Research Article

Related Fixed Point of Set-Valued Mappings in Three Menger Spaces

Ismat Beg¹ and Sunny Chauhan²

¹ Centre for Applicable Mathematics and Statistics, University of Central Punjab, Lahore 54770, Pakistan

² R. H. Government Postgraduate College, Kashipur, Udham Singh Nagar, Uttarakhand 244 713, India

Correspondence should be addressed to Ismat Beg; ibeg@lums.edu.pk

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We prove a related fixed point theorem for set-valued mappings in three Menger spaces. Some examples are also furnished to support the results. Our main result generalizes and extends several known results in the literature.

1. Introduction

Fisher [1] initiated the study of conditions for the existence of a relation connecting the fixed points of two mappings in two different metric spaces (also see [2]). Afterward, Fisher and Türkoğlu [3] proved a related fixed point theorem for set-valued mappings in two metric spaces. In 2003, Chourasia and Fisher [4] established a related fixed point theorem for two pairs of set valued mappings in two metric spaces. Several mathematicians have extensively developed metrical related fixed point theorems (see, e.g., [5–8]). Later on, Pant [9] firstly studied related fixed point theorems for single valued mappings in two complete Menger spaces and generalized the results of Fisher [1, 2]. Pant and Kumar [10] extended the results of Namdeo et al. [7] to two pairs of single-valued mappings.

In this paper, we prove a related fixed point theorem for set-valued mappings in three complete Menger spaces. Our result generalizes the result of Jain et al. [6] and extends the result of Fisher and Türkoğlu [3].

2. Preliminaries

Definition 1 (see [11]). A mapping $\Delta : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a t -norm if

- (i) $\Delta(a, 1) = a, \Delta(0, 0) = 0$,
- (ii) $\Delta(a, b) = \Delta(b, a)$,

(iii) $\Delta(a, b) \leq \Delta(c, d)$, whenever $a \leq c$ and $b \leq d$,

(iv) $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$, for all $a, b, c, d \in [0, 1]$.

Examples of t -norms are $\Delta_M(a, b) = \min\{a, b\}$, $\Delta_P(a, b) = ab$, and $\Delta_L(a, b) = \max\{a + b - 1, 0\}$.

Definition 2 (see [11]). A mapping $F : \mathbb{R} \rightarrow \mathbb{R}^+$ is called distribution function if it is nondecreasing, left continuous with $\inf\{F(t) : t \in \mathbb{R}\} = 0$ and $\sup\{F(t) : t \in \mathbb{R}\} = 1$.

Let $\mathfrak{F}$ be the set of all distribution functions whereas H stands for the specific distribution function (also known as Heaviside function) defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0; \\ 1, & \text{if } t > 0. \end{cases} \quad (1)$$

Definition 3 (see [11]). Let X be a nonempty set. An ordered pair $(X, \mathcal{F})$ is called a PM-space where $\mathcal{F}$ is a mapping from $\mathcal{F} : X \times X \rightarrow \mathfrak{F}$ satisfying the following conditions:

- (i) $F_{x,y}(t) = H(t)$ if and only if $x = y$,
- (ii) $F_{x,y}(t) = F_{y,x}(t)$,
- (iii) $F_{x,z}(t) = 1$ and $F_{z,y}(s) = 1$; then $F_{x,y}(t + s) = 1$ for all $x, y, z \in X$ and $t, s > 0$.

Every metric space (X, d) can be realized as a PM-space by considering $\mathcal{F} : X \times X \rightarrow \mathfrak{F}$ defined by

$F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in X$. So PM-spaces offer a wider framework (than that of the metric spaces) and are general enough to cover even wider statistical situations.

Definition 4 (see [12]). A Menger space $(X, \mathcal{F}, \Delta)$ is a triplet where $(X, \mathcal{F})$ is a PM-space and Δ is a t -norm satisfying the following condition:

$$F_{x,y}(t+s) \geq \Delta(F_{x,z}(t), F_{z,y}(s)), \quad (2)$$

for all $x, y, z \in X$ and $t, s > 0$.

Definition 5 (see [11]). Let $(X, \mathcal{F}, \Delta)$ be a Menger space and Δ a continuous t -norm. A sequence $\{x_n\}$ in X is said to be

- (i) convergent to a point x in X if and only if for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer $N(\varepsilon, \lambda)$ such that $F_{x_n, x}(\varepsilon) > 1 - \lambda$ for all $n \geq N(\varepsilon, \lambda)$.
- (ii) Cauchy if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer $N(\varepsilon, \lambda)$ such that $F_{x_n, x_m}(\varepsilon) > 1 - \lambda$ for all $n, m \geq N(\varepsilon, \lambda)$.

A Menger space in which every Cauchy sequence is convergent is said to be complete.

Lemma 6 (see [13]). Let $(X, \mathcal{F}, \Delta)$ be a Menger space. If there exists a constant $k \in (0, 1)$ such that

$$F_{x,y}(kt) \geq F_{x,y}(t), \quad (3)$$

for all $t > 0$ with fixed $x, y \in X$, then $x = y$.

We need the following definition due to Chang et al. [14] for subsequent use in next section.

Definition 7 (see [14]). Let $(X, \mathcal{F}, \Delta)$ be a Menger space and $\mathcal{U}$ a nonempty subset of X . Then $\mathcal{U}$ is said to be probabilistically bounded if $\sup_{t>0} \inf_{x,y \in \mathcal{U}} F_{x,y}(t) = 1$.

If X is itself probabilistically bounded, then X is said to be a probabilistically bounded space.

Let $\mathcal{CB}(X)$ be a family of nonempty closed and bounded subsets of a Menger space $(X, \mathcal{F}, \Delta)$. For all $\mathcal{U}, \mathcal{V} \in \mathcal{CB}(X)$ and for every $t > 0$, we define

$$\begin{aligned} {}_D F_{\mathcal{U}, \mathcal{V}}(t) &= \sup \{F_{a,b}(t); a \in \mathcal{U}, b \in \mathcal{V}\}, \\ {}_\delta F_{\mathcal{U}, \mathcal{V}}(t) &= \inf \{F_{a,b}(t); a \in \mathcal{U}, b \in \mathcal{V}\}. \end{aligned} \quad (4)$$

If the set $\mathcal{U}$ consists of a single point a , we write ${}_D F_{\mathcal{U}, \mathcal{V}}(t) = {}_D F_{a, \mathcal{V}}(t)$. Also if the set $\mathcal{V}$ consists of a single point b , we write ${}_D F_{\mathcal{U}, \mathcal{V}}(t) = {}_D F_{\mathcal{U}, b}(t)$. It follows immediately from the definition that

- (i) ${}_D F_{\mathcal{U}, \mathcal{V}}(t) = {}_\delta F_{\mathcal{V}, \mathcal{U}}(t) \geq 0$,
- (ii) ${}_D F_{\mathcal{U}, \mathcal{V}}(t) = 1$ if and only if $\mathcal{U} = \mathcal{V} = \{a\}$, for all $\mathcal{U}, \mathcal{V} \in \mathcal{CB}(X)$.

3. Main Result

Now we prove a related fixed point theorem for set-valued mappings in three complete Menger spaces.

Theorem 8. Let $(X, \mathcal{F}, \Delta)$, $(Y, \mathcal{G}, \Delta)$, and (Z, H, Δ) be three complete Menger spaces, where Δ is a continuous t -norm. If P is a continuous mapping of X into $\mathcal{CB}(Y)$, Q is a continuous mapping of Y into $\mathcal{CB}(Z)$, and R is a mapping of Z into $\mathcal{CB}(X)$ satisfying the inequalities

$$\begin{aligned} {}_\delta F_{RQP_x, RQP_{x'}}(kt) &\geq \min \{F_{x, x'}(t), {}_\delta F_{x, RQP_x}(t), \\ &{}_\delta F_{x', RQP_{x'}}(t), {}_\delta G_{P_x, P_{x'}}(t), \\ &{}_\delta H_{QP_x, QP_{x'}}(t)\}, \end{aligned} \quad (5)$$

$$\begin{aligned} {}_\delta G_{PRQ_y, PRQ_{y'}}(kt) &\geq \min \{G_{y, y'}(t), {}_\delta G_{y, PRQ_y}(t), \\ &{}_\delta G_{y', PRQ_{y'}}(t), {}_\delta H_{Q_y, Q_{y'}}(t), \\ &{}_\delta F_{RQ_y, RQ_{y'}}(t)\}, \end{aligned} \quad (6)$$

$$\begin{aligned} {}_\delta H_{QPR_z, QPR_{z'}}(kt) &\geq \min \{H_{z, z'}(t), {}_\delta H_{z, QPR_z}(t), \\ &{}_\delta H_{z', QPR_{z'}}(t), {}_\delta F_{Rz, Rz'}(t), \\ &{}_\delta G_{PR_z, PR_{z'}}(t)\} \end{aligned} \quad (7)$$

for all $x, x' \in X$, $y, y' \in Y$, $z, z' \in Z$, $k \in (0, 1)$ and $t > 0$. Then RQP has a unique fixed point u in X , PRQ has a unique fixed point v in Y , and QPR has a unique fixed point w in Z . Further $Pu = \{v\}$, $Qv = \{w\}$, and $Rw = \{u\}$.

Proof. Let $x = x_1$ be an arbitrary point in X . We define the sequences $\{x_n\}$ in X , $\{y_n\}$ in Y , and $\{z_n\}$ in Z inductively as follows. We choose a point y_1 in Px_1 , then a point z_1 in Qy_1 , and then a point x_2 in Rz_1 . In general, having chosen x_n in X , y_n in Y , and z_n in Z , choose a point x_{n+1} in Rz_n , then a point y_{n+1} in Px_{n+1} , and then a point z_{n+1} in Qy_{n+1} for $n = 1, 2, \dots$. Applying inequality (5), we get

$$\begin{aligned} F_{x_{n+1}, x_{n+2}}(kt) &\geq {}_\delta F_{RQP_{x_n}, RQP_{x_{n+1}}}(kt) \\ &\geq \min \{F_{x_n, x_{n+1}}(t), {}_\delta F_{x_n, RQP_{x_n}}(t), \\ &{}_\delta F_{x_{n+1}, RQP_{x_{n+1}}}(t), {}_\delta G_{P_{x_n}, P_{x_{n+1}}}(t), \\ &{}_\delta H_{QP_{x_n}, QP_{x_{n+1}}}(t)\}, \\ &\geq \min \{{}_\delta F_{RQP_{x_{n-1}}, RQP_{x_n}}(t), \\ &{}_\delta F_{RQP_{x_n}, RQP_{x_{n+1}}}(t), \\ &{}_\delta G_{PRQ_{y_{n-1}}, PRQ_{y_n}}(t), \\ &{}_\delta H_{QPR_{z_{n-1}}, QPR_{z_n}}(t)\}, \\ &\geq \min \{{}_\delta F_{RQP_{x_{n-1}}, RQP_{x_n}}(t), \\ &{}_\delta G_{PRQ_{y_{n-1}}, PRQ_{y_n}}(t), \\ &{}_\delta H_{QPR_{z_{n-1}}, QPR_{z_n}}(t)\}. \end{aligned} \quad (8)$$

Using inequality (6), we get

$$\begin{aligned}
 G_{y_{n+1}, y_{n+2}}(kt) &\geq \delta G_{PRQy_n, PRQy_{n+1}}(kt) \\
 &\geq \min \left\{ G_{y_n, y_{n+1}}(t), \delta G_{y_n, PRQy_n}(t), \right. \\
 &\quad \delta G_{y_{n+1}, PRQy_{n+1}}(t), \delta H_{Qy_n, Qy_{n+1}}(t), \\
 &\quad \left. \delta F_{RQy_n, RQy_{n+1}}(t) \right\}, \\
 &\geq \min \left\{ \delta G_{PRQy_{n-1}, PRQy_n}(t), \right. \\
 &\quad \delta G_{PRQy_n, PRQy_{n+1}}(t), \\
 &\quad \delta H_{QPRz_{n-1}, QPRz_n}(t), \\
 &\quad \left. \delta F_{RQPX_n, RQPX_{n+1}}(t) \right\}, \\
 &\geq \min \left\{ \delta G_{PRQy_{n-1}, PRQy_n}(t), \right. \\
 &\quad \delta H_{QPRz_{n-1}, QPRz_n}(t), \\
 &\quad \left. \delta F_{RQPX_n, RQPX_{n+1}}(t) \right\}, \\
 &\geq \min \left\{ \delta G_{PRQy_{n-1}, PRQy_n}(t), \right. \\
 &\quad \delta H_{QPRz_{n-1}, QPRz_n}(t), \\
 &\quad \left. \delta F_{RQPX_n, RQPX_{n+1}}(t) \right\},
 \end{aligned} \tag{9}$$

on using inequality (8). Further, on using inequality (7), we have

$$\begin{aligned}
 H_{z_{n+1}, z_{n+2}}(kt) &\geq \delta H_{QPRz_n, QPRz_{n+1}}(kt) \\
 &\geq \min \left\{ H_{z_n, z_{n+1}}(t), \delta H_{z_n, QPRz_n}(t), \right. \\
 &\quad \delta H_{z_{n+1}, QPRz_{n+1}}(t), \delta F_{Rz_n, Rz_{n+1}}(t), \\
 &\quad \left. \delta G_{PRz_n, PRz_{n+1}}(t) \right\}, \\
 &\geq \min \left\{ \delta H_{QPRz_{n-1}, QPRz_n}(t), \right. \\
 &\quad \delta H_{QPRz_n, QPRz_{n+1}}(t), \\
 &\quad \delta F_{RQPX_n, RQPX_{n+1}}(t), \\
 &\quad \left. \delta G_{PRQy_n, PRQy_{n+1}}(t) \right\}, \\
 &\geq \min \left\{ \delta H_{QPRz_{n-1}, QPRz_n}(t), \right. \\
 &\quad \delta F_{RQPX_n, RQPX_{n+1}}(t), \\
 &\quad \left. \delta G_{PRQy_{n-1}, PRQy_n}(t) \right\},
 \end{aligned} \tag{10}$$

on using inequalities (8) and (9). It follows easily by induction on using inequalities (8), (9), and (10) that

$$\begin{aligned}
 F_{x_{n+1}, x_{n+2}}(t) &\geq \min \left\{ \delta F_{RQPX_1, RQPX_2} \left(\frac{t}{k^{n-1}} \right), \right. \\
 &\quad \delta G_{PRQy_1, PRQy_2} \left(\frac{t}{k^{n-1}} \right), \\
 &\quad \left. \delta H_{QPRz_1, QPRz_2} \left(\frac{t}{k^{n-1}} \right) \right\}, \\
 G_{y_{n+1}, y_{n+2}}(t) &\geq \min \left\{ \delta F_{RQPX_1, RQPX_2} \left(\frac{t}{k^{n-1}} \right), \right. \\
 &\quad \delta G_{PRQy_1, PRQy_2} \left(\frac{t}{k^{n-1}} \right), \\
 &\quad \left. \delta H_{QPRz_1, QPRz_2} \left(\frac{t}{k^{n-1}} \right) \right\}, \\
 H_{z_{n+1}, z_{n+2}}(t) &\geq \min \left\{ \delta F_{RQPX_1, RQPX_2} \left(\frac{t}{k^{n-1}} \right), \right. \\
 &\quad \delta G_{PRQy_1, PRQy_2} \left(\frac{t}{k^{n-1}} \right), \\
 &\quad \left. \delta H_{QPRz_1, QPRz_2} \left(\frac{t}{k^{n-1}} \right) \right\}
 \end{aligned} \tag{11}$$

for all $n = 1, 2, \dots$. Since $\delta F_{RQPX_1, RQPX_2}(t/k^{n-1}) \rightarrow 1$, $\delta G_{PRQy_1, PRQy_2}(t/k^{n-1}) \rightarrow 1$, and $\delta H_{QPRz_1, QPRz_2}(t/k^{n-1}) \rightarrow 1$ as $n \rightarrow \infty$, it follows that $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are Cauchy sequences in complete Menger space X and so one has a limit u in X . Similarly, the sequences $\{y_n\}$ and $\{z_n\}$ are also Cauchy sequences with limits v and w in the complete Menger spaces Y and Z , respectively. Then we have

$$\lim_{n \rightarrow \infty} RQPX_n = \{u\} = \lim_{n \rightarrow \infty} RQy_n, \tag{12}$$

$$\lim_{n \rightarrow \infty} PRQy_n = \{v\} = \lim_{n \rightarrow \infty} PRz_n, \tag{13}$$

$$\lim_{n \rightarrow \infty} QPRz_n = \{w\} = \lim_{n \rightarrow \infty} QPX_n.$$

Using the continuity of P and Q , we have

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} Px_n = Pu = \{v\}, \tag{14}$$

$$\lim_{n \rightarrow \infty} z_n = \lim_{n \rightarrow \infty} Qy_n = Qv = \{w\},$$

and then we see that

$$QPu = \{w\}. \tag{15}$$

Now we show that u is a fixed point of RQP . Applying inequality (5), we have

$$\begin{aligned}
 \delta F_{RQPu, RQPX_n}(kt) &\geq \delta F_{RQPu, RQPX_{n-1}}(kt) \\
 &\geq \min \left\{ F_{u, x_{n-1}}(t), \delta F_{u, RQPu}(t), \right. \\
 &\quad \delta F_{x_{n-1}, RQPX_{n-1}}(t), \delta G_{Pu, Px_{n-1}}(t), \\
 &\quad \left. \delta H_{QPu, QPX_{n-1}}(t) \right\}.
 \end{aligned} \tag{16}$$

Since P and Q are continuous, it follows on letting $n \rightarrow \infty$ and using (12) that

$$\delta F_{RQPu,u}(kt) \geq \delta F_{u,RQPu}(t). \quad (17)$$

Owing to Lemma 6, $RQP = \{u\}$ and so u is a fixed point of RQP . We now have

$$RQPu = Ru = \{u\}, \quad (18)$$

on using (15) and then

$$\begin{aligned} PRQv &= PRQPu = \{v\}, \\ QPRw &= QPRQv = Qv = \{w\}, \end{aligned} \quad (19)$$

on using (14). Hence v and w are fixed points of PRQ and QPR , respectively. Further, we see that

$$RQv = \{u\}, \quad PRw = \{v\}. \quad (20)$$

To prove the uniqueness of u , suppose that RQP has another fixed point u' . Then, using inequality (5), we have

$$\begin{aligned} \delta F_{u',RQPu'}(kt) &\geq \delta F_{RQPu',RQPu'}(kt) \\ &\geq \min \{F_{u',u'}(t), \delta F_{u',RQPu'}(t), \\ &\quad \delta G_{Pu',Pu'}(t), \delta H_{QPu',QPu'}(t)\} \\ &\geq \min \{\delta G_{Pu',Pu'}(t), \delta H_{QPu',QPu'}(t)\}. \end{aligned} \quad (21)$$

Next, using inequality (6), we have

$$\begin{aligned} \delta G_{Pu',Pu'}(kt) &\geq \delta G_{PRQPu',PRQPu'}(kt) \\ &\geq \min \{G_{Pu',Pu'}(t), \delta G_{Pu',PRQPu'}(t), \\ &\quad \delta H_{QPu',QPu'}(t), \delta F_{RQPu',RQPu'}(t)\} \\ &\geq \min \{\delta H_{QPu',QPu'}(t), \delta F_{RQPu',RQPu'}(t)\}, \end{aligned} \quad (22)$$

and using inequality (7), we obtain

$$\begin{aligned} \delta H_{QPu',QPu'}(kt) &\geq \delta H_{QPRQPu',QPRQPu'}(kt) \\ &\geq \min \{H_{QPu',QPu'}(t), \delta H_{QPu',QPRQPu'}(t), \\ &\quad \delta F_{RQPu',RQPu'}(t), \\ &\quad \delta G_{PRQPu',PRQPu'}(t)\} \\ &\geq \min \{\delta F_{RQPu',RQPu'}(t), \\ &\quad \delta G_{PRQPu',PRQPu'}(t)\}. \end{aligned} \quad (23)$$

It now follows easily from inequalities (22) and (23) that

$$\delta G_{PRQPu',PRQPu'}(t) \geq \delta F_{RQPu',RQPu'}\left(\frac{t}{k}\right), \quad (24)$$

and then

$$\delta H_{QPu',QPu'}(t) \geq \delta F_{RQPu',RQPu'}\left(\frac{t}{k}\right). \quad (25)$$

Using inequalities (21), (24), and (25), we obtain

$$\begin{aligned} \delta F_{u',RQPu'}(t) &\geq \delta F_{RQPu',RQPu'}(t) \\ &\geq \delta F_{RQPu',RQPu'}\left(\frac{t}{k}\right) \\ &\geq \cdots \geq \delta F_{RQPu',RQPu'}\left(\frac{t}{k^{2n}}\right), \end{aligned} \quad (26)$$

for all $n = 1, 2, \dots$. Since $\delta F_{RQPu',RQPu'}(t/k^{2n}) \rightarrow 1$ as $n \rightarrow \infty$, it follows that $RQPu' = u'$ and so $RQPu'$ is a singleton. It then follows from inequality (25) that QPu' is a singleton and from inequality (24) that Pu' is a singleton. Using inequality (5) again, we have

$$\begin{aligned} F_{u,u'}(kt) &= \delta F_{RQPu,RQPu'}(kt) \\ &\geq \min \{F_{u,u'}(t), F_{u,u}(t), F_{u',RQPu'}(t), \\ &\quad G_{Pu,Pu'}(t), H_{QPu,QPu'}(t)\} \\ &\geq \min \{G_{Pu,Pu'}(t), H_{QPu,QPu'}(t)\}. \end{aligned} \quad (27)$$

Next using inequality (6), we have

$$\begin{aligned} G_{Pu,Pu'}(kt) &= \delta G_{PRQPu,PRQPu'}(kt) \\ &\geq \min \{G_{Pu,Pu'}(t), G_{Pu,PRQPu}(t), \\ &\quad G_{Pu',PRQPu'}(t), H_{QPu,QPu'}(t), \\ &\quad F_{RQPu,RQPu'}(t)\} \\ &= \min \{H_{QPu,QPu'}(t), F_{u,u'}(t)\}, \end{aligned} \quad (28)$$

and using inequality (7), we have

$$\begin{aligned} H_{QPu,QPu'}(kt) &\geq \delta H_{QPRQPu,QPRQPu'}(kt) \\ &\geq \min \{H_{QPu,QPu'}(t), H_{QPu,QPRQPu}(t), \\ &\quad H_{QPu',QPRQPu'}(t), F_{RQPu,RQPu'}(t), \\ &\quad G_{PRQPu,PRQPu'}(t)\} \\ &= \min \{F_{u,u'}(t), G_{Pu,Pu'}(t)\}. \end{aligned} \quad (29)$$

It follows from inequalities (27), (28), and (29) that $F_{u,u'}(t) = 1$, which shows the uniqueness of u . The uniqueness of v and w can be verified similarly. $\square$

Example 9. Consider $X = [0, 2]$, $Y = [1, 5]$, and $Z = [0, 10]$ and define

$$\begin{aligned} F_{x,y}(t) &= G_{x,y}(t) = H_{x,y}(t) \\ &= \begin{cases} \frac{t}{t + |x - y|}, & \text{if } t > 0; \\ 0, & \text{if } t = 0, \end{cases} \end{aligned} \quad (30)$$

for all $x, y \in X$. Then $(X, \mathcal{F}, \Delta)$, $(Y, \mathcal{G}, \Delta)$, and (Z, H, Δ) are three complete Menger spaces with continuous t -norm $\Delta(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$. Define the set-valued mappings $P : X \rightarrow \mathcal{CB}(Y)$, $Q : Y \rightarrow \mathcal{CB}(Z)$ and $R : Z \rightarrow \mathcal{CB}(X)$ by $Px = [1 + x, 2]$ for all $x \in X$, $Qy = [2y + 1, 5]$ for all $y \in Y$ and

$$Rz = \begin{cases} \left[\frac{z}{5}, 1 \right], & \text{if } z \in [0, 5]; \\ \left(1, \frac{z}{5} \right], & \text{if } z \in (5, 10]. \end{cases} \quad (31)$$

We can easily verify the inequalities with $k = 1/2 \in (0, 1)$. Thus all the conditions of Theorem 8 are satisfied and RQP has a unique fixed point $\{1\}$ in $\mathcal{CB}(X)$, PRQ has a unique fixed point $\{2\}$ in $\mathcal{CB}(Y)$, and QPR has a unique fixed point $\{5\}$ in $\mathcal{CB}(Z)$. Hence, $P(1) = \{2\}$, $Q(2) = \{5\}$, and $R(5) = \{1\}$, and so $RQP(1) = RQ(2) = R(5) = \{1\}$, $PRQ(2) = PR(5) = P(1) = \{2\}$, and $QPR(5) = QP(1) = Q(2) = \{5\}$.

Corollary 10. Let $(X, \mathcal{F}, \Delta)$, $(Y, \mathcal{G}, \Delta)$ and (Z, H, Δ) be three complete Menger spaces, where Δ is a continuous t -norm. If P is a continuous mapping of X into Y , Q is a continuous mapping of Y into Z and R is a mapping of Z into X satisfying the inequalities

$$\begin{aligned} F_{RQP_x, RQP_{x'}}(kt) &\geq \min \{F_{x, x'}(t), F_{x, RQP_x}(t), \\ &\quad F_{x', RQP_{x'}}(t), G_{Px, Px'}(t), \\ &\quad H_{QP_x, QP_{x'}}(t)\} \\ G_{PRQ_y, PRQ_{y'}}(kt) &\geq \min \{G_{y, y'}(t), G_{y, PRQ_y}(t), \\ &\quad G_{y', PRQ_{y'}}(t), H_{Qy, Qy'}(t), \\ &\quad F_{RQ_y, RQ_{y'}}(t)\} \\ H_{QPR_z, QPR_{z'}}(kt) &\geq \min \{H_{z, z'}(t), H_{z, QPR_z}(t), \\ &\quad H_{z', QPR_{z'}}(t), F_{Rz, Rz'}(t), \\ &\quad G_{PR_z, PR_{z'}}(t)\} \end{aligned} \quad (32)$$

for all $x, x' \in X$, $y, y' \in Y$, $z, z' \in Z$, $k \in (0, 1)$, and $t > 0$. Then RQP has a unique fixed point u in X , PRQ has a unique fixed point v in Y and QPR has a unique fixed point w in Z . Further, $Pu = v$, $Qv = w$ and $Rw = u$.

Example 11. Let $X = [0, 2]$, $Y = [1, 3]$, $Z = [3, 5]$ and $F_{x, y}(t)$, $G_{x, y}(t)$ and $H_{x, y}(t)$ be defined as Example 9. Then $(X, \mathcal{F}, \Delta)$, $(Y, \mathcal{G}, \Delta)$, and (Z, H, Δ) be three complete Menger spaces with continuous t -norm $\Delta(a, b) = \min\{a, b\}$ for all $a, b \in [0, 1]$. Define the mappings $P : X \rightarrow Y$, $Q : Y \rightarrow Z$ and $R : Z \rightarrow X$ by $Px = x + 1$ for all $x \in X$, $Qy = y + 2$ for all $y \in Y$ and

$$Rz = \begin{cases} \frac{z}{4}, & \text{if } z \in [3, 4]; \\ \frac{z+1}{3}, & \text{if } z \in (4, 5]. \end{cases} \quad (33)$$

Also the inequalities (32) can be verified with $k = 1/2 \in (0, 1)$. Thus all the conditions of Corollary 10 are satisfied and RQP has a unique fixed point 1 in X , PRQ has a unique fixed point 2 in Y , and QPR has a unique fixed point 4 in Z . Further, $P(1) = 2$, $Q(2) = 4$, and $R(4) = 1$; that is, $RQP(1) = RQ(2) = R(4) = 1$, $PRQ(2) = PR(4) = P(1) = 2$, and $QPR(4) = QP(1) = Q(2) = 4$.

The following corollaries are the natural results due to Theorem 8 and Corollary 10.

Corollary 12. Let $(X, \mathcal{F}, \Delta)$ and $(Y, \mathcal{G}, \Delta)$ be two complete Menger spaces, where Δ is a continuous t -norm. If P is a continuous mapping of X into $\mathcal{CB}(Y)$, Q is a mapping of Y into $\mathcal{CB}(X)$ satisfying the inequalities

$$\begin{aligned} \delta F_{QP_x, QP_{x'}}(kt) &\geq \min \{F_{x, x'}(t), \delta F_{x, QP_x}(t), \\ &\quad \delta F_{x', QP_{x'}}(t), \delta G_{Px, Px'}(t)\} \\ \delta G_{PQ_y, PQ_{y'}}(kt) &\geq \min \{G_{y, y'}(t), \delta G_{y, PQ_y}(t), \\ &\quad \delta G_{y', PQ_{y'}}(t), \delta F_{Qy, Qy'}(t)\} \end{aligned} \quad (34)$$

for all $x, x' \in X$, $y, y' \in Y$, $k \in (0, 1)$, and $t > 0$. Then QP has a unique fixed point u in X and PQ has a unique fixed point v in Y .

Corollary 13. Let $(X, \mathcal{F}, \Delta)$ and $(Y, \mathcal{G}, \Delta)$ be two complete Menger spaces, where Δ is a continuous t -norm. If P is a continuous mapping of X into Y , Q is a mapping of Y into X satisfying the inequalities

$$\begin{aligned} F_{QP_x, QP_{x'}}(kt) &\geq \min \{F_{x, x'}(t), F_{x, QP_x}(t), \\ &\quad F_{x', QP_{x'}}(t), G_{Px, Px'}(t)\}, \\ G_{PQ_y, PQ_{y'}}(kt) &\geq \min \{G_{y, y'}(t), G_{y, PQ_y}(t), \\ &\quad G_{y', PQ_{y'}}(t), F_{Qy, Qy'}(t)\} \end{aligned} \quad (35)$$

for all $x, x' \in X$, $y, y' \in Y$, $k \in (0, 1)$, and $t > 0$. Then QP has a unique fixed point u in X and PQ has a unique fixed point v in Y .

Remark 14. Theorem 8 generalizes the result of Jain et al. [6] and Fisher [2]. Corollaries 10–13 extend the results of Fisher and Türkoğlu [3, Theorem 1, Corollary 1] to PM-spaces.

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