Research Article

Single Peak Solitons for the Boussinesq-Like $B(2, 2)$ Equation

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The nonlinear dispersive Boussinesq-like $B(2, 2)$ equation $u_{tt} + (u^2)_{xx} - (u^2)_{xxxx} = 0$, which exhibits single peak solitons, is investigated. Peakons, cuspons and smooth soliton solutions are obtained by setting the $B(2, 2)$ equation under inhomogeneous boundary condition. Asymptotic behavior and numerical simulations are provided for these three types of single peak soliton solutions of the $B(2, 2)$ equation.

1. Introduction

The $B(2, 2)$ interest inspired by the well-known Camassa-Holm (CH) equation and its singular peakon solutions [1] prompted search for other integrable equations with nonsmooth solitons. An integrable CH-type equation with cubic nonlinearity

$$m_t + [m(u^2 - u_x^2)]_x = 0, \quad m = u - u_{xx}, \quad (1)$$

was derived independently by Fokas [2], by Fuchssteiner [3], by Olver and Rosenau [4], and by Qiao [5]. It is shown in [5–7] that (1) admits Lax pair and bi-Hamiltonian structures and possesses the M/W-shape soliton solution and a new type of cusped soliton solution. Another peakon equation with cubic nonlinearity has been recently discovered by Novikov [8]. In the work by Hone and Wang [9], it is shown that Novikov's equation admits peakon solutions like the CH equation. Also, it has a Lax pair in matrix form and a bi-Hamiltonian structure.

The Boussinesq-like $B(m, n)$ equation with nonlinear dispersion is given by

$$u_{tt} + a(u^n)_{xx} + b(u^m)_{xxxx} = 0, \quad (2)$$

where $a, b \in \mathbb{R}$ and $m, n \in \mathbb{Z}^+$. This equation is the generalized form of the Boussinesq equation, where, in particular, the

case $(m, n) = (1, 2)$ leads to the Boussinesq equation. Equation (2), for $ab > 0$, is the major equation for compactons (solitons with compact support). Abundant compactons [10–13] are developed by the Adomian decomposition method. For $ab < 0$, exact solutions with solitary patterns of Boussinesq-like $B(m, n)$ equations are obtained in the works by Shang [14] and Zhang et al. [15] by extending sinh-cosh method and by using the integral approach, respectively.

A natural question is that whether the Boussinesq-like $B(m, n)$ equation (2) has nonsmooth solitons such as peakons or cuspons. The present paper focuses on the following Boussinesq-like $B(2, 2)$ equation:

$$u_{tt} + (u^2)_{xx} - (u^2)_{xxxx} = 0. \quad (3)$$

We give all possible single peak soliton solutions of (3) through setting the traveling wave solution under the inhomogeneous boundary condition $u \rightarrow A$ (A is a nonzero constant) as $x \rightarrow \pm\infty$. New cusped soliton solutions, and smooth soliton solutions are obtained. Asymptotic analysis and numerical simulations are provided for peaked solitons, cusped solitons and smooth solitons of the $B(2, 2)$ equation. The method used here is based on the phase portrait analysis technique which is similar to that in [16–18].

2. Asymptotic Behavior of Solutions

In this section, we first introduce some notations. Let $C^k(\Omega)$ denote the set of all k times continuously differential functions on the open set Ω . $L^p_{\text{loc}}(R)$ refers to the set of all functions whose restriction on any compact subset is L^p integrable. $H^1_{\text{loc}}(R)$ stands for $H^1_{\text{loc}}(R) = \{u \in L^2_{\text{loc}}(R) \mid u' \in L^2_{\text{loc}}(R)\}$.

Let us consider the traveling wave solution of the $B(2, 2)$ equation (3) through the setting $u(x, t) = U(x - ct)$, where c is the wave speed. Let $\xi = x - ct$; then $u(x, t) = U(\xi)$. Substituting it into (3) yields

$$c^2 U'' + (U^2)'' - (U^2)'''' = 0, \quad (4)$$

where “ $'$ ” is the derivative with respect to ξ . Integrating (4) once and neglecting the integration constant, we have

$$c^2 U' + (U^2)' - (U^2)''' = 0. \quad (5)$$

Integrating (5) once again, we obtain

$$c^2 U + U^2 - 2U'^2 - 2UU'' = g_1, \quad (6)$$

where g_1 is an integration constant. Furthermore, we get

$$U'^2 = \frac{U^2}{4} + \frac{c^2 U}{3} - \frac{g_1}{2} + \frac{g_2}{U^2}, \quad (7)$$

where g_2 is also an integration constant.

To seek exact solutions with solitary patterns for (7), we impose the boundary condition

$$\lim_{\xi \rightarrow \pm\infty} U(\xi) = A, \quad (8)$$

where A is a nonzero constant. Equation (7) can be cast into the following ordinary differential equation:

$$U'^2 = \frac{(U - A)^2 [3U^2 + (6A + 4c^2)U + 3A^2 + 2Ac^2]}{12U^2}. \quad (9)$$

The fact that both sides of (9) are nonnegative implies that $3U^2 + (6A + 4c^2)U + 3A^2 + 2Ac^2 \geq 0$. If $3A + 2c^2 \geq 0$, then (9) reduces to

$$U'^2 = \frac{(U - A)^2 (U - B_1)(U - B_2)}{4U^2}, \quad (10)$$

where

$$B_1 = -\frac{3A + 2c^2}{3} + \frac{\sqrt{2c^2(3A + 2c^2)}}{3}, \quad (11)$$

$$B_2 = -\frac{3A + 2c^2}{3} - \frac{\sqrt{2c^2(3A + 2c^2)}}{3}.$$

Obviously, $B_1 \geq B_2$.

Definition 1. A function $u(x, t) = U(x - ct)$ is said to be a single peak soliton solution for the $B(2, 2)$ equation (3) if $U(\xi)$ satisfies the following conditions.

(A1) $U(\xi)$ is continuous on R and has a unique peak point ξ_0 , where $u(\xi)$ attains its global maximum or minimum value.

(A2) $U(\xi) \in C^4(R - \{\xi_0\})$ satisfies (8) on $R - \{\xi_0\}$.

(A3) $\lim_{\xi \rightarrow \pm\infty} U(\xi) = A$.

Definition 2. A wave function U is called peakon if U is smooth locally on either side of ξ_0 and $\lim_{\xi \uparrow \xi_0} U'(\xi) = -\lim_{\xi \downarrow \xi_0} U'(\xi) = a$, $a \neq 0$, $a \neq \pm\infty$.

Definition 3. A wave function U is called cuspon if U is smooth locally on either side of ξ_0 and $\lim_{\xi \uparrow \xi_0} U'(\xi) = -\lim_{\xi \downarrow \xi_0} U'(\xi) = \pm\infty$.

Without any loss of generality, we choose the peak point ξ_0 as vanishing, $\xi_0 = 0$.

Theorem 4. Suppose that $U(\xi)$ is a single peak soliton solution for the $B(2, 2)$ equation (3) at the peak point $\xi_0 = 0$. Then one has the following.

(i) If $A < -2c^2/3$, then $U(0) = 0$.

(ii) If $A \geq -2c^2/3$, then $U(0) = 0$ or $U(0) = B_1$ or $U(0) = B_2$.

Proof. If $U(0) \neq 0$, then $U(\xi) \neq 0$ for any $\xi \in R$ since $U(\xi) \in C^4(R - \{0\})$. Differentiating both sides of (9) yields $U(\xi) \in C^\infty(R)$.

(i) For $A < -2c^2/3$, if $U(0) \neq 0$, then $U \in C^\infty(R)$. By the definition of single peak soliton, we have $U'(0) = 0$. However, by (9) we must have $U(0) = A$, which contradicts the fact that 0 is the unique peak point.

(ii) For $A \geq -2c^2/3$, if $U(0) \neq 0$, by (9) we know that $U'(0)$ exists. According to the definition of peak point, we have $U'(0) = 0$. Thus we obtain $U(0) = B_1$ or $U(0) = B_2$ from (10) since $U(0) = A$ contradicts the fact that 0 is the unique peak point. $\square$

Theorem 5. Suppose that $U(\xi)$ is a single peak soliton solution for the $B(2, 2)$ equation (3) at the peak point $\xi_0 = 0$. Then one has the following solutions classification and asymptotic behavior.

(i) If $U(0) \neq 0$, then $U(\xi)$ is a smooth soliton solution.

(ii) If $U(0) = 0$ and $A = -2c^2/3$, then $U(\xi)$ gives the peaked soliton solution $(-2c^2/3)(1 - e^{-|x-ct|/2})$.

(iii) If $U(0) = 0$ and $A \neq -2c^2/3$, then $U(\xi)$ is a cusped soliton solution and

$$U(\xi) = \mu|\xi|^{1/2} + O(|\xi|), \quad \xi \rightarrow 0, \quad (12)$$

$$U'(\xi) = \frac{\mu}{2}|\xi|^{-1/2} \text{sgn}(\xi) + O(1), \quad \xi \rightarrow 0,$$

where $\mu = \pm(1/3)\sqrt{3|A|\sqrt{9A^2 + 6Ac^2}}$. Thus $U(\xi) \notin H^1_{\text{loc}}(R)$.

Proof. (i) If $U(0) \neq 0$, then $U(\xi) \in C^\infty(R)$ for any $\xi \in R$, and so $U(\xi)$ is a smooth soliton solution.

(ii) If $U(0) = 0$ and $A = -2c^2/3$, then (9) becomes

$$U'^2 = \frac{1}{4} \left(U + \frac{2c^2}{3} \right)^2. \quad (13)$$

Solving (13), we obtain the peaked soliton solution

$$U(\xi) = -\frac{2c^2}{3} \left(1 - e^{-|x-ct|/2} \right). \quad (14)$$

(iii) If $U(0) = 0$ and $A \neq -2c^2/3$, then by the definition of single peak soliton solution we have $A \neq 0$; thus, $3U^2 + (6A + 4c^2)U + 3A^2 + 2Ac^2$ does not contain the factor U . From (9), we obtain

$$U' = -\operatorname{sgn}(A) \frac{U-A}{U} \times \sqrt{\frac{3U^2 + (6A + 4c^2)U + 3A^2 + 2Ac^2}{12}} \operatorname{sgn}(\xi). \quad (15)$$

Let $\frac{h(U)}{U\sqrt{3U^2 + (6A + 4c^2)U + 3A^2 + 2Ac^2}} = \frac{2\sqrt{3}\operatorname{sgn}(A)/(A - 2\sqrt{3}/|A|\sqrt{3A^2 + 2Ac^2})}{2\sqrt{3}/|A|\sqrt{3A^2 + 2Ac^2}}$; then $h(0) = 2\sqrt{3}/|A|\sqrt{3A^2 + 2Ac^2}$ and

$$\int U h(U) dU = \int \operatorname{sgn}(\xi) d\xi. \quad (16)$$

Inserting $h(U) = h(0) + O(U)$ into (16) and using the initial condition $U(0) = 0$, we obtain

$$\frac{h(0)}{2} U^2 (1 + O(U)) = |\xi|. \quad (17)$$

Thus

$$\begin{aligned} U &= \pm \sqrt{\frac{2}{h(0)}} |\xi|^{1/2} (1 + O(U))^{-1/2} \\ &= \pm \sqrt{\frac{2}{h(0)}} |\xi|^{1/2} (1 + O(U)), \end{aligned} \quad (18)$$

which implies that $U = O(|\xi|^{1/2})$. Therefore, we have

$$\begin{aligned} U(\xi) &= \mu |\xi|^{1/2} + O(|\xi|), \quad \xi \rightarrow 0, \\ U'(\xi) &= \frac{\mu}{2} |\xi|^{-1/2} \operatorname{sgn}(\xi) + O(1), \quad \xi \rightarrow 0, \end{aligned} \quad (19)$$

where $\mu = \pm(1/3)\sqrt{3|A|\sqrt{9A^2 + 6Ac^2}}$. Thus $U(\xi) \notin H_{\text{loc}}^1(R)$. $\square$

3. Peakons, Cuspons, and Smooth Soliton Solutions

Theorem 5 gives a classification for all single peak soliton solutions for the $B(2, 2)$ equation (3). In this section, we will

present all possible single peak soliton solutions. We should discuss three cases: $A > -2c^2/3$, $A = -2c^2/3$, and $A < -2c^2/3$.

Case I ($A > -2c^2/3$). By virtue of Theorems 4 and 5, any single peak soliton solution for the $B(2, 2)$ equation (3) must satisfy the following initial and boundary values problem:

$$\begin{aligned} U'^2 &= \frac{(U-A)^2 (U-B_1)(U-B_2)}{4U^2}, \\ U(0) &\in \{0, B_1, B_2\}, \\ \lim_{\xi \rightarrow \pm\infty} U(\xi) &= A. \end{aligned} \quad (20)$$

Equation (20) implies that

$$\begin{aligned} U &\geq B_1, \quad \text{or} \quad U \leq B_2, \\ (A-B_1)(A-B_2) &\geq 0. \end{aligned} \quad (21)$$

Since $3A + 2c^2 \neq 0$, introducing the constant $\alpha = (A/(3A + 2c^2))$ yields

$$\alpha(\alpha + 1) \geq 0, \quad (22)$$

which implies that

$$\alpha < -1, \quad \alpha = -1, \quad \alpha = 0, \quad \alpha > 0. \quad (23)$$

From the standard phase analysis, we know that if $U(\xi)$ is a single peak soliton solution of the $B(2, 2)$ equation (3), then

$$U' = -\operatorname{sgn}(A) \frac{U-A}{2U} \sqrt{(U-B_1)(U-B_2)} \operatorname{sgn}(\xi), \quad (24)$$

$$U(0) = \begin{cases} \max(0, B_1), & \text{if } U(0) \text{ is a minimum,} \\ \min(0, B_2), & \text{if } U(0) \text{ is a maximum.} \end{cases} \quad (25)$$

Taking the integration of both sides of (24) leads to

$$\int \phi(U) dU = -\frac{|\xi|}{2}, \quad (26)$$

where $\phi(U) = -\operatorname{sgn}(A)(U/(U-A)\sqrt{(U-B_1)(U-B_2)})$.

When $\alpha \neq 0, -1$, that is, $2A(2A + c^2) \neq 0$, we obtain

$$\begin{aligned} \int \phi(U) dU &= \operatorname{sgn}(A) I_1(U) + \frac{|A|}{\sqrt{4A^2 + 2Ac^2}} I_2(U) - K \\ &= \Phi(U) - K, \end{aligned} \quad (27)$$

with

$$\begin{aligned} I_1(U) &= \ln \left| 2U - B_1 - B_2 + \sqrt{(U-B_1)(U-B_2)} \right|, \\ I_2(U) &= \ln \left(\left| (A-B_1)(U-B_2) + (A-B_2)(U-B_1) \right. \right. \\ &\quad \left. \left. - 2\sqrt{(A-B_1)(A-B_2)(U-B_1)(U-B_2)} \right| \right) \\ &\quad \times (U-A)^{-1} \Bigg), \end{aligned} \quad (28)$$

and K is an integration constant. Thus we obtain the implicit solution $U(\xi)$ defined by

$$\Phi(U) = -\frac{|\xi|}{2} + K. \quad (29)$$

Obviously,

$$\begin{aligned} I_1(B_1) &= I_1(B_2) = \ln |B_1 - B_2| = \ln \frac{\sqrt{6Ac^2 + 4c^4}}{3}, \\ I_2(B_1) &= I_2(B_2) = \ln |B_1 - B_2| = \ln \frac{\sqrt{6Ac^2 + 4c^4}}{3}. \end{aligned} \quad (30)$$

So, for $U(0) = B_1$ or B_2 , the constant $K_0 = \Phi(U(0))$ is defined by

$$\begin{aligned} K_0 &= \operatorname{sgn}(A) \ln \frac{\sqrt{6Ac^2 + 4c^4}}{3} + \frac{|A|}{\sqrt{6Ac^2 + 4c^4}} \\ &\quad \times \ln \frac{\sqrt{6Ac^2 + 4c^4}}{3}, \end{aligned} \quad (31)$$

and for $U(0) = 0$,

$$K_0 = I_1(0) - \frac{2A}{\sqrt{4A^2 - 2Ac}} I_2(0). \quad (32)$$

(1) $\alpha < -1$.

If $\alpha < -1$, then

$$A < B_2 < 0 < B_1, \quad U(0) = B_2, \quad A < U \leq B_2. \quad (33)$$

From $\phi(U) < 0$, we know that $\Phi(U)$ is strictly decreasing on $(A, B_2]$, and

$$\Phi_1(U) = \Phi|_{(A, B_2]}(U) \quad (34)$$

has the inverse denoted by $U_1(\xi) = \Phi_1^{-1}(-(|\xi|/2) + K_0)$. $U_1(\xi)$ gives a smooth soliton solution satisfying

$$U_1(0) = B_2, \quad \lim_{\xi \rightarrow \pm\infty} U_1(\xi) = A, \quad U_1'(0) = 0. \quad (35)$$

The profile of smooth soliton solution is shown in Figure 1(a).

(2) $\alpha = -1$.

If $\alpha = -1$, then $A = B_2 = -c^2/2$ and $B_1 = c^2/6$; there is no single peak soliton solution.

(3) $\alpha = 0$.

In case $A = 0$, (9) becomes

$$U' = -\frac{1}{2} \sqrt{U \left(U + \frac{4c^4}{3} \right)} \operatorname{sgn}(\xi), \quad U(\pm\infty) = 0. \quad (36)$$

Thus there is no single peak soliton solution for the previous boundary condition (8).

(4) $\alpha > 0$.

If $\alpha > 0$, then $A > 0$ and

$$B_2 < B_1 < 0 < A, \quad U(0) = 0, \quad 0 \leq U < A. \quad (37)$$

From $\phi(U) > 0$, we know that $\Phi(U)$ is strictly increasing on $[0, A)$, and

$$\Phi_2(U) = \Phi|_{[0, A)}(U) \quad (38)$$

gives a unique cusped soliton solution. Therefore, $U_2(\xi) = \Phi_2^{-1}(-(|\xi|/2) + K_0)$ is the solution satisfying

$$U_2(0) = 0, \quad \lim_{\xi \rightarrow \pm\infty} U_2(\xi) = A, \quad (39)$$

$$U_2'(0+) = +\infty, \quad U_2'(0-) = -\infty.$$

The profile of cuspon is shown in Figure 1(b).

Case II ($A = -2c^2/3$). If $A = -2c^2/3$, then the only possible single peak soliton solution is the peakon

$$U(\xi) = -\frac{2c^2}{3} \left(1 - e^{-|x-ct|/2} \right). \quad (40)$$

The profile of peaked soliton is shown in Figure 1(c).

Case III ($A < -2c^2/3$). In this case, according to Theorem 4 and standard phase portrait analytical technique, we have $U(0) = 0$, $A < U \leq 0$, and

$$U' = \frac{U - A}{U} \sqrt{\frac{3U^2 + (6A + 4c^2)U + 3A^2 + 2Ac^2}{12}} \operatorname{sgn}(\xi). \quad (41)$$

Let

$$X = U + a_1,$$

$$a_1 = \frac{3A + 2c^2}{3}, \quad (42)$$

$$a_2^2 = -\frac{2}{9} (3A + 2c^2),$$

and then (41) becomes

$$\varphi(X) dX = \frac{X - a_1}{(X - a_1 - A) \sqrt{X^2 + a_2^2}} dX = \frac{1}{2} \operatorname{sgn}(\xi) d\xi. \quad (43)$$

Integration of both sides of (43) gives

$$\Psi(X) = \frac{|\xi|}{2} + K, \quad (44)$$

where

$$\Psi(X) = J_1(X) + \frac{A}{\sqrt{(a_1 + A)^2 + a_2^2}} J_2(X),$$

$$J_1(X) = \ln \left| X + \sqrt{X^2 + a_2^2} \right|,$$

$$J_2(X)$$

$$= \ln \left| \frac{2 \left[a_2^2 + (a_1 + A)X - \sqrt{(a_1 + A)^2 + a_2^2} \sqrt{X^2 + a_2^2} \right]}{X - a_1 - A} \right|. \quad (45)$$

FIGURE 1: The profiles of waves. (a) Smooth soliton, $A = -5/2$, $c = -2$. (b) Cuspon, $A = 2$, $c = 1$. (c) Peakon, $A = -2/3$, $c = 1$. (d) Cuspon, $A = -1$, $c = -1$.

$\Psi(X)$ is strictly decreasing on the interval $[a_1, a_1 + A)$. Define

$$\Psi_1(X) = \Psi|_{[a_1, a_1 + A)}(X). \quad (46)$$

Then

$$\Psi_1(X) = \frac{|\xi|}{2} + K_0, \quad (47)$$

where

$$K_0 = J_1(a_1) + \frac{A}{\sqrt{(a_1 + A)^2 + a_2^2}} J_2(a_1). \quad (48)$$

Since Ψ_1 is a strictly decreasing function, we can solve for X uniquely from (47) and obtain

$$U(\xi) = \Psi_1^{-1}\left(\frac{|\xi|}{2} + K_0\right) - a_1, \quad (49)$$

which satisfies

$$\begin{aligned} U(0) &= 0, & \lim_{\xi \rightarrow \pm\infty} U(\xi) &= A, \\ U'(0+) &= -\infty, & U'(0-) &= +\infty. \end{aligned} \quad (50)$$

Therefore, the solution U defined by (49) is a cusped soliton solution for the $B(2, 2)$ equation (3). The profile of cuspon is shown in Figure 1(d).

Let us summarize our results in the following theorem.

Theorem 6. Suppose that $U(x - ct)$ is a single peak soliton solution for the $B(2, 2)$ equation (3) at the peak point $\xi_0 = 0$, which satisfies the inhomogeneous boundary condition (8). Then one has the following.

(1) For $3A + 2c^2 \neq 0$, let $\alpha = A/(3A + 2c^2)$; then

- (i) if $-1 \leq \alpha \leq 0$, there is no soliton for the $B(2, 2)$ equation (3);
- (ii) if $\alpha < 0$ and $A < 0$, the $B(2, 2)$ equation (3) has the smooth soliton solution

$$U(\xi) = \Phi_1^{-1} \left(-\frac{|\xi|}{2} + K_0 \right), \quad (51)$$

with the following properties:

$$U(0) = B_2, \quad U(\pm\infty) = A, \quad U'(0) = 0; \quad (52)$$

- (iii) if $\alpha > 0$, the $B(2, 2)$ equation (3) has the cusped soliton solution

$$U(x, t) = \Phi_2^{-1} \left(-\frac{|\xi|}{2} + K_0 \right), \quad (53)$$

with the following properties:

$$\begin{aligned} U(0) &= 0, & U(\pm\infty) &= A, \\ U'(0+) &= +\infty, & U'(0-) &= -\infty; \end{aligned} \quad (54)$$

- (iv) if $\alpha < 0$ and $A > 0$, the $B(2, 2)$ equation (3) has the cusped soliton solution

$$U(x, t) = \Psi_1^{-1} \left(\frac{|\xi|}{2} + K_0 \right) - a_1, \quad (55)$$

with the following properties:

$$\begin{aligned} U(0) &= 0, & U(\pm\infty) &= A, \\ U'(0+) &= -\infty, & U'(0-) &= +\infty. \end{aligned} \quad (56)$$

(2) When $3A + 2c^2 = 0$, then U is the peakon

$$U(x, t) = -\frac{2c^2}{3} \left(1 - e^{-|x-ct|/2} \right), \quad (57)$$

with the following properties:

$$\begin{aligned} U(0) &= 0, & U(\pm\infty) &= A, \\ U'(0+) &= -\frac{c^2}{3}, & U'(0-) &= \frac{c^2}{3}. \end{aligned} \quad (58)$$

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