Research Article

Some Properties of Convolved k -Fibonacci Numbers

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Received 30 April 2013; Accepted 28 May 2013

Academic Editors: S. Falcon and Y. Zhang

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We define the convolved k -Fibonacci numbers as an extension of the classical convolved Fibonacci numbers. Then we give some combinatorial formulas involving the k -Fibonacci and k -Lucas numbers. Moreover we obtain the convolved k -Fibonacci numbers from a family of Hessenberg matrices.

1. Introduction

Fibonacci numbers and their generalizations have many interesting properties and applications to almost every field of science and art (e.g., see [1]). The Fibonacci numbers F_n are the terms of the sequence 0, 1, 1, 2, 3, 5, ..., wherein each term is the sum of the two previous terms, beginning with the values $F_0 = 0$ and $F_1 = 1$.

Besides the usual Fibonacci numbers many kinds of generalizations of these numbers have been presented in the literature. In particular, a generalization is the k -Fibonacci numbers.

For any positive real number k , the k -Fibonacci sequence, say $\{F_{k,n}\}_{n \in \mathbb{N}}$, is defined recurrently by

$$\begin{aligned} F_{k,0} &= 0, & F_{k,1} &= 1, \\ F_{k,n+1} &= kF_{k,n} + F_{k,n-1}, & n &\geq 1. \end{aligned} \quad (1)$$

In [2], k -Fibonacci numbers were found by studying the recursive application of two geometrical transformations used in the four-triangle longest-edge (4TLE) partition. These numbers have been studied in several papers; see [2–7].

The convolved Fibonacci numbers $F_j^{(r)}$ are defined by

$$(1 - x - x^2)^{-r} = \sum_{j=0}^{\infty} F_{j+1}^{(r)} x^j, \quad r \in \mathbb{Z}^+. \quad (2)$$

If $r = 1$ we have classical Fibonacci numbers. Note that

$$F_{m+1}^{(r)} = \sum_{j_1+j_2+\dots+j_r=m} F_{j_1+1} F_{j_2+1} \cdots F_{j_r+1}. \quad (3)$$

Moreover, using a result of Gould [8, page 699] on Humbert polynomials (with $n = j$, $m = 2$, $x = 1/2$, $y = -1$, $p = -r$, and $C = 1$), we have

$$F_{j+1}^{(r)} = \sum_{k=0}^{\lfloor j/2 \rfloor} \binom{j+r-k-1}{j-k} \binom{j-k}{k}. \quad (4)$$

It seems that convolved Fibonacci numbers first appeared in the classical Riordan's book [9]. These numbers have been studied in several papers; see [10–12]. In this paper, we obtain new identities for convolved k -Fibonacci numbers.

2. Some Properties of k -Fibonacci Numbers and k -Lucas Numbers

The characteristic equation associated with the recurrence relation (1) is $x^2 = kx + 1$. The roots of this equation are

$$r_1 = \frac{k + \sqrt{k^2 + 4}}{2}, \quad r_2 = \frac{k - \sqrt{k^2 + 4}}{2}. \quad (5)$$

Then we have the following basic identities:

$$\begin{aligned} r_1 + r_2 &= k, & r_1 - r_2 &= \sqrt{k^2 + 4}, \\ r_1 r_2 &= -1. \end{aligned} \quad (6)$$

TABLE 1: $F_{1,n}^{(r)}$, with $r = 1, 2, 3, 4, 5, 6, 7, 8, \dots$

n	$F_{1,n}^{(1)}$	$F_{1,n}^{(2)}$	$F_{1,n}^{(3)}$	$F_{1,n}^{(4)}$	$F_{1,n}^{(5)}$	$F_{1,n}^{(6)}$	$F_{1,n}^{(7)}$	$F_{1,n}^{(8)}$
0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8
3	2	5	9	14	20	27	35	44
4	3	10	22	40	65	98	140	192
5	5	20	51	105	190	315	490	726
6	8	38	111	256	511	924	1554	2472
7	13	71	233	594	1295	2534	4578	7776
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE 2: $F_{2,n}^{(r)}$, with $r = 1, 2, 3, 4, 5, 6, 7, 8, \dots$

n	$F_{2,n}^{(1)}$	$F_{2,n}^{(2)}$	$F_{2,n}^{(3)}$	$F_{2,n}^{(4)}$	$F_{2,n}^{(5)}$	$F_{2,n}^{(6)}$	$F_{2,n}^{(7)}$	$F_{2,n}^{(8)}$
0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1
2	2	4	6	8	10	12	14	16
3	5	14	27	44	65	90	119	152
4	12	44	104	200	340	532	784	1104
5	29	131	366	810	1555	2709	4396	6756
6	70	376	1212	3032	6482	12432	22008	36624
7	169	1052	3842	10716	25235	52808	101220	181224
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Some of the properties that the k -Fibonacci numbers verify are summarized below (see [2, 6] for the proofs).

- (i) Binet formula: $F_{k,n} = (r_1^n - r_2^n)/(r_1 - r_2)$.
- (ii) Combinatorial formula: $F_{k,n} = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-i}{i} k^{n-1-2i}$.
- (iii) Generating function: $f_k(x) = x/(1 - kx - x^2)$.

Definition 1. For any positive real number k , the k -Lucas sequence, say $\{L_{k,n}\}_{n \in \mathbb{N}}$, is defined recurrently by

$$\begin{aligned} L_{k,0} &= 2, & L_{k,1} &= k, \\ L_{k,n+1} &= kL_{k,n} + L_{k,n-1}. \end{aligned} \quad (7)$$

If $k = 1$ we have the classical Lucas numbers. Some properties that the k -Lucas numbers verify are summarized below (see [13] for the proofs).

- (i) Binet formula: $L_{k,n} = r_1^n + r_2^n$.
- (ii) Relation with k -Fibonacci numbers: $L_{k,n} = F_{k,n-1} + F_{k,n+1}$, $n \geq 1$.

3. Convolved k -Fibonacci Numbers

Definition 2. The convolved k -Fibonacci numbers $F_{k,j}^{(r)}$ are defined by

$$f_k^{(r)}(x) = (1 - kx - x^2)^{-r} = \sum_{j=0}^{\infty} F_{k,j+1}^{(r)} x^j, \quad r \in \mathbb{Z}^+. \quad (8)$$

Note that

$$F_{k,m+1}^{(r)} = \sum_{j_1+j_2+\dots+j_r=m} F_{k,j_1+1} F_{k,j_2+1} \cdots F_{k,j_r+1}. \quad (9)$$

Moreover, from Humbert polynomials (with $n = j$, $m = 2$, $x = k/2$, $y = -1$, $p = -r$, and $C = 1$), we have

$$F_{k,j+1}^{(r)} = \sum_{l=0}^{\lfloor j/2 \rfloor} \binom{j+r-l-1}{j-l} \binom{j-l}{l} k^{j-2l}. \quad (10)$$

If $r = 1$ we obtain the combinatorial formula of k -Fibonacci numbers. In Tables 1, 2, and 3 some values of convolved k -Fibonacci numbers are provided. The purpose of this paper is to investigate the properties of these numbers.

Theorem 3. The following identities hold:

- (1) $F_{k,2}^{(r)} = kr$,
- (2) $F_{k,n}^{(r)} = F_{k,n}^{(r-1)} + kF_{k,n-1}^{(r)} + F_{k,n-2}^{(r)}$, $n \geq 2$,
- (3) $nF_{k,n+1}^{(r)} = r(kF_{k,n}^{(r+1)} + 2F_{k,n-1}^{(r+1)})$, $n \geq 1$.

Proof. (1) Taking $j = 1$ in (10), we obtain

$$F_{k,2}^{(r)} = \binom{r}{1} \binom{1}{0} k = rk. \quad (11)$$

(2) This identity is obtained from observing that

$$\sum_{j=0}^{\infty} F_{k,j+1}^{(r)} x^j = (kx + x^2) \sum_{j=0}^{\infty} F_{k,j+1}^{(r)} x^j + \sum_{j=0}^{\infty} F_{k,j+1}^{(r-1)} x^j. \quad (12)$$

TABLE 3: $F_{3,n}^{(r)}$, with $r = 1, 2, 3, 4, 5, 6, 7, 8, \dots$

n	$F_{3,n}^{(1)}$	$F_{3,n}^{(2)}$	$F_{3,n}^{(3)}$	$F_{3,n}^{(4)}$	$F_{3,n}^{(5)}$	$F_{3,n}^{(6)}$	$F_{3,n}^{(7)}$	$F_{3,n}^{(8)}$
0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1
2	3	6	9	12	15	18	21	24
3	10	29	57	94	140	195	259	332
4	33	126	306	600	1035	1638	2436	3456
5	109	516	1491	3385	6630	11739	19306	30006
6	360	2034	6813	17568	38493	75348	135702	229176
7	1189	7807	29737	85826	207935	445718	872130	1589664
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

(3) Taking the first derivative of $f_k^{(r)}(x) = (1 - kx - x^2)^{-r}$, we obtain

$$\begin{aligned} (f_k^{(r)}(x))' &= \sum_{j=1}^{\infty} F_{k,j+1}^{(r)} j x^{j-1} \\ &= r \left(\frac{1}{1 - kx - x^2} \right)^{r-1} \left(\frac{k + 2x}{(1 - kx - x^2)^2} \right) \quad (13) \\ &= r(k + 2x) f_k^{(r+1)}(x). \end{aligned}$$

Therefore the identity is clear. $\square$

In the next theorem we show that the convolved k -Fibonacci numbers can be expressed in terms of k -Fibonacci and k -Lucas numbers. This theorem generalizes Theorem 4 of [11].

Theorem 4. Let $j \geq 0$ and $r \geq 1$. We have

$$\begin{aligned} F_{k,j+1}^{(r)} &= \sum_{\substack{l=0 \\ r+l \equiv 0 \pmod{2}}}^{r-1} \binom{r+l-1}{l} \binom{r-l+j-1}{j} \\ &\quad \times \frac{1}{(k^2 + 4)^{(r+l)/2}} L_{k,r+j-l}^{(r)} \\ &\quad + \sum_{\substack{l=0 \\ r+l \equiv 1 \pmod{2}}}^{r-1} \binom{r+l-1}{l} \binom{r-l+j-1}{j} \\ &\quad \times \frac{1}{(k^2 + 4)^{(r+l-1)/2}} F_{k,r+j-l}^{(r)}. \end{aligned} \quad (14)$$

Proof. Given $\alpha, \beta \in \mathbb{C}$, such that $\alpha\beta \neq 0$ and $\alpha \neq \beta$. Then we have the following partial fraction decomposition:

$$\begin{aligned} (1 - \alpha z)^{-r} (1 - \beta z)^{-r} &= \sum_{l=0}^{r-1} \binom{-r}{l} \frac{\alpha^r \beta^l}{(\alpha - \beta)^{r+l}} (1 - \alpha z)^{l-r} \\ &\quad + \sum_{l=0}^{r-1} \binom{-r}{l} \frac{\beta^r \alpha^l}{(\beta - \alpha)^{r+l}} (1 - \beta z)^{l-r}, \end{aligned} \quad (15)$$

where $\binom{t}{0} = 1$ and $\binom{t}{l} = (t(t-1)\cdots(t-l+1))/l!$ with $t \in \mathbb{R}$. Using the Taylor expansion

$$(1 - z)^t = \sum_{j=0}^{\infty} (-1)^j \binom{t}{j} z^j. \quad (16)$$

Then $(1 - \alpha z)^{-r} (1 - \beta z)^{-r} = \sum_{j=0}^{\infty} \gamma(j) z^j$, where

$$\begin{aligned} \gamma(j) &= \sum_{l=0}^{r-1} \binom{-r}{l} \frac{\alpha^r \beta^l}{(\alpha - \beta)^{r+l}} (-1)^j \binom{l-r}{j} \alpha^j \\ &\quad + \sum_{l=0}^{r-1} \binom{-r}{l} \frac{\beta^r \alpha^l}{(\beta - \alpha)^{r+l}} (-1)^j \binom{l-r}{j} \beta^j. \end{aligned} \quad (17)$$

Note that $1 - kz - z^2 = (1 - r_1 z)(1 - r_2 z)$. On substituting these values of $\alpha = r_1$ and $\beta = r_2$ and using the identities (6), we obtain

$$\begin{aligned} F_{k,j+1}^{(r)} &= \sum_{l=0}^{r-1} \binom{-r}{l} \frac{(-1)^l \alpha^{r-l}}{(k^2 + 4)^{(r+l)/2}} (-1)^j \binom{l-r}{j} \alpha^j \\ &\quad + \sum_{l=0}^{r-1} \binom{-r}{l} \frac{(-1)^l \beta^{r-l}}{(k^2 - 4)^{(r+l)/2}} (-1)^j \binom{l-r}{j} \beta^j \\ &= \sum_{l=0}^{r-1} (-1)^l \binom{-r}{l} (-1)^j \binom{l-r}{j} \\ &\quad \times \frac{1}{(k^2 + 4)^{(r+l)/2}} (\alpha^{r+j-l} + (-1)^{r+l} \beta^{r+j-l}). \end{aligned} \quad (18)$$

Since that $(-1)^l \binom{-r}{l} = \binom{r+l-1}{l}$ and $(-1)^j \binom{l-r}{j} = \binom{r-l+j-1}{j}$, then

$$\begin{aligned} F_{k,j+1}^{(r)} &= \binom{r+l-1}{l} \binom{r-l+j-1}{j} \\ &\quad \times \frac{1}{(k^2 + 4)^{(r+l)/2}} (\alpha^{r+j-l} + (-1)^{r+l} \beta^{r+j-l}). \end{aligned} \quad (19)$$

From the above equality and Binet formula, we obtain (14). $\square$

4. Hessenberg Matrices and Convolved k -Fibonacci Numbers

An upper Hessenberg matrix, A_n , is an $n \times n$ matrix, where $a_{i,j} = 0$ whenever $i > j + 1$ and $a_{j+1,j} \neq 0$ for some j . That is, all entries below the superdiagonal are 0 but the matrix is not upper triangular:

$$A_n = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n-1} & a_{2,n} \\ 0 & a_{3,2} & a_{3,3} & \cdots & a_{3,n-1} & a_{3,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & a_{n,n-1} & a_{n,n} \end{pmatrix}. \quad (20)$$

We consider a type of upper Hessenberg matrix whose determinants are k -Fibonacci numbers. Some results about Fibonacci numbers and Hessenberg can be found in [14]. The following known result about upper Hessenberg matrices will be used.

Theorem 5. Let $a_1, p_{i,j}$ ($i \leq j$) be arbitrary elements of a commutative ring R , and let the sequence $a_1, a_2, \dots$ be defined by

$$a_{n+1} = \sum_{i=1}^n p_{i,n} a_i, \quad (n = 1, 2, \dots). \quad (21)$$

If

$$A_n = \begin{pmatrix} p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,n-1} & p_{1,n} \\ -1 & p_{2,2} & p_{2,3} & \cdots & p_{2,n-1} & p_{2,n} \\ 0 & -1 & p_{3,3} & \cdots & p_{3,n-1} & p_{3,n} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & -1 & a_{n,n} \end{pmatrix}, \quad (22)$$

then

$$a_{n+1} = a_1 \det A_n. \quad (23)$$

In particular, if

$$F_n^{(k)} = \begin{pmatrix} k & 1 & 0 & \cdots & 0 & 0 \\ -1 & k & 1 & \cdots & 0 & 0 \\ 0 & -1 & k & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & k & 1 \\ 0 & 0 & 0 & \cdots & -1 & k \end{pmatrix}, \quad (24)$$

then from Theorem 5 we have that

$$\det F_n^{(k)} = F_{k,n+1}, \quad (n = 1, 2, \dots). \quad (25)$$

It is clear that the principal minor $M^{(k)}(i)$ of $F_n^{(k)}$ is equal to $F_{k,i} F_{k,n-i+1}$. It follows that the principal minor $M^{(k)}(i_1, i_2, \dots, i_l)$ of the matrix $F_n^{(k)}$ is obtained by deleting rows and columns with indices $1 \leq i_1 < i_2 < \cdots < i_l \leq n$:

$$M^{(k)}(i_1, i_2, \dots, i_l) = F_{k,i_1} F_{k,i_2-i_1} \cdots F_{k,i_l-i_{l-1}} F_{k,n-i_l+1}. \quad (26)$$

Then we have the following theorem.

Theorem 6. Let $S_{n-l}^{(k)}$, ($l = 0, 1, 2, \dots, n-1$) be the sum of all principal minors of $F_n^{(k)}$ of order $n-l$. Then

$$\begin{aligned} S_{n-l}^{(k)} &= \sum_{j_1+j_2+\cdots+j_{l+1}=n-l} F_{k,j_1+1} F_{k,j_2+1} \cdots F_{k,j_{l+1}+1} \\ &= F_{k,n-l+1}^{(l+1)}. \end{aligned} \quad (27)$$

Since the coefficients of the characteristic polynomial of a matrix are, up to the sign, sums of principal minors of the matrix, then we have the following.

Corollary 7. The convolved k -Fibonacci number $F_{k,n-l+1}^{(l+1)}$ is equal, up to the sign, to the coefficient of t^l in the characteristic polynomial $p_n(t)$ of $F_n^{(k)}$.

Corollary 8. The following identity holds:

$$F_{k,n-l+1}^{(l+1)} = \sum_{i=0}^{\lfloor (n-l)/2 \rfloor} \binom{n-i}{i} \binom{n-2i}{l} k^{n-2i-l}. \quad (28)$$

Proof. The characteristic matrix of $F_n^{(k)}$ has the form

$$\begin{pmatrix} t-k & 1 & 0 & \cdots & 0 & 0 \\ -1 & t-k & 1 & \cdots & 0 & 0 \\ 0 & -1 & t-k & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & t-k & 1 \\ 0 & 0 & 0 & \cdots & -1 & t-k \end{pmatrix}. \quad (29)$$

Then $p_n(t) = f_{n+1}(t-k)$, where $f_{n+1}(t)$ is a Fibonacci polynomial. Then from Corollary 8 and the following identity for Fibonacci polynomial [5]:

$$f_{n+1}(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} x^{n-2i}, \quad (30)$$

we obtain that

$$\begin{aligned} f_{n+1}(t-k) &= \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} \sum_{l=0}^{n-2i} \binom{n-2i}{l} \\ &\quad \times (-1)^{n-l} k^{n-2i-l} t^l. \end{aligned} \quad (31)$$

Therefore the corollary is obtained. $\square$

Acknowledgments

The author would like to thank the anonymous referees for their helpful comments. The author was partially supported by Universidad Sergio Arboleda under Grant no. USA-II-2012-14.

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