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Research Article

A New Criterion for Affineness

Jing Zhang

Department of Mathematics and Statistics, University at Albany, SUNY, Albany, NY 12222, USA

Correspondence should be addressed to Jing Zhang; jzhang@albany.edu

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We show that an irreducible quasiprojective variety Y of dimension $d \geq 1$ defined over an algebraically closed field with characteristic zero is an affine variety if and only if $H^i(Y, \mathcal{O}_Y) = 0$ and $H^i(Y, \mathcal{O}_Y(-Z)) = 0$ for all $i > 0$, $Z = H \cap Y$, where H is any hypersurface with sufficiently large degree. A direct application is that an irreducible quasiprojective variety Y over $\mathbb{C}$ is a Stein variety if it satisfies the two vanishing conditions. Here, all sheaves are algebraic.

1. Introduction

We work over an algebraically closed field with characteristic zero.

Affine varieties are important in algebraic geometry. J.-P. Serre introduced sheaf and cohomology techniques to algebraic geometry and discovered his well-known cohomology criterion ([1], [2, Chapter 2, Theorem 1.1]): a variety (or a Noetherian scheme) Y is an affine variety if and only if for all coherent sheaves F on Y and all $i > 0$, $H^i(Y, F) = 0$. Goodman and Hartshorne proved that Y is an affine variety if and only if Y contains no complete curves and the dimension $h^1(Y, F)$ of the linear space $H^1(Y, F)$ is bounded for all coherent sheaves F on Y [3]. Let X be the completion of Y. In 1969, Goodman also proved that Y is affine if and only if after suitable blowing up of the closed subvariety on the boundary $X - Y$, the new boundary $X' - Y$ is a support of an ample divisor, where $X' \to X$ is the blowing up with center in $X - Y$ ([4], [2, Chapter 2, Theorem 6.1]). For any quasiprojective variety Y, we may assume that the boundary $X - Y$ is the support of an effective divisor D with simple normal crossings by blowing up the closed subvariety in $X - Y$. Y is affine if D is ample. So, if we can show the ampleness of D, then Y is affine. There are two important criteria for ampleness according to Nakai-Moishezon and Kleiman ([5], [6, Chapter 1, Section 1.5]). Another sufficient condition is that if Y contains no complete curves and the linear system $|nD|$ is base point free, then Y is affine [2, Chapter 2, Page 64]. Therefore, we can apply base point free theorem if we know the numerical condition of D [6, Chapter 3, Page 75, Theorem 3.3]. Neeman proved that if Y is a quasicompact Zariski open subset of an affine scheme Spec$A$, then Y is affine if and only if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$ [7]. The significance of Neeman's theorem is that it is not assumed that the ring A is Noetherian.

In [8], we show that if a quasiprojective variety Y is Stein, $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, and Y has $d = \dim X$ algebraically independent nonconstant regular functions, then Y is an affine variety. In this note, we give a new criterion for affineness.

Theorem 1. *An irreducible quasiprojective variety Y of dimension $d \geq 1$ is an affine variety if and only if for all $i > 0$, $H^i(Y, \mathcal{O}_Y) = 0$, and $H^i(Y, \mathcal{O}_Y(-Z)) = 0$, where H is any hypersurface with sufficiently large degree and $Z = H \cap Y$.*

By Cartan's Theorem B, an analytic variety Y is a Stein variety if and only if for all analytic coherent sheaf F on Y, $H^i(Y, F) = 0$ for all $i > 0$. Since an algebraic affine variety over $\mathbb{C}$ is a Stein variety [2, Page 232], we have the following application.

Corollary 2. *An irreducible quasiprojective variety Y of dimension $d \geq 1$ over $\mathbb{C}$ is a Stein variety if for all $i > 0$, $H^i(Y, \mathcal{O}_Y) = 0$, and $H^i(Y, \mathcal{O}_Y(-Z)) = 0$, where H is any hypersurface with sufficiently large degree, $Z = H \cap Y$, and all sheaves are algebraic.*

Notice that in Theorem 1, the two vanishing conditions imply that any hypersurface section of Y is an affine variety. On the other hand, if every hypersurface section of Y is affine, Y may not be affine (Example 12).

If Y is an affine variety, then the ring $\Gamma(Y, \mathcal{O}_Y)$ is finitely generated [9, Page 20]. However, in our proof of Theorem 1, we do not directly check the finitely generated property of this ring, which is very hard in general. And the question of a quasiprojective variety Y to be affine is different from the behavior of the boundary divisor D, in particular, the numerical condition of D like nefness and finitely generated property of the graded ring

$$\bigoplus_{n=0}^{\infty} H^0\left(X, \mathcal{O}_X(nD)\right). \tag{1}$$

The reason is that

$$\Gamma\left(Y, \mathcal{O}_Y\right) \neq \bigoplus_{n=0}^{\infty} H^0\left(X, \mathcal{O}_X(nD)\right). \tag{2}$$

We will give two examples to demonstrate this difference in Section 3. One example (due to Zariski) is an affine surface $Y = X - D$ such that the the corresponding graded ring $\bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nD))$ is not finitely generated for an effective divisor D. The other example [2, Page 232] is a surface $Y = X - D$ such that

$$H^0\left(Y, \mathcal{O}_Y\right) = H^0\left(X, \mathcal{O}_X(nD)\right) = \mathbb{C}. \tag{3}$$

A necessary condition for the affineness of Y with dimension d is that Y has plenty of nonconstant regular functions. More precisely, Y has $d = \dim Y$ algebraically independent nonconstant regular functions. This means that the corresponding effective boundary divisor D must be big, that is,

$$h^0\left(X, \mathcal{O}_X(nD)\right) \geq an^d \tag{4}$$

for some positive number a and $n \gg 0$. So, this surface Y is not affine but $\bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nD))$ is finitely generated.

We will prove Theorem 1 in Section 2 and give examples in Section 3.

2. Proof of the Theorem

Recall our notation: Y is an open subset of a projective variety X with dimension $d \geq 1$ and D is the effective boundary divisor with support $X - Y$. We may assume that D has simple normal crossings by further blowing up suitable closed subvariety of $X - Y$.

In the following lemmas, Y is irreducible and satisfies $H^i(Y, \mathcal{O}_Y) = 0$ and $H^i(Y, \mathcal{O}_Y(-Z)) = 0$ for all $i > 0$ and $Z = H \cap Y$, where H is any hypersurface with sufficiently large degree in the ambient projective space containing X.

Lemma 3. *Let H be a hypersurface in Theorem 1, $Z = H \cap Y$, then Z satisfies the same vanishing conditions: $H^i(Z, \mathcal{O}_Z) = 0$ and $H^i(Z, \mathcal{O}_Z(-Z')) = 0$ for all $i > 0$, where Z' is any hypersurface section on Y with sufficiently large degree.*

Proof. There is a short exact sequence,

$$0 \longrightarrow \mathcal{O}_Y(-Z) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_Z \longrightarrow 0, \tag{5}$$

where Z is considered as a Cartier divisor on Y. By the assumption, we have $H^i(Y, \mathcal{O}_Y) = 0$ and $H^i(Y, \mathcal{O}_Y(-Z)) = 0$ for all $i > 0$. The corresponding long exact sequence gives $H^i(Z, \mathcal{O}_Z) = 0$ for all $i > 0$. Similarly, for any hypersurfaces H and H', $Z' = H' \cap Y$, from the short exact sequence

$$0 \longrightarrow \mathcal{O}_Y\left(-Z - Z'\right) \longrightarrow \mathcal{O}_Y\left(-Z'\right) \longrightarrow \mathcal{O}_Z\left(-Z'\right) \longrightarrow 0, \tag{6}$$

we have $H^i(Z, \mathcal{O}_Z(-Z')) = 0$ for all $i > 0$. □

Lemma 4. *If Z is a curve with $H^1(Z, \mathcal{O}_Z(-Z')) = 0$ for all hypersurface sections $Z' = H' \cap Z$ with sufficiently large degree, then Z is an affine curve.*

Proof. First, we assume that Z is irreducible.

If Z is complete, then by Riemann-Roch for singular curves [9, Page 298], we have

$$h^0\left(Z, \mathcal{O}_Z\left(-Z'\right)\right) - h^1\left(Z, \mathcal{O}_Z\left(-Z'\right)\right) = \deg D + 1 - p_a, \tag{7}$$

where p_a is the arithmetic genus of Z and D is the divisor with support in the set Z_{reg} of smooth points of Z given by $-Z'$. Choose the hypersurface H' with sufficiently large degree such that

$$\deg D + 1 - g < 0, \tag{8}$$

then $h^1(Z, \mathcal{O}_Z(-Z')) > 0$. This is a contradiction. Therefore, Z is not a complete but an affine curve [2, Page 62].

If Z is not irreducible, then Z is still an affine curve. Assume that Z has two irreducible components Z_1 and Z_2, $Z_1 \neq Z_2$. Then, the dimension of $Z_1 \cap Z_2$ is at most 0. So,

$$H^1\left(Z_1 \cap Z_2, \mathcal{O}_Z\left(-Z'\right)\right) = 0. \tag{9}$$

By Mayer-Vietoris sequence, for the ideal sheaf $\mathcal{O}_Z(-Z')$, we have

$$\begin{aligned} &H^1\left(Z, \mathcal{O}_Z\left(-Z'\right)\right) \longrightarrow H^1\left(Z_1, \mathcal{O}_Z\left(-Z'\right)\right) \\ &\quad \bigoplus H^1\left(Z_2, \mathcal{O}_Z\left(-Z'\right)\right) \longrightarrow H^1\left(Z_1 \cap Z_2, \mathcal{O}_Z\left(-Z'\right)\right), \end{aligned} \tag{10}$$

where the first and last terms vanish. This gives

$$H^1\left(Z_1, \mathcal{O}_Z\left(-Z'\right)\right) = H^1\left(Z_2, \mathcal{O}_Z\left(-Z'\right)\right) = 0. \tag{11}$$

Now Z_1 and Z_2 are affine curves. If Z has more than 2 irreducible components, then by using mathematical induction, every irreducible component of Z is an affine curve. Thus Z is an affine curve. □

By Lemmas 3 and 4 and mathematical induction, we may assume that every hypersurface section of Y is an affine variety.

Lemma 5. *If Z is an irreducible surface with $H^i(Z, \mathcal{O}_Z) = 0$ and for all $i > 0$ $H^i(Z, \mathcal{O}_Z(-Z')) = 0$ for all hypersurface sections Z' with sufficiently large degree, then Z is an affine surface.*

Proof. By Lemma 3, any hypersurface section A with sufficiently large degree on Z satisfies the same vanishing condition. So, A is an affine curve by Lemma 4. Since A is closed in Z, Z is not complete.

Let C be an irreducible curve on Z, then we may choose A such that A is irreducible [10] and $C \cap A$ contains more than two points [10]. Let P_1 and P_2 be two distinct points on $C \cap A$, then there is a regular function ϕ on A such that $\phi(P_1) \neq \phi(P_2)$ since A is an affine curve. From the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(-A) \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{O}_A \longrightarrow 0, \tag{12}$$

and $H^1(Z, \mathcal{O}_Z(-A)) = 0$, we have a surjective map from $H^0(Z, \mathcal{O}_Z)$ to $H^0(A, \mathcal{O}_A)$. Lift ϕ from A to Z, we have a regular function on Z such that it is not a constant on C. By Goodman and Hartshorne's theorem [3], Z is a quasiaffine variety. By Neeman's theorem [7], Z is an affine surface. □

Let X be an irreducible normal complete variety and let D be a Cartier divisor on X. If $H^0(X, \mathcal{O}_X(nD)) = 0$ for all $n > 0$, then we define the D-dimension $\kappa(D, X)$ to be $-\infty$. Otherwise, we define

$$\kappa(D, X) = \operatorname{tr} \cdot \deg_k \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)) - 1. \tag{13}$$

If X is not normal, we define $\kappa(D, X) = \kappa(\pi^* D, X')$, where $\pi : X' \to X$ is the normalization. From the definition, we see that if D is an effective divisor, then $0 \leq \kappa(D, X) \leq d$, where d is the dimension of X. An effective divisor D is defined to be big if $\kappa(D, X) = d$.

Lemma 6. *Let $d = \dim X > 2$ and H be a hypersurface such that $\overline{Z} = H \cap X$ is irreducible, then D is a big divisor on $\overline{Z}$.*

Proof. Let $Z = H \cap Y$ be the open irreducible hypersurface section, then it satisfies the same condition in Theorem 1 by the above lemmas. We may assume that Z is an affine variety by Lemmas 4 and 5 and inductive assumption. So, the closure $\overline{Z}$ in X has $d - 1$ algebraically independent nonconstant rational functions which are regular on Z. This implies that D is a big divisor on $\overline{Z} = X \cap H$, that is, $\kappa(D|_{\overline{Z}}, \overline{Z}) = d - 1$. □

Lemma 7. *Y has no complete curves.*

Proof. If Y has an irreducible complete curve C, choose a hypersurface H such that $Z = H \cap Y$ is irreducible [10] and H intersects C at more than 2 distinct points. Let $P_1, P_2 \in C \cap H$, $P_1 \neq P_2$. By Lemmas 4 and 5, we may assume that any irreducible hypersurface section of dimension $d - 1$ with sufficiently large degree is an affine variety. By inductive assumption, Z is an affine variety. So, there is a regular function $\phi \in H^0(Z, \mathcal{O}_Z)$ such that $\phi(P_1) \neq \phi(P_2)$. Since

$$\begin{aligned} 0 &\longrightarrow H^0(Y, \mathcal{O}_Y(-Z)) \longrightarrow H^0(Y, \mathcal{O}_Y) \\ &\longrightarrow H^0(Z, \mathcal{O}_Z) \longrightarrow 0, \end{aligned} \tag{14}$$

we can lift ϕ to Y. So there is a regular function f on Y such that $f|_C$ is not a constant. This is not possible since C is complete. The contradiction shows that Y has no complete curves. □

By Lemma 4 and inductive assumption, if Z is not irreducible, the proof still works since Z is affine.

Lemma 8. *For any irreducible curve C on Y, there is a regular function f on Y such that f is not a constant on C.*

Proof. Let $\overline{C}$ be an irreducible complete curve in X containing C. Then, $\overline{C} - C$ is a finite set and a general hypersurface does not contain any point in $\overline{C} - C$. Let H be a hypersurface away from $\overline{C} - C$ such that H intersects C at more than two distinct points and $Z = H \cap Y$ is irreducible [10]. Then, Z is an affine variety and $H \cap \overline{C} = Z \cap C$. Let P_1 and P_2 be two distinct points in $Z \cap C$, then there is a regular function ϕ on Z such that $\phi(P_1) \neq \phi(P_2)$. Lift ϕ to Y, we find a regular function f on Y such that $f|_Z = \phi$ and $f|_C$ is not a constant. □

Theorem 9. *An irreducible quasiprojective variety Y is an affine variety if and only if $H^i(Y, \mathcal{O}_Y) = 0$ and $H^i(Y, \mathcal{O}_Y(-Z)) = 0$ for all $i > 0$, $Z = H \cap Y$, where H is any hypersurface with sufficiently large degree.*

Proof. By Serre's criterion, one direction is trivial: if Y is affine, then it satisfies $H^i(Y, \mathcal{O}_Y) = 0$ and $H^i(Y, \mathcal{O}_Y(-Z)) = 0$ for all $i > 0$ and $Z = H \cap Y$, where H is any hypersurface.

We assume that Y satisfies $H^i(Y, \mathcal{O}_Y) = 0$ and $H^i(Y, \mathcal{O}_Y(-Z)) = 0$ for all $i > 0$.

By Lemma 8 and Goodman and Hartshorne's theorem [3], Y is a quasiaffine variety since for every irreducible curve C on Y, there is a global regular function f on Y such that $f|_Y$ is not a constant. By Neeman's result [7], a quasiaffine variety Y is affine if and only if $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$. So, Y is an affine variety. □

If Y is not irreducible, then Theorem 9 still holds since the proof works by Lemma 4 and mathematical induction.

Corollary 10. *An irreducible quasiprojective variety Y of dimension $d \geq 1$ over $\mathbb{C}$ is a Stein variety if for all $i > 0$, $Z = H \cap Y$, $H^i(Y, \mathcal{O}_Y) = 0$, and $H^i(Y, \mathcal{O}_Y(-Z)) = 0$, where H is any hypersurface with sufficiently large degree, $Z = H \cap Y$ and all sheaves are algebraic.*

3. Examples

Again Y is an irreducible open variety contained in a projective variety X such that $Y = X - D$, where D is an effective boundary divisor with support $X - Y$. In this section, we assume that the ground field is $\mathbb{C}$.

Example 11. There is an affine surface Y such that the graded ring

$$\bigoplus_{n=0}^{\infty} H^0\left(X, \mathcal{O}_X(nD)\right) \tag{15}$$

is not finitely generated. This example is according to Zariski [11, Pages 562–564].

Let C be a smooth curve of degree 3 in $\mathbb{P}^2$. Let Λ be a divisor class cut out on C by a curve of degree 4 in $\mathbb{P}^2$. There exist 12 distinct points $p_1, p_2, \ldots, p_{12}$ on C such that

$$m\left(p_1 + p_2 + \cdots + p_{12}\right) \notin m\Lambda \tag{16}$$

for all positive integers m. Let X be a surface obtained by blowing up $\mathbb{P}^2$ at these 12 points $p_1, p_2, \ldots, p_{12}$. Let $\overline{C}$ be the strict transformation of C (i.e., the closure of the inverse image of $C - \{p_1, p_2, \ldots, p_{12}\}$ in X). Let L be a line not passing through any point p_i in these 12 points. Let $\overline{L}$ be the strict transform of L. Then, the complete linear system

$$\left|m\left(\overline{C} + \overline{L}\right)\right| \tag{17}$$

has a fixed locus $\overline{C}$ for all $m \geq 1$ and

$$\left|m\overline{C} + (m-1)\overline{L}\right| \tag{18}$$

has no fixed components and is base point free. By Nakai-Moishenzon's ampleness criterion [9, Chapter V, Section 1], the divisor

$$m\overline{C} + (m-1)\overline{L} \tag{19}$$

is ample. Hence, the complement $Y = X - (m\overline{C} + (m-1)\overline{L})$ is affine but the graded ring

$$R = \bigoplus_{m=0}^{\infty} H^0\left(X, \mathcal{O}_X\left(m\left(\overline{C} + \overline{L}\right)\right)\right) \tag{20}$$

is not finitely generated.

Example 12. A nonaffine surface Y such that the graded ring

$$\bigoplus_{n=0}^{\infty} H^0\left(X, \mathcal{O}_X(nD)\right) \tag{21}$$

is finitely generated.

Let C be an elliptic curve and E the unique nonsplit extension of $\mathcal{O}_C$ by itself. Let $X = \mathbb{P}_C(E)$ and D be the canonical section, then $Y = X - D$ is not affine and $H^0(X, \mathcal{O}_X(nD)) = \mathbb{C}$ [2, Page 232]. So,

$$\bigoplus_{n=0}^{\infty} H^0\left(X, \mathcal{O}_X(nD)\right) \tag{22}$$

is finitely generated.

Since the surface Y has no complete curves, any hypersurface section of Y is an affine curve. But Y is not affine since $H^0(Y, \mathcal{O}_Y) = \mathbb{C}$ [2, Page 232]. It shows that if every hypersurface section of Y is affine, Y may not be affine.

The above two examples demonstrate that the affineness of Y and the finitely generated property of the graded ring

$$\bigoplus_{n=0}^{\infty} H^0\left(X, \mathcal{O}_X(nD)\right) \tag{23}$$

are different in nature. The reason is that

$$\Gamma\left(Y, \mathcal{O}_Y\right) \neq \bigoplus_{n=0}^{\infty} H^0\left(X, \mathcal{O}_X(nD)\right). \tag{24}$$

In fact, we have the following.

Lemma 13 (see [3]). *Let V be a scheme and D be an effective Cartier divisor on V. Let $U = V - \mathrm{Supp}D$ and F be any coherent sheaf on V, then, for every $i \geq 0$,*

$$\lim_{\vec{n}} H^i\left(V, F \otimes \mathcal{O}(nD)\right) \cong H^i\left(U, F|_U\right). \tag{25}$$

So, we have

$$\Gamma\left(Y, \mathcal{O}_Y\right) \cong \lim_{\vec{n}} H^0\left(X, \mathcal{O}_X(nD)\right). \tag{26}$$

The direct limit is the quotient of the direct sum and its subring, so it is much "smaller" than direct sum [12, Chapter II, Section 10]. And even though Y is affine, the boundary divisor can be very bad. For example, D may not be nef. It is easy to see this by blowing up $\mathbb{P}^2$ at a point. Let L be a line in $\mathbb{P}^2$, let O be a point on L. Let $\pi : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at O. Let E be the exceptional divisor and $D = \pi^{-1}(L) + mE$, where $\pi^{-1}(L)$ is the strict transform of L and m is a large positive integer. Then, $D \cdot E = 1 - m < 0$ [2, Chapter V, Corollary 3.7]. Therefore, D is not nef.

Example 14. If a smooth threefold Y such that Y contains no complete curves, then $H^i(Y, \Omega_Y^j) = 0$ for all $i > 0$ and $j \geq 0$ but is not affine.

Let C_t be a smooth projective elliptic curve defined by $y^2 = x(x-1)(x-t)$, $t \neq 0, 1$. Let Z be the elliptic surface defined by the same equation, then we have surjective morphism from Z to $C = \mathbb{C} - \{0, 1\}$ such that for every $t \in C$, the fiber $f^{-1}(t) = C_t$. In [13], we proved that there is a rank 2 vector bundle E on Z such that when restricted to C_t, $E|_{C_t} = E_t$ is the unique nonsplit extension of $\mathcal{O}_{C_t}$ by $\mathcal{O}_{C_t}$, where f is the morphism from Z to C. We also proved that there is a divisor D on $X = \mathbb{P}_Z(E)$ such that when restricted to $X_t = \mathbb{P}_{C_t}(E_t)$, $D|_{X_t} = D_t$ is the canonical section of X_t. Let $Y = X - D$, we have $H^i(Y, \Omega_Y^j) = 0$ for all $i > 0$ and $j \geq 0$. We know that this threefold Y contains no complete curves [13] and $\kappa(D, X) = 1$. So, Y is not affine.

Example 15. A surface Y without complete curves such that $\kappa(D, X) = 2$ but is not affine.

Remove a line L from $\mathbb{P}^2$, then we have $\mathbb{C}^2 = \mathbb{P}^2 - L$. Remove the origin O from $\mathbb{C}^2$, let $Y = \mathbb{C}^2 - \{O\}$. Then, Y is

not affine since the boundary is not connected [2, Chapter II, Section 3 and Section 6]. Blow up $\mathbb{P}^2$ with center O, let E be the exceptional divisor and $\pi : X \to \mathbb{P}^2$ be the blowup. Let $D = \pi^{-1}(L) + E$, where $\pi^{-1}(L)$ is the strict transformation of L. Then by Iitaka's result, on X, $\kappa(D, X) = 2$ and $X - D \cong Y$ has no complete curves, but Y is not affine.

Lemma 16 (see [6]). *Let $\pi : X \to Z$ be a surjective map between projective varieties, X smooth, Z normal. Let F be the geometric generic fiber of π and assume that F is connected. The following two statements are equivalent:*

(i) *$R^i\pi_*\mathcal{O}_X = 0$ for all $i > 0$;*

(ii) *Z has rational singularities and $H^i(F, \mathcal{O}_F) = 0$ for all $i > 0$.*

Example 17. A smooth variety Y of dimension $d \geq 1$ with $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$ and $\kappa(D, X) = d$ but is not affine.

Let X be the smooth projective variety obtained by blowing up a point O in $\mathbb{P}^d$. Let $\pi : X \to \mathbb{P}^d$ be the blowup. Let H be a hyperplane not passing through O. Let $D = \pi^*(H)$ and $Y = X - D$. Then, $\kappa(D, X) = d$ [14, Chapter 2, Theorem 5.13]. Let E be the exceptional divisor on X, then $E \cong \mathbb{P}^{d-1}$. So, $H^i(E, \mathcal{O}_E) = 0$ for all $i > 0$. By Lemma 16, we have $R^i\pi_*\mathcal{O}_X = 0$ for all $i > 0$.

Let $U = \mathbb{P}^d - H$, then $U \cong \mathbb{C}^d$. For the global sections on affine space U, we have [9, Chapter III, Proposition 8.1, 8.5 and Chapter II, Proposition 5.1(d)]

$$0 = R^i\pi_*\mathcal{O}_X(U) = H^i(Y, \mathcal{O}_Y), \tag{27}$$

for all $i > 0$.

It is obvious that Y is not affine since it contains a projective space $\mathbb{P}^{d-1}$.

Example 18. A threefold Y satisfies the following three conditions but is not affine:

(1) Y contains no complete curves;

(2) the boundary $X - Y$ is connected;

(3) $\kappa(D, X) = 3$.

Let H be a hyperplane in $\mathbb{P}^3$. Let L be a line not contained in H. Blow up $\mathbb{P}^3$ along L, let $\pi : X \to \mathbb{P}^3$ be the blowup. Define a divisor D on X such that $D = \pi^{-1}H + E$, where E is the exceptional divisor on X. Let $Y = X - D$, then $Y \cong \mathbb{P}^3 - H - L$.

It is easy to see that the above three conditions are satisfied. Y is an open subset of $\mathbb{C}^3$ and $\mathbb{C}^3 - Y$ is a line, which is not of codimension 1. So, Y is not affine [2, 7].

Example 19. A quasiprojective variety Y with a surjective morphism $f : Y \to U$ such that U is affine, a general fiber is affine and $H^i(Y, \mathcal{O}_Y) = 0$ for all $i > 0$, but Y is not affine.

Let Y, U be the varieties defined in Example 17. Then the fiber space $\pi : Y \to U$ satisfied the above requirements. Y is not affine because it has a projective space $\mathbb{P}^{d-1}$.

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