beta\lambda}^\gamma &= \mathbf{R}_{\alpha\beta\lambda}^\gamma - \left\{ \left\{ M_\lambda^\kappa \frac{\partial F_{\alpha\beta}^\gamma}{\partial v^\kappa} + \frac{\delta^t M_{\alpha\beta}^\gamma}{\delta^t u^\lambda} + F_{\lambda\mu}^\gamma M_{\alpha\beta}^\mu + M_{\lambda\mu}^\gamma F_{\alpha\beta}^\mu \right. \right. \\
&\quad - M_{\lambda\mu}^\gamma M_{\alpha\beta}^\mu + \frac{\delta^t N_{\alpha\lambda}^\gamma}{\delta^t u^\beta} + F_{\beta\mu}^\gamma N_{\alpha\lambda}^\mu \\
&\quad + N_{\beta\mu}^\gamma F_{\alpha\lambda}^\mu + N_{\beta\mu}^\gamma N_{\alpha\lambda}^\mu + N_{\lambda\mu}^\gamma M_{\alpha\beta}^\mu \\
&\quad + N_{\alpha\beta}^\mu M_{\lambda\mu}^\gamma + (g^{rs} \delta_\lambda^\gamma f_s - f g^{rs} \mathbf{G}_s^\kappa C_{\lambda\kappa}^\gamma) \\
&\quad \left. \times (g_{\alpha\beta} f_r - f \mathbf{G}_r^\mu C_{\alpha\beta\mu}) \right\} - \mathfrak{E}_\lambda^\beta, \quad (94)
\end{aligned}$$

where

$$\begin{aligned}
M_{jk}^i &= M_k^r C_{jr}^i + M_j^r C_{kr}^i - M_h^r g^{ih} C_{jkr}, \\
M_{\alpha\beta}^\gamma &= M_\beta^\mu C_{\alpha\mu}^\gamma + M_\alpha^\mu C_{\beta\mu}^\gamma - M_\nu^\mu g^{\gamma\nu} C_{\alpha\beta\mu}, \quad (95) \\
N_{\alpha\beta}^\gamma &= f^{-1} (f_\beta \delta_\alpha^\gamma + f_\alpha \delta_\beta^\gamma - f_\lambda g^{\gamma\lambda} g_{\alpha\beta})
\end{aligned}$$

and $\mathfrak{E}_j^i$ denotes the interchange of indices i, j , and subtraction.

By Theorem 18, we have the following.

Theorem 16. Let $(M_1 \times_f M_2, F)$ be a flat twisted product Finsler manifold, and let (M_1, F_1) be Riemannian. If f is a function on M_2 , only, then (M_1, F_1) is locally flat.

Similarly, we get the following.

Theorem 17. Let $(M_1 \times_f M_2, F)$ be a flat twisted product Finsler manifold, and let (M_2, F_2) be Riemannian. If f is a function on M_1 , only, then (M_2, F_2) is a space of positive constant curvature $\|grad f\|^2$.

Proof. Since M_2 is Riemannian and f is a function on M_1 , then by (94), we obtain

$$\mathbf{R}_{\alpha\beta\lambda}^\gamma = R_{\alpha\beta\lambda}^\gamma + \|\text{grad}f\|^2 (\delta_\lambda^\gamma g_{\alpha\beta} - \delta_\beta^\gamma g_{\alpha\lambda}). \quad (96)$$

Since $(M_1 \times_f M_2, F)$ is flat, then $\mathbf{R}_{\alpha\beta\lambda}^\gamma = 0$. Thus the proof is complete. $\square$

Theorem 18. *Let $(M_1 \times_f M_2, F)$ be a twisted product Riemannian manifold, and let f be a function on M_2 , only. Then $(M_1 \times_f M_2, F)$ is flat, if and only if (M_1, F_1) is flat, and the Riemannian curvature of (M_2, F_2) satisfies in the following equation:*

$$R_{\alpha\beta\lambda}^\gamma = \left\{ \frac{\delta^t N_{\alpha\lambda}^\gamma}{\delta^t u^\beta} + F_{\beta\mu}^\gamma N_{\alpha\lambda}^\mu + N_{\beta\mu}^\gamma F_{\alpha\lambda}^\mu + N_{\beta\mu}^\gamma N_{\alpha\lambda}^\mu \right\} - \mathfrak{G}_{\lambda}^\beta. \quad (97)$$

5. Twisted Product Finsler Manifolds with Non-Riemannian Curvature Properties

There are several important non-Riemannian quantities such as the Berwald curvature $\mathbf{B}$, the mean Berwald curvature $\mathbf{E}$, and the Landsberg curvature $\mathbf{L}$ [22]. They all vanish for Riemannian metrics, hence they are said to be non-Riemannian. In this section, we find some necessary and sufficient conditions under which a twisted product Riemannian manifold are Berwaldian, of isotropic Berwald curvature, of isotropic mean Berwald curvature. First, we prove the following.

Lemma 19. *Let $(M_1 \times_f M_2, F)$ be a twisted product Finsler manifold. Then the coefficients of Berwald curvature are as follows:*

$$\begin{aligned} \mathbf{B}_{\alpha\beta\lambda}^\gamma &= B_{\alpha\beta\lambda}^\gamma + f^{-1} \left(C_{\lambda;\alpha;\beta}^{\gamma\gamma} f_\nu F_2^2 + 2C_{\alpha;\beta}^{\gamma\gamma} f_\nu v_\lambda \right. \\ &\quad + 2C_{\alpha;\lambda}^{\gamma\gamma} f_\nu v_\beta + 2C_{\alpha}^{\gamma\gamma} f_\nu g_{\lambda\beta} \\ &\quad + 2C_{\lambda;\beta}^{\gamma\gamma} f_\nu v_\alpha + 2C_{\beta}^{\gamma\gamma} f_\nu g_{\lambda\alpha} \\ &\quad \left. + 2C_{\lambda}^{\gamma\gamma} f_\nu g_{\alpha\beta} - 2C_{\alpha\beta\lambda} f^\gamma \right), \end{aligned} \quad (98)$$

$$\mathbf{B}_{ijl}^k = B_{ijl}^k + f C_{l;ji}^{kh} f_h F_2^2, \quad (99)$$

$$\mathbf{B}_{i\beta l}^k = 2f C_{i;\beta}^{kh} f_h v_\beta, \quad (100)$$

$$\mathbf{B}_{\alpha\beta l}^k = 2f g_{\alpha\beta} C_l^{kh} f_h, \quad (101)$$

$$\mathbf{B}_{\alpha\beta\lambda}^k = -2f C_{\alpha\beta\lambda} f^k, \quad (102)$$

$$\mathbf{B}_{i\beta\lambda}^\gamma = \mathbf{B}_{ij\lambda}^\gamma = \mathbf{B}_{ijk}^\gamma = 0. \quad (103)$$

Let $(M_1 \times_f M_2, F)$ is a Berwald manifold. Then we have $\mathbf{B}_{abc}^d = 0$. By using (102), we get

$$C_{\alpha\beta\lambda} f^k = 0. \quad (104)$$

Multiplying this equation in g_{kr} , we obtain

$$C_{\alpha\beta\lambda} f_r = 0. \quad (105)$$

Thus if f is not constant on M_1 , then we have $C_{\alpha\beta\lambda} = 0$. Also, from (101), we result that

$$C_l^{kh} f_h = 0. \quad (106)$$

Differentiating this equation with respect to y^j gives us

$$C_{l;j}^{kh} f_h = 0. \quad (107)$$

Similarly we obtain

$$C_{l;ji}^{kh} f_h = 0. \quad (108)$$

Setting the last equation in (99) implies that $B_{ijl}^k = 0$; that is, (M_1, F_1) is Berwaldian. These explanations give us the following theorem.

Theorem 20. *Let $(M_1 \times_f M_2, F)$ be a twisted product Finsler manifold, and let f be not constant on M_1 . Then $(M_1 \times_f M_2, F)$ is Berwaldian if and only if (M_1, F_1) is Berwaldian, (M_2, F_2) is Riemannian, and the equation $C_l^{kh} f_h = 0$ is hold.*

But if f is constant on M_1 , that is, $f_i = 0$, then we get the following.

Theorem 21. *Let $(M_1 \times_f M_2, F)$ be a twisted product Finsler manifold, and f is constant on M_1 . Then $(M_1 \times_f M_2, F)$ is Berwaldian if and only if (M_1, F_1) is Berwaldian and the Berwald curvature of (M_2, F_2) satisfies in the following equation:*

$$\begin{aligned} B_{\alpha\beta\lambda}^\gamma &= -f^{-1} \left(C_{\beta;\alpha;\lambda}^{\gamma\gamma} f_\nu F_2^2 + 2C_{\beta;\alpha}^{\gamma\gamma} f_\nu v_\lambda + 2C_{\lambda;\alpha}^{\gamma\gamma} f_\nu v_\beta \right. \\ &\quad + 2C_{\alpha}^{\gamma\gamma} f_\nu g_{\lambda\beta} + 2C_{\beta;\lambda}^{\gamma\gamma} f_\nu v_\alpha \\ &\quad + 2C_{\beta}^{\gamma\gamma} f_\nu g_{\lambda\alpha} + 2C_{\lambda}^{\gamma\gamma} f_\nu g_{\alpha\beta} \\ &\quad \left. - 2g^{\gamma\gamma} C_{\alpha\beta\lambda} f_\gamma \right). \end{aligned} \quad (109)$$

Here, we consider twisted product Finsler manifold $(M_1 \times_f M_2, F)$ of isotropic Berwald curvature.

Theorem 22. *Every isotropic Berwald twisted product Finsler manifold $(M_1 \times_f M_2, F)$ is a Berwald manifold.*

Proof. Let $(M_1 \times_f M_2, F)$ be an isotropic Berwald manifold. Then we have

$$\mathbf{B}_{abc}^d = cF^{-1} \left\{ \mathbf{h}_a^d \mathbf{h}_{bc} + \mathbf{h}_b^d \mathbf{h}_{ac} + \mathbf{h}_c^d \mathbf{h}_{ab} + 2\mathbf{C}_{abc} \mathbf{y}^d \right\}, \quad (110)$$

where $c = c(\mathbf{x})$ is a function on M . Setting $a = j$, $b = k$, $c = l$, and $d = \gamma$ and using (103) imply that

$$cF^{-1} \left\{ \frac{3}{F^2} y_j y_k y_l v^\gamma - v^\gamma (y_j g_{kl} + y_k g_{jl} + y_l g_{jk}) \right\} = 0. \quad (111)$$

Multiplying the previous equation in $y^j y^k$, we derive that $c f^2 F_1^2 F_2^2 = 0$. Thus we have $c = 0$; that is, $(M_1 \times_f M_2)$ is Berwaldian. $\square$

Now, we are going to study twisted product Finsler manifold of isotropic mean Berwald curvature. For this work, we must compute the coefficients of mean Berwald curvature of a twisted product Finsler manifold.

Lemma 23. *Let $(M_1 \times_f M_2, F)$ be a twisted product Finsler manifold. Then the coefficients of mean Berwald curvature are as follows:*

$$E_{\alpha\beta} = E_{\alpha\beta} + fg_{\alpha\beta}I^h f_h + \frac{1}{2}fI_{\alpha;\beta}^\nu f_\nu F_2^2 + f^{-1}f_\nu \times \left(C_{\alpha;\beta}^{\gamma\nu} \nu_\gamma + I_{\alpha;\beta}^\nu \nu_\beta + I_{\beta;\beta}^\nu \nu_\alpha + C_{\alpha\beta}^\nu + I^\nu g_{\alpha\beta} \right), \quad (112)$$

$$E_{ij} = E_{ij} + \frac{1}{2}fI_{i;j}^h f_h F_2^2, \quad (113)$$

$$E_{i\beta} = fI_{i;\beta}^h f_h \nu_\beta, \quad (114)$$

where E_{ij} and $E_{\alpha\beta}$ are the coefficients of mean Berwald curvature of (M_1, F_1) and (M_2, F_2) , respectively.

Proof. By definition and Lemma 19, we get the proof. $\square$

Theorem 24. *The twisted product Finsler manifold $(M_1 \times_f M_2, F)$ is weakly Berwald if and only if (M_1, F_1) is weakly Berwald, $I^h f_h = 0$, and the following hold:*

$$E_{\alpha\beta} = -\frac{1}{2}fI_{\alpha;\beta}^\nu f_\nu F_2^2 - f^{-1}f_\nu \times \left(C_{\alpha;\beta}^{\gamma\nu} \nu_\gamma + I_{\alpha;\beta}^\nu \nu_\beta + I_{\beta;\beta}^\nu \nu_\alpha + C_{\alpha\beta}^\nu + I^\nu g_{\alpha\beta} \right). \quad (115)$$

Proof. If $(M_1 \times_f M_2)$ be a weakly Berwald manifold, then we have

$$E_{\alpha\beta} = E_{ij} = E_{i\beta} = 0. \quad (116)$$

Thus by using (114), we result that $I_{i;\beta}^h f_h = 0$. This equation implies that

$$I_{i;j\beta}^h f_h = 0, \quad I^h f_h = 0. \quad (117)$$

By setting these equations in (112) and (113), we conclude that $E_{ij} = 0$ and $E_{\alpha\beta}$ satisfies in (115). $\square$

Now, if f is constant on M_2 , then (115) implies that $E_{\alpha\beta} = 0$. Thus we conclude the following.

Corollary 25. *Let $(M_1 \times_f M_2, F)$ be a twisted product Finsler manifold, and let f be a function on M_1 , only. Then $(M_1 \times_f M_2, F)$ is weakly Berwald if and only if (M_1, F_1) and (M_2, F_2) are weakly Berwald manifolds and $I^h f_h = 0$.*

Now, we consider twisted product Finsler manifolds with isotropic mean Berwald curvature. It is remarkable that as a consequence of Lemma 23, we have the following.

Lemma 26. *Twisted product Finsler manifold $(M_1 \times_f M_2, F)$ is isotropic mean Berwald manifold if and only if*

$$E_{\alpha\beta} + fg_{\alpha\beta}I^h f_h + \frac{1}{2}fI_{\alpha;\beta}^\nu f_\nu F_2^2 + f^{-1}f_\nu \times \left(C_{\alpha;\beta}^{\gamma\nu} \nu_\gamma + I_{\alpha;\beta}^\nu \nu_\beta + I_{\beta;\beta}^\nu \nu_\alpha + C_{\alpha\beta}^\nu + I^\nu g_{\alpha\beta} \right) - \frac{n+1}{2}cf^2F^{-1} \left(g_{\alpha\beta} - \frac{f^2}{F^2} \nu_\alpha \nu_\beta \right) = 0, \quad (118)$$

$$E_{ij} + \frac{1}{2}fI_{i;j}^h f_h F_2^2 - \frac{n+1}{2}cF^{-1} \left(g_{ij} - \frac{1}{F^2} y_i y_j \right) = 0, \quad (119)$$

$$c(n+1)F^{-3}y_i + fI_{i;\beta}^h f_h = 0, \quad (120)$$

where $c = c(\mathbf{x})$ is a scalar function on M .

Theorem 27. *Every twisted product Finsler manifold $(M_1 \times_f M_2, F)$ with isotropic mean Berwald curvature is a weakly Berwald manifold.*

Proof. Suppose that F is isotropic mean Berwald twisted product Finsler metric. Then differentiating (120) with respect to ν^y gives us

$$c(n+1)f^2F^{-5}\nu_y y_i = 0. \quad (121)$$

Thus, we conclude that $c = 0$. This implies that F reduces to a weakly Berwald metric. $\square$

6. Locally Dually Flat Twisted Product Finsler Manifolds

In [23], Amari and Nagaoka introduced the notion of dually flat Riemannian metrics when they study the information geometry on Riemannian manifolds. Information geometry has emerged from investigating the geometrical structure of a family of probability distributions and has been applied successfully to various areas including statistical inference, control system theory, and multiterminal information theory. In Finsler geometry, Shen extends the notion of locally dually flatness for Finsler metrics [17]. Dually flat Finsler metrics form a special and valuable class of Finsler metrics in Finsler information geometry, which play a very important role in studying flat Finsler information structure [24, 25].

In this section, we study locally dually flat twisted product Finsler metrics. It is remarkable that a Finsler metric $F = F(\mathbf{x}, \mathbf{y})$ on a manifold M is said to be locally dually flat if at any point there is a standard coordinate system $(\mathbf{x}^a, \mathbf{y}^a)$ in TM such that it satisfies

$$\frac{\partial^2 F^2}{\partial \mathbf{x}^b \partial \mathbf{y}^a} \mathbf{y}^b = 2 \frac{\partial F^2}{\partial \mathbf{x}^a}. \quad (122)$$

In this case, the coordinate $(\mathbf{x}^a)$ is called an adapted local coordinate system. By using (122), we can obtain the following lemma.

Lemma 28. Let $(M_1 \times_f M_2, F)$ be a twisted product Finsler manifold. Then F is locally dually flat if and only if F_1 and F_2 satisfy in the following equations:

$$\frac{\partial^2 F_1^2}{\partial x^k \partial y^l} y^k = 2 \frac{\partial F_1^2}{\partial x^l} + 4 f f_l F_2^2, \quad (123)$$

$$4 f_k v_\beta y^k + f \frac{\partial^2 F_2^2}{\partial u^\alpha \partial v^\beta} v^\alpha + 4 f_\alpha v_\beta v^\alpha = 2 f \frac{\partial F_2^2}{\partial u^\beta} + 4 f_\beta F_2^2. \quad (124)$$

Now, let F be a locally dually flat Finsler metric. Taking derivative with respect to v^j from (123) yields $f_l = 0$, which means that f is a constant function on M_1 . In this case, the relations (123) and (124) reduce to the following:

$$\frac{\partial^2 F_1^2}{\partial x^k \partial y^l} y^k = 2 \frac{\partial F_1^2}{\partial x^l}, \quad (125)$$

$$f \frac{\partial^2 F_2^2}{\partial u^\alpha \partial v^\beta} v^\alpha + 4 f_\alpha v_\beta v^\alpha = 2 f \frac{\partial F_2^2}{\partial u^\beta} + 4 f_\beta F_2^2. \quad (126)$$

By (125), we deduce that F_1 is locally dually flat.

Now, we assume that F_1 and F_2 are locally dually flat Finsler metrics. Then we have

$$\begin{aligned} \frac{\partial^2 F_1^2}{\partial x^k \partial y^l} y^k &= 2 \frac{\partial F_1^2}{\partial x^l}, \\ \frac{\partial^2 F_2^2}{\partial u^\alpha \partial v^\beta} v^\alpha &= 2 \frac{\partial F_2^2}{\partial u^\beta}. \end{aligned} \quad (127)$$

By (127), we derive that (123) and (124) are hold if and only if the following hold:

$$f_l = 0, \quad f_\alpha v_\beta v^\alpha = f_\beta F_2^2. \quad (128)$$

Therefore we can conclude the following.

Theorem 29. Let $(M_1 \times_f M_2, F)$ be a twisted product Finsler manifold.

- (i) If F is locally dually flat, then F_1 is locally dually flat, f is a function with respect to (u^α) only, and F_2 satisfies in (126).
- (ii) If F_1 and F_2 are locally dually flat, then F is locally dually flat if and only if f is a function with respect (u^α) only and F_2 satisfies in (128).

By Theorem 29, we conclude the following.

Corollary 30. There is not exist any locally dually flat proper twisted product Finsler manifold.

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