Research Article

Some New Results on Distance $k$-Domination in Graphs

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We determine the distance $k$-domination number for the total graph, shadow graph, and middle graph of path $P_n$.

1. Introduction

We begin with finite, connected, and undirected graphs, $G = (V(G), E(G))$ without loops or multiple edges. A dominating set $D$ of a graph $G$ is a set of vertices of $G$ such that every vertex of $V(G) - D$ is adjacent to some vertex of $D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. Further, the open neighbourhood of $v \in V(G)$ is the set $N(v) = \{u \in V(G)/uv \in E(G)\}$. The closed neighbourhood $N[v]$ of $v \in V(G)$ is the set $N[v] = N(v) \cup \{v\}$. The distance $d(u, v)$ between two vertices $u$ and $v$ is the length of shortest path between $u$ and $v$ in $G$, if exists otherwise, $d(u, v) = \infty$. The open $k$-neighbourhood set $N_k(v)$ of vertex $v \in V(G)$ is the set of all vertices of $G$ which are different from $v$ and at distance at most $k$ from $v$ in $G$, that is, $N_k(v) = \{u \in V(G)/d(u, v) \leq k\}$. The closed $k$-neighbourhood set $N_k[v]$ of $v$ is defined as $N_k[v] = N_k(v) \cup \{v\}$. Obviously, $N[1] = N_1(v)$.

The total graph $T(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in $G$.

The Shadow graph $D_2(G)$ of a connected graph $G$ is obtained by taking two copies of $G$, say $G'$ and $G''$. Join each vertex $u'$ in $G'$ to the neighbours of corresponding vertex $u''$ in $G''$.

The middle graph $M(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent whenever either they are adjacent edges of $G$ or one is a vertex of $G$ and the other is an edge incident with it.

For standard terminology and notations we rely upon Balakrishnan and Ranganathan [1] and Haynes et al. [2].

The concept of distance dominating set was initiated by Slater [3] while the term distance $k$-dominating set was coined by Henning et al. [4]. For an integer $k \geq 1$, a $D \subseteq V(G)$ is a $k$-dominating set of $G$ if every vertex in $V(G) - D$ is within distance $k$ from some vertex $v \in D$. That is, $N_k[D] = V(G)$. The minimum cardinality among all $k$-dominating sets of $G$ is called the $k$-domination number of $G$ and it is denoted by $\gamma_k(G)$. It is obvious that $\gamma(G) = \gamma_1(G)$. A $k$-dominating set of cardinality $\gamma_k(G)$ is called a $\gamma_k$-set. The distance domination in the context of spanning tree is discussed by Griggs and Hutchinson [5] while bounds on the distance two-domination number and the classes of graphs attaining these bounds are reported in the work of Sridharan et al. [6]. In [7] Topp and Volkmann have discussed distance $k$-domination as $k$-covering and characterized connected graphs of order $(k + 1)n$ with distance $k$-domination ($k$-covering). Application of distance domination in Ad Hoc wireless networking is briefly discussed by Wu and Li [8]. More details and bibliographic references on distance $k$-domination can be found in a survey paper by Henning [9].

2. Some Definitions and Main Results

Proposition 1 (see [9]). For $k \geq 1$, let $D$ be a $k$-dominating set of a graph $G$. Then $D$ is a minimal $k$-dominating set of $G$ if and only if each $d \in D$ has at least one of the following properties.

(1) There exists a vertex $v \in V(G) - D$ such that $N_k(v) \cap D = \{d\}$.

(2) The vertex $d$ is at distance at least $k + 1$ from every other vertex $d' \in D$ in $G$. 
Theorem 2. For \( n \leq 2k \), \( \gamma_k(T(P_n)) = 1 \).

Proof. Let \( v_1, v_2, \ldots, v_n \) and \( e_1, e_2, \ldots, e_{n-1} \) be the vertices and the edges of \( P_n \), respectively. Then \( v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1} \) will be the vertices of \( T(P_n) \). Then \( D = \{v_{n/2}\} \) is distance \( k \)-dominating set of \( T(P_n) \) as \( n \leq 2k \). The set \( D \) being a singleton set it is obviously a minimal distance \( k \)-dominating set of \( T(P_n) \).

Theorem 3. For \( n > 2k \),

\[
\gamma_k(T(P_n)) = \begin{cases} \left\lfloor \frac{n}{2k} \right\rfloor & \text{for } n \equiv 0,1 \pmod{2k} \\ \left\lfloor \frac{n}{2k} \right\rfloor + 1 & \text{for } n \equiv 2,3,\ldots,2k-1 \pmod{2k}. \end{cases}
\]

Proof. Let \( v_1, v_2, \ldots, v_n \) and \( e_1, e_2, \ldots, e_{n-1} \) be the vertices and the edges of \( P_n \), respectively. Then \( v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1} \) will be the vertices of \( T(P_n) \). Now every vertex from \( v_{k+1}, v_{k+2}, \ldots, v_{n-k} \) dominates \( 2k+1 \) vertices of \( v_i \) and \( 2k \) vertices of \( e_j \) at a distance \( k \) while every vertex from \( e_{k+1}, e_{k+2}, \ldots, e_{n-k} \) dominates \( 2k \) vertices of \( v_i \) and \( 2k+1 \) vertices of \( e_j \) at a distance \( k \). Therefore at least one vertex from \( \{v_{k+1+(2k)}j, v_{k+2+(2k)}j, \ldots, v_{n-k+(2k)}j \} \) must belongs to any distance \( k \)-dominating set \( D \) of \( T(P_n) \).

Hence,

\[ \gamma_k(T(P_n)) \geq \left\lfloor \frac{n}{2k} \right\rfloor. \] (2)

Now depending upon the number of vertices of \( P_n \), consider the following subsets.

For \( n \equiv 0,1 \pmod{2k} \),

\[ D = \{v_{(k+1)+(2k)}j \mid 0 \leq j < \left\lfloor \frac{n}{2k} \right\rfloor\}, \quad |D| = \left\lfloor \frac{n}{2k} \right\rfloor, \] (3)

for \( n \equiv 2,3,\ldots,k \pmod{2k} \),

\[ D = \{v_{(k+1)+(2k)}j, v_{n-1} \mid 0 \leq j < \left\lfloor \frac{n}{2k} \right\rfloor\}, \quad |D| = \left\lfloor \frac{n}{2k} \right\rfloor + 1, \] (4)

for \( n \equiv k+1, k+2, \ldots, 2k-1 \pmod{2k} \),

\[ D = \{v_{(k+1)+(2k)}j \mid 0 \leq j \leq \left\lfloor \frac{n}{2k} \right\rfloor\}, \quad |D| = \left\lfloor \frac{n}{2k} \right\rfloor + 1. \] (5)

We claim that each \( D \) is a distance \( k \) dominating set because

\[ d(v_{(k+1)+(2k)}j, v_{i+(2k)}j) \leq k, \]

\[ d(v_{(k+1)+(2k)}j, e_{i+(2k)}j) \leq k, \] (6)

where \( 1 \leq i \leq 2k, d(v_n, v_{n-1}) \leq k \) and \( d(v_n, e_{n-1}) \leq k \), where, \( 1 \leq i \leq k \).

Therefore

\[ N_k \{v_{(k+1)+(2k)}j\} = \{v_{1+(2k)}j, v_{2+(2k)}j, \ldots, v_{k+(2k)}j, v_{(k+2)+(2k)}j, \ldots, v_{k+(2k)}j, e_{1+(2k)}j, e_{2+(2k)}j, \ldots, e_{k+(2k)}j, e_{(k+1)+(2k)}j, \ldots, e_{(k+2)+(2k)}j\}. \]

(7)

This implies that \( N_k(D) = V(T(P_n)) \) for \( n \equiv 1,2,\ldots,2(k \mod(2k)) \). Now from the nature of \( T(P_n) \), one can observe that every vertex \( d \) of \( D \) is a distance at least \( k+1 \) apart from every other vertex of \( D \) in \( T(P_n) \).

Thus by Proposition 1, above defined \( D \) is a minimal distance \( k \)-dominating set of \( T(P_n) \). Hence, from (2), for \( n > 2k \),

\[ \gamma_k(T(P_n)) = \begin{cases} \left\lfloor \frac{n}{2k} \right\rfloor & \text{for } n \equiv 0,1 \pmod{2k} \\ \left\lfloor \frac{n}{2k} \right\rfloor + 1 & \text{for } n \equiv 2,3,\ldots,2k-1 \pmod{2k}. \end{cases} \] (8)

Theorem 4. For \( n \leq 2k+1 \), \( \gamma_k(D_2(P_n)) = 1 \).

Proof. Consider two copies of path \( P_n \). Let \( v_1, v_2, \ldots, v_n \) be the vertices of first copy of path \( P_n \) and \( u_1, u_2, \ldots, u_n \) be the vertices of second copy of path \( P_n \). Then \( D = \{v_{n/2}\} \) is distance \( k \)-dominating set as \( n \leq 2k+1 \). The set \( D \) being a singleton set it is obviously a minimal distance \( k \)-dominating set of \( D_2(P_n) \).

Theorem 5. For \( n > 2k+1 \),

\[ \gamma_k(D_2(P_n)) = \begin{cases} \left\lfloor \frac{n}{2k + 1} \right\rfloor + 1 & \text{for } n \equiv 1,2,\ldots,2k \mod(2k+1) \\ \left\lfloor \frac{n}{2k + 1} \right\rfloor & \text{for } n \equiv 0 \mod(2k+1). \end{cases} \] (9)

Proof. Consider two copies of path \( P_n \). Let \( v_1, v_2, \ldots, v_n \) be the vertices of first copy of path \( P_n \) and \( u_1, u_2, \ldots, u_n \) be the vertices of second copy of path \( P_n \). Now every vertex from \( v_{k+1}, \ldots, v_{n-k} \) dominates \( 2k+1 \) vertices of \( v_i \) and \( 2k+1 \) vertices of \( u_i \) at a distance \( k \) while every vertex from \( u_{k+1}, u_{k+2}, \ldots, u_{n-k} \) dominates \( 2k+1 \) vertices of \( v_i \) and \( 2k+1 \) vertices of \( u_i \) at a distance \( k \). Therefore at least one vertex from \( \{v_{(k+1)+(2k)}j, v_{(k+2)+(2k)}j, \ldots, v_{(k+1)+(2k)}j, v_{(k+2)+(2k)}j, \ldots, v_{(k+1)+(2k)}j\} \) must belongs to any distance \( k \)-dominating set \( D \) of \( D_2(P_n) \).

Hence

\[ \gamma_k(D_2(P_n)) \geq \left\lfloor \frac{n}{2k+1} \right\rfloor. \] (10)

Now depending upon the number of vertices of \( P_n \), consider the following subsets.
For \( n \equiv 1, 2, \ldots, k \pmod{2k+1} \),
\[
D = \left\{ v_{(k+1)j+2k+1}, v_n | 0 \leq j < \left\lfloor \frac{n}{2k+1} \right\rfloor \right\},
\]  
\[|D| = \left\lfloor \frac{n}{2k+1} \right\rfloor + 1,
\]  
for \( n \equiv k+1, k+2, \ldots, 2k \pmod{2k+1} \),
\[
D = \left\{ v_{(k+1)j+2k+1}, v_n | 0 \leq j < \left\lfloor \frac{n}{2k+1} \right\rfloor \right\},
\]  
\[|D| = \left\lfloor \frac{n}{2k+1} \right\rfloor + 1,
\]
for \( n \equiv 0 \pmod{2k+1} \),
\[
D = \left\{ v_{(k+1)j+2k+1}, v_n | 0 \leq j < \left\lfloor \frac{n}{2k+1} \right\rfloor \right\},
\]  
\[|D| = \left\lfloor \frac{n}{2k+1} \right\rfloor + 1,
\]
where \( 1 \leq i \leq 2k + 1 \), \( d(v_i, v_{i+1}) \leq k \), and \( d(v_i, v_n) \leq k \), where \( 1 \leq n \leq 2k \).

\[d(e_k + (2k)j, v_{i+(2k)j}) \leq k,\]  
\[d(e_1, e_n-1) \leq k,\]  
\[d(e_n-1, v_{n-1}) \leq k,\]  
where \( 1 \leq i \leq k \).

**Theorem 7.** For \( n > 2k \),
\[
\gamma_k(M(P_n)) = \begin{cases} \frac{n}{2k} + 1 & \text{for } n \equiv 1, 2, \ldots, 2k - 1 \pmod{2k} \, (17) \\ \frac{n}{2k} & \text{for } n \equiv 0 \pmod{2k} \end{cases}
\]

**Proof.** Let \( v_1, v_2, \ldots, v_n \) and \( e_1, e_2, \ldots, e_{n-1} \) be the vertices and the edges of \( P_n \) respectively. Then \( v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1} \) will be the vertices of \( M(P_n) \). Then \( D = \{v_{n/2}\} \) is distance \( k \)-dominating set of \( M(P_n) \) as \( n \leq 2k \). The set \( D \) being a singleton set, it is obviously a minimal distance \( k \)-dominating set of \( M(P_n) \).

**Theorem 6.** For \( n \leq 2k \), \( \gamma_k(M(P_n)) = 1 \).

**Proof.** Let \( v_1, v_2, \ldots, v_n \) and \( e_1, e_2, \ldots, e_{n-1} \) be the vertices and the edges of \( P_n \) respectively. Then \( v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1} \) will be the vertices of \( M(P_n) \). Now every vertex from \( v_k, v_{k+1}, \ldots, v_{n-k} \) dominates \( 2k-1 \) vertices of \( v_i \)'s and \( 2k \) vertices of \( e_i \)'s at a distance \( k \) while every vertex from \( e_k, e_{k+1}, \ldots, e_{n-k} \) dominates \( 2k \) vertices of \( v_i \)'s and \( 2k+1 \) vertices of \( e_i \)'s at a distance \( k \). Therefore at least one vertex from \( \{e_1, e_2, e_3, \ldots, e_{2k}, e_{2k+1}, \ldots, e_{2k+(2k-2)}\} \) must belong to any distance \( k \)-dominating set \( D \) of \( M(P_n) \).

Hence,
\[
\gamma_k(M(P_n)) = \left\lfloor \frac{n}{2k} \right\rfloor + 1,
\]

Now depending upon the number of vertices of \( P_n \), consider the following subsets.
For \( n \equiv 1, 2, \ldots, k \pmod{2k} \),
\[
D = \left\{ e_{k+2k}j, e_{n-1} | 0 \leq j < \left\lfloor \frac{n}{2k} \right\rfloor \right\},
\]  
\[|D| = \left\lfloor \frac{n}{2k} \right\rfloor + 1,
\]
for \( n \equiv k+1, k+2, \ldots, 2k-1 \pmod{2k} \),
\[
D = \left\{ e_{k+2k}j, e_{n-1} | 0 \leq j < \left\lfloor \frac{n}{2k} \right\rfloor \right\},
\]  
\[|D| = \left\lfloor \frac{n}{2k} \right\rfloor + 1,
\]
for \( n \equiv 0 \pmod{2k} \),
\[
D = \left\{ e_{k+2k}j, e_{n-1} | 0 \leq j < \left\lfloor \frac{n}{2k} \right\rfloor \right\},
\]  
\[|D| = \left\lfloor \frac{n}{2k} \right\rfloor + 1,
\]
where \( 1 \leq i \leq 2k, d(e_i, v_{i-1}) \leq k \), \( d(e_{n-1}, v_{n-1}) \leq k \), \( d(e_{n-1}, v_{n-1}) \leq k \), where \( 1 \leq l \leq k \).
Therefore
\[ N_k(e_{k+(2k)j}) = \{ e_{1+(2k)j}, e_{2+(2k)j}, \ldots, e_{(k-1)+(2k)j}, \ldots, e_{k+(2k)j} \} \]

This implies that \( N_k[D] = V(M(P_n)) \) for \( n > 2k \). Now from the nature of \( M(P_n) \), one can observe that every vertex \( d \) of \( D \) is at a distance at least \( k + 1 \) apart from every other vertex of \( D \) in \( M(P_n) \).

Thus by Proposition 1, above defined \( D \) is a minimal distance-\( k \)-dominating set of \( M(P_n) \). Hence from (18), for \( n > 2k \),
\[
\gamma_k(M(P_n)) = \begin{cases} 
\left\lceil \frac{n}{2k} \right\rceil + 1 & \text{for } n \equiv 1, 2, \ldots, 2k - 1 \pmod{2k} \\
\frac{n}{2k} & \text{for } n \equiv 0 \pmod{2k}.
\end{cases}
\]

References
