Research Article

Kantowski-Sachs String Cosmological Model with Bulk Viscosity in General Scalar Tensor Theory of Gravitation

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A new class of spatially homogeneous Kantowski-Sachs string cosmological models with bulk viscosity in Nordtvedt (1970) general scalar-tensor theory of gravitation with the help of a special case proposed by Schwinger (1970) is obtained. In this paper we have presented anisotropic as well as isotropic cosmological models. Some important features of the models, thus obtained, have been discussed. These exact models are new and more general and represent not only the early stages of evolution but also the present universe.

1. Introduction

Nordtvedt [1] proposed a general class of scalar-tensor gravitational theories in which the parameter ω of the BD theory is allowed to be an arbitrary (positive definite) function of the scalar field [$\omega \rightarrow \omega(\phi)$]. This general class of scalar-tensor gravitational theories includes the Jordan [2] and Brans-Dicke [3] theories as special cases.

Barker [4], Ruban and Finkelstein [5], Banerjee and Santos [6, 7], and Shanti and Rao [8, 9] are some of the authors who have investigated several aspects of Nordtvedt general scalar-tensor theory. Rao and Kumari [10] have discussed a cosmological model with negative constant deceleration parameter in this theory, and very recently Rao et al. [11] have obtained Kaluza-Klein radiating model in a general scalar-tensor theory of gravitation.

The field equations of general scalar-tensor theory proposed by Nordtvedt are

$$\begin{aligned}
 R_{ij} - \frac{1}{2}g_{ij}R = & -8\pi\phi^{-1}T_{ij} - \omega\phi^{-2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}\right) \\
 & - \phi^{-1}\left(\phi_{i,j} - g_{ij}\phi_{,k}^{,k}\right), \\
 \phi_{,k}^{,k} = & \frac{8\pi T}{3+2\omega} - \frac{1}{(3+2\omega)}\frac{d\omega}{d\phi}\phi_{,i}\phi^{,i},
 \end{aligned} \tag{1}$$

where R_{ij} is the Ricci tensor, R is the scalar curvature, T_{ij} is the stress energy tensor of the matter, and comma and semicolon denote partial and covariant differentiation, respectively.

Also, we have

$$T_{ij}^{ij} = 0 \tag{2}$$

which is a consequence of the field equations (1).

Beside the Bianchi type metrics, the Kantowski-Sachs [12] models are also describing spatially homogeneous universes. For a review of Kantowski-Sachs metrics one can refer to MacCallum [13]. These metrics represent homogeneous but anisotropically expanding (or contracting) cosmologies and provide models where the effects of anisotropy can be estimated and compared with all well-known Friedmann-Robertson-Walker class of cosmologies. Wang [14] has obtained Kantowski-Sachs string cosmological model with bulk viscosity in general relativity. Kandalkar et al. [15] have discussed Kantowski-Sachs viscous fluid cosmological model with a varying Λ . Kandalkar et al. [16] have obtained string cosmology in Kantowski-Sachs space-time with bulk viscosity and magnetic field. Rao et al. [17] have studied various Bianchi type string cosmological models in the presence of bulk viscosity. Recently Rao et al. [18] and Rao and Sireesha [19] have obtained Bianchi types II, VIII, and IX string cosmological models with bulk viscosity in Lyra [20] and Brans and Dicke [3] theory of gravitation, respectively.

Motivated by the above investigations, we study spatially homogeneous and anisotropic Kantowski-Sachs string cosmological model with bulk viscosity in Nordtvedt [1] general scalar-tensor theory with the help of a special case proposed by Schwinger [21], that is, $3 + 2\omega(\phi) = 1/\lambda\phi$, where λ is a constant.

2. Metric and Energy Momentum Tensor

We consider a spatially homogeneous Kantowski-Sachs metric of the form

$$ds^2 = dt^2 - A^2 dr^2 - B^2 (d\theta^2 + \sin^2\theta d\phi^2), \quad (3)$$

where A and B are the functions of time t only.

The energy momentum tensor for a bulk viscous fluid containing one-dimensional string is

$$T_{ij} = (\rho + \bar{p}) u_i u_j - \bar{p} g_{ij} - \lambda_s x_i x_j, \quad (4)$$

$$\bar{p} = p - 3\xi H, \quad (5)$$

where $p = \epsilon_0 \rho$ ($0 \leq \epsilon_0 \leq 1$). Here $\bar{p}$ is the total pressure which includes the proper pressure p , ρ is the rest energy density of the system, λ_s is the tension in the string, $\xi(t)$ is the coefficient of bulk viscosity, $3\xi H$ is usually known as bulk viscous pressure, H is the Hubble parameter, u^i is the four velocity vector, and x^i is a space-like vector which represents the anisotropic directions of the string.

Here u^i and x^i satisfy the equations

$$\begin{aligned} g_{ij} u^i u^j &= 1, \\ g_{ij} x^i x^j &= -1, \\ u^i x_i &= 0. \end{aligned} \quad (6)$$

We assume that the string is to be lying along the x -axis. The one-dimensional strings are assumed to be loaded with particles, and the particle energy density is

$$\rho_p = \rho - \lambda_s. \quad (7)$$

In a comoving coordinate system, we get

$$T_1^1 = \lambda_s - \bar{p}, \quad T_2^2 = T_3^3 = -\bar{p}, \quad T_4^4 = \rho, \quad (8)$$

where ρ , λ_s , $\bar{p}$, and ϕ are functions of time t only.

3. Solution of the Field Equations

Using comoving coordinates, the field equations (1) to (2) for the metric (3) with the help of (4) to (8) can be written as

$$\begin{aligned} 2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{1}{B^2} + \frac{\omega\dot{\phi}^2}{2\phi^2} + \frac{\ddot{\phi}}{\phi} + 2\frac{\dot{\phi}\dot{B}}{\phi B} &= \frac{8\pi(\lambda_s - \bar{p})}{\phi}, \\ \frac{\ddot{A}}{A} + \frac{\dot{B}}{B} + \frac{A\dot{B}}{AB} + \frac{\omega\dot{\phi}^2}{2\phi^2} + \frac{\ddot{\phi}}{\phi} + \frac{\dot{\phi}}{\phi} \left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) &= -\frac{8\pi\bar{p}}{\phi}, \end{aligned}$$

$$2\frac{A\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{1}{B^2} - \frac{\omega\dot{\phi}^2}{2\phi^2} + \frac{\dot{\phi}}{\phi} \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} \right) = \frac{8\pi\rho}{\phi},$$

$$\ddot{\phi} + \dot{\phi} \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} \right) = \frac{8\pi}{(3+2\omega)} (\rho + \lambda_s - 3\bar{p})$$

$$- \frac{1}{(3+2\omega)} \frac{d\omega}{d\phi} \dot{\phi}^2,$$

$$\dot{\rho} + (\rho + \bar{p}) \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B} \right) - \lambda_s \frac{\dot{A}}{A} = 0.$$

(9)

Here the overhead dot denotes differentiation with respect to t .

By using the transformation $dt = AB^2 dT$, the above field equations (9) can be written as

$$2\frac{B''}{B} - 3\frac{B'^2}{B^2} - 2\frac{A'B'}{AB} + A^2 B^2 + \frac{\omega}{2} \frac{\phi'^2}{\phi^2} + \frac{\phi''}{\phi} - \frac{\phi'}{\phi} \frac{A'}{A} \quad (10)$$

$$= \frac{8\pi}{\phi} (\lambda_s - \bar{p}) (A^2 B^4),$$

$$\begin{aligned} \frac{A''}{A} - \frac{A'^2}{A^2} + \frac{B''}{B} - 2\frac{B'^2}{B^2} - 2\frac{A'B'}{AB} + \frac{\omega}{2} \frac{\phi'^2}{\phi^2} + \frac{\phi''}{\phi} - \frac{\phi'}{\phi} \frac{B'}{B} \\ = \frac{-8\pi\bar{p}}{\phi} (A^2 B^4), \end{aligned} \quad (11)$$

$$\begin{aligned} 2\frac{A'B'}{AB} + \frac{B'^2}{B^2} + A^2 B^2 - \frac{\omega}{2} \frac{\phi'^2}{\phi^2} + \frac{\phi'}{\phi} \left(\frac{A'}{A} + 2\frac{B'}{B} \right) \\ = \frac{8\pi}{\phi} \rho (A^2 B^4), \end{aligned} \quad (12)$$

$$(3+2\omega)\phi'' = 8\pi(\rho - 3\bar{p} + \lambda_s)(A^2 B^4) - \phi'^2 \frac{d\omega}{d\phi}, \quad (13)$$

$$\rho' + (\rho + \bar{p}) \left(\frac{A'}{A} + 2\frac{B'}{B} \right) - \lambda_s \left(\frac{A'}{A} \right) = 0. \quad (14)$$

From hereafter the overhead dash denotes differentiation with respect to T .

The field equations (10) to (13) are four independent equations with seven unknowns A , B , ω , ϕ , ρ , $\bar{p}$, and λ_s .

From (10) to (13), we have

$$\begin{aligned} (3+2\omega)\phi'' + \frac{d\omega}{d\phi}\phi'^2 \\ = 2\phi \left[\frac{A''}{A} - \frac{A'^2}{A^2} + 2\frac{B''}{B} - 3\frac{B'^2}{B^2} - 2\frac{A'B'}{AB} + A^2 B^2 \right] \\ + \omega \frac{\phi'^2}{\phi} + 3\phi''. \end{aligned} \quad (15)$$

Here we obtain string cosmological model with bulk viscosity in Nordtvedt's general scalar-tensor theory with the help of a special case proposed by Schwinger [21] in the form

$$3 + 2\omega(\phi) = \frac{1}{\lambda\phi}, \quad \lambda = \text{constant}. \quad (16)$$

From (15) and (16), we get

$$\begin{aligned} & \frac{1}{\lambda} \left[\frac{\phi''}{\phi} - \frac{\phi'^2}{\phi^2} \right] + \frac{3}{2} \frac{\phi'^2}{\phi} - 3\phi'' \\ &= 2\phi \left[\frac{A''}{A} - \frac{A'^2}{A^2} + 2\frac{B''}{B} - 3\frac{B'^2}{B^2} - 2\frac{A'B'}{AB} + A^2 B^2 \right]. \end{aligned} \quad (17)$$

In order to solve the above equation completely, we assume that the expansion scalar is proportional to shear scalar. This condition leads to

$$A = B^n, \quad n \neq 0. \quad (18)$$

From (17) and (18), we get

$$\phi = e^{(k_1 T + k_2)}, \quad (19)$$

where k_1 and k_2 are arbitrary constants and

$$\frac{B''}{B} - \frac{3(n+1)}{n+2} \frac{B'^2}{B^2} = \frac{-1}{(n+2)} B^{2n+2} - \frac{3}{4(n+2)} k_1^2. \quad (20)$$

From (20), we get

$$B = \left[\left(\frac{k_4}{k_3} \right) \text{sech}(n+1) k_4 T \right]^{1/(n+1)}, \quad n \neq -1, \quad (21)$$

where $k_3^2 = 1/(n^2 + n + 1)$ and $k_4^2 = 3k_1^2/4(2n+1)$.

From (18) and (21), we get

$$A = \left[\left(\frac{k_4}{k_3} \right) \text{sech}(n+1) k_4 T \right]^{n/(n+1)}. \quad (22)$$

From (12), we get the string energy density

$$\begin{aligned} \rho &= \frac{1}{8\pi k_5} [\cosh(n+1) k_4 T]^{2(n+2)/(n+1)} \\ &\times \left\{ e^{(k_1 T + k_2)} \left[(2n+1) k_4^2 \tanh^2(n+1) k_4 T \right. \right. \\ &\quad \left. \left. + \frac{k_4^2}{k_3^2} \text{sech}^2(n+1) k_4 T \right. \right. \\ &\quad \left. \left. - (n+2) k_1 k_4 \tanh(n+1) k_4 T + \frac{3}{4} k_1^2 \right] \right. \\ &\quad \left. - \frac{k_1^2}{4\lambda} \right\}, \end{aligned} \quad (23)$$

where $k_5 = (k_4/k_3)^{2(n+2)/(n+1)}$.

From (11), we get the total pressure

$$\begin{aligned} \bar{p} &= \frac{1}{8\pi k_5} [\cosh(n+1) k_4 T]^{2(n+2)/(n+1)} \\ &\times \left\{ e^{(k_1 T + k_2)} \left[(2n+1) k_4^2 \tanh^2(n+1) k_4 T \right. \right. \\ &\quad \left. \left. + (n+1)^2 k_4^2 \text{sech}^2(n+1) k_4 T \right. \right. \\ &\quad \left. \left. - k_1 k_4 \tanh(n+1) k_4 T - \frac{1}{4} k_1^2 \right] \right. \\ &\quad \left. - \frac{k_1^2}{4\lambda} \right\}. \end{aligned} \quad (24)$$

The proper pressure is given by

$$\begin{aligned} p &= \frac{\epsilon_0}{8\pi k_5} [\cosh(n+1) k_4 T]^{2(n+2)/(n+1)} \\ &\times \left\{ e^{(k_1 T + k_2)} \left[(2n+1) k_4^2 \tanh^2(n+1) k_4 T \right. \right. \\ &\quad \left. \left. + \frac{k_4^2}{k_3^2} \text{sech}^2(n+1) k_4 T \right. \right. \\ &\quad \left. \left. - (n+2) k_1 k_4 \tanh(n+1) k_4 T + \frac{3}{4} k_1^2 \right] \right. \\ &\quad \left. - \frac{k_1^2}{4\lambda} \right\}. \end{aligned} \quad (25)$$

The coefficient of bulk viscosity is given by

$$\begin{aligned} \xi &= \frac{1}{8\pi k_5 (n+2)} [\cosh(n+1) k_4 T]^{2(n+2)/(n+1)} \\ &\times \left\{ e^{(k_1 T + k_2)} \left[[(n^2 + 2)(1 - \epsilon_0) + n(4 - 3\epsilon_0)] \right. \right. \\ &\quad \left. \left. \times k_4 \tanh(n+1) k_4 T \right. \right. \\ &\quad \left. \left. - \frac{(3\epsilon_0 + 1) k_1^2}{4 k_4} \coth(n+1) k_4 T \right. \right. \\ &\quad \left. \left. + (\epsilon_0 (n+2) - 1) k_4 \right] \right. \\ &\quad \left. + \frac{(\epsilon_0 - 1) k_1^2}{4\lambda k_4} \coth(n+1) k_4 T \right\}. \end{aligned} \quad (26)$$

From (10) and (11), we get the string tension density

$$\begin{aligned} \lambda_s &= \frac{1}{8\pi k_5} [\cosh(n+1) k_4 T]^{2(n+2)/(n+1)} \\ &\times \left\{ e^{(k_1 T + k_2)} \left[n(2n+1) k_4^2 \text{sech}^2(n+1) k_4 T \right. \right. \\ &\quad \left. \left. + (n-1) k_1 k_4 \tanh(n+1) k_4 T \right] \right\}. \end{aligned} \quad (27)$$

The metric (3) can now be written as

$$ds^2 = \frac{1}{k_5} [\cosh(n+1)k_4T]^{2(n+2)/(n+1)} dT^2 - \left[\left(\frac{k_4}{k_3} \right) \operatorname{sech}(n+1)k_4T \right]^{2n/(n+1)} dr^2 - \left[\left(\frac{k_4}{k_3} \right) \operatorname{sech}(n+1)k_4T \right]^{2/(n+1)} (d\theta^2 + \sin^2\theta d\phi^2). \quad (28)$$

Thus, for $n \neq 1$, the metric (28) together with (19) and (23) to (27) constitutes a Kantowski-Sachs anisotropic string cosmological model with bulk viscosity in Nordtvedt's general scalar-tensor theory of gravitation with a special case proposed by Schwinger.

3.1. Isotropic Cosmological Model. For $n = 1$, the metric (28) together with (19) and (23) to (27) constitutes isotropic string cosmological model with bulk viscosity in Nordtvedt's general scalar-tensor theory of gravitation with a special case proposed by Schwinger.

4. Some Important Features of the Model

The volume element of the model (28) is given by

$$V = (-g)^{1/2} = \sqrt{k_5} [\operatorname{sech}(n+1)k_4T]^{(n+2)/(n+1)} \sin\theta. \quad (29)$$

The expression for the expansion scalar θ is given by

$$\theta = u^i_{;i} = -(n+2)k_4 \tanh(n+1)k_4T, \quad (30)$$

and the shear σ is given by

$$\sigma^2 = \frac{1}{2} \sigma^{ij} \sigma_{ij} = \frac{7}{18} (n+2)^2 k_4^2 \tanh^2(n+1)k_4T. \quad (31)$$

The deceleration parameter q is given by

$$q = -3\theta^{-2} \left(\theta_{;i} u^i + \frac{1}{3} \theta^2 \right), \quad (32)$$

$$q = - \left(\frac{-12(n+1)}{(n+2)e^{2(n+1)k_4T} (1 - e^{-2(n+1)k_4T})^2} + 1 \right).$$

The Hubble parameter H is given by

$$H = \frac{-(n+2)k_4 \tanh(n+1)k_4T}{3}. \quad (33)$$

The tensor of rotation $w_{ij} = u_{i;j} - u_{j;i}$ is identically zero, and hence this universe is nonrotational.

5. Conclusions

In this paper we have presented Kantowski-Sachs string cosmological models with bulk viscosity in Nordtvedt [1] general scalar-tensor theory with the help of a special case

proposed by Schwinger [21]. The models presented here are free from singularities, and the spatial volume vanishes as $T \rightarrow \infty$. We observe that, as T approaches to infinity, the expansion scalar θ leads to a constant value. This shows that the universe expands homogeneously. We also observe that the shear scalar σ and the Hubble parameter H approach to constant value as T tends to infinity. The energy density, total pressure, string tension density, and coefficient of bulk viscosity diverge with the increase of time. For this model, the deceleration parameter q is negative for large values of T . Hence the model represents accelerating universe and conformed the well-known fact that the scalar field and the bulk viscosity will play a vital role in getting an accelerated universe. For $n = 1$, the model represents isotropic universe. These anisotropic as well as isotropic exact models are new, more general, and free from singularities. Also the models presented here represent not only the early stages of evolution but also the present universe.

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