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Research Article

Some New Results on Distance k-Domination in Graphs

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We determine the distance k-domination number for the total graph, shadow graph, and middle graph of path P_n.

1. Introduction

We begin with finite, connected, and undirected graphs, $G = (V(G), E(G))$ without loops or multiple edges. A dominating set D of a graph G is a set of vertices of G such that every vertex of $V(G) - D$ is adjacent to some vertex of D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. Further, the open neighbourhood of $v \in V(G)$ is the set $N(v) = \{u \in V(G)/uv \in E(G)\}$. The closed neighbourhood $N[v]$ of $v \in V(G)$ is the set $N[v] = N(v) \cup \{v\}$. The distance $d(u, v)$ between two vertices u and v is the length of shortest path between u and v in G, if exists otherwise, $d(u, v) = \infty$. The open k-neighbourhood set $N_k(v)$ of vertex $v \in V(G)$ is the set of all vertices of G which are different from v and at distance at most k from v in G, that is, $N_k(v) = \{u \in V(G)/d(u, v) \leq k\}$. The closed k-neighbourhood set $N_k[v]$ of v is defined as $N_k[v] = N_k(v) \cup \{v\}$. Obviously $N(v) = N_1(v)$.

The *total graph* $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.

The *Shadow graph* $D_2(G)$ of a connected graph G is obtained by taking two copies of G, say G' and G''. Join each vertex u' in G' to the neighbours of corresponding vertex u'' in G''.

The middle graph $M(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent whenever either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.

For standard terminology and notations we rely upon Balakrishnan and Ranganathan [1] and Haynes et al. [2].

The concept of distance dominating set was initiated by Slater [3] while the term distance k-dominating set was coined by Henning et al. [4]. For an integer $k \geq 1$, a $D \subseteq V(G)$ is a k-dominating set of G if every vertex in $V(G) - D$ is within distance k from some vertex $v \in D$. That is, $N_k[D] = V(G)$. The minimum cardinality among all k-dominating sets of G is called the k-domination number of G and it is denoted by $\gamma_k(G)$. It is obvious that $\gamma(G) = \gamma_1(G)$. A k-dominating set of cardinality $\gamma_k(G)$ is called a γ_k-set. The distance domination in the context of spanning tree is discussed by Griggs and Hutchinson [5] while bounds on the distance two-domination number and the classes of graphs attaining these bounds are reported in the work of Sridharan et al. [6]. In [7] Topp and Volkmann have discussed distance k-domination as k-covering and characterized connected graphs of order $(k + 1)n$ with distance k-domination (k-covering). Application of distance domination in Ad Hoc wireless networking is briefly discussed by Wu and Li [8]. More details and bibliographic references on distance k-domination can be found in a survey paper by Henning [9].

2. Some Definitions and Main Results

Proposition 1 (see [9]). *For $k \geq 1$, let D be a k-dominating set of a graph G. Then D is a minimal k-dominating set of G if and only if each $d \in D$ has at least one of the following properties.*

(1) *There exists a vertex $v \in V(G) - D$ such that $N_k(v) \cap D = \{d\}$.*

(2) *The vertex d is at distance at least $k+1$ from every other vertex d of D in G.*

Theorem 2. *For $n \le 2k$, $\gamma_k(T(P_n)) = 1$.*

Proof. Let $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots e_{n-1}$ be the vertices and the edges of P_n, respectively. Then $v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}$ will be the vertices of $T(P_n)$. Then $D = \{v_{\lfloor n/2 \rfloor}\}$ is distance k-dominating set of $T(P_n)$ as $n \le 2k$. The set D being a singleton set it is obviously a minimal distance k-dominating set of $T(P_n)$. □

Theorem 3. *For $n > 2k$,*

$$\gamma_k(T(P_n)) = \begin{cases} \left\lfloor \dfrac{n}{2k} \right\rfloor & \text{for } n \equiv 0, 1 \pmod{2k} \\ \left\lfloor \dfrac{n}{2k} \right\rfloor + 1 & \text{for } n \equiv 2, 3, \ldots, 2k-1 \pmod{2k}. \end{cases} \tag{1}$$

Proof. Let $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots, e_{n-1}$ be the vertices and the edges of P_n respectively. Then $v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}$ will be the vertices of $T(P_n)$. Now every vertex from v_{k+1}, $v_{k+2}, \ldots, v_{n-k}$ dominates $2k+1$ vertices of v_i's and $2k$ vertices of e_i's at a distance k while every vertex from $e_k, e_{k+1}, \ldots, e_{n-k}$ dominates $2k$ vertices of v_i's and $2k+1$ vertices of e_i's at a distance k. Therefore at least one vertex from $\{v_{1+(2k)j}, v_{2+(2k)j}, \ldots, v_{(k+1)+(2k)j}, v_{(k+2)+(2k)j}, \ldots, v_{(2k)+(2k)j}\}$ must belongs to any distance k dominating set D of $T(P_n)$.

Hence,

$$\gamma_k(T(P_n)) \ge \left\lfloor \frac{n}{2k} \right\rfloor. \tag{2}$$

Now depending upon the number of vertices of P_n, consider the following subsets.

For $n \equiv 0, 1 \pmod{(2k)}$,

$$D = \left\{ v_{(k+1)+(2k)j} \mid 0 \le j < \left\lfloor \frac{n}{2k} \right\rfloor \right\}, \quad |D| = \left\lfloor \frac{n}{2k} \right\rfloor, \tag{3}$$

for $n \equiv 2, 3, \ldots, k \pmod{(2k)}$,

$$D = \left\{ v_{(k+1)+(2k)j}, v_n \mid 0 \le j < \left\lfloor \frac{n}{2k} \right\rfloor \right\}, \quad |D| = \left\lfloor \frac{n}{2k} \right\rfloor + 1, \tag{4}$$

for $n \equiv k+1, k+2, \ldots, 2k-1 \pmod{(2k)}$,

$$D = \left\{ v_{(k+1)+(2k)j} \mid 0 \le j \le \left\lfloor \frac{n}{2k} \right\rfloor \right\}, \quad |D| = \left\lfloor \frac{n}{2k} \right\rfloor + 1. \tag{5}$$

We claim that each D is a distance k dominating set because

$$\begin{aligned} d\left(v_{(k+1)+(2k)j}, v_{i+(2k)j}\right) &\le k, \\ d\left(v_{(k+1)+(2k)j}, e_{i+(2k)j}\right) &\le k, \end{aligned} \tag{6}$$

where $1 \le i \le 2k$, $d(v_n, v_{n-l}) \le k$, and $d(v_n, e_{n-l}) \le k$, where, $1 \le l \le k$.

Therefore

$$\begin{aligned} N_k\left(v_{(k+1)+(2k)j}\right) = \{ & v_{1+(2k)j}, v_{2+(2k)j}, \ldots, v_{k+(2k)j}, \\ & v_{(k+2)+(2k)j}, \ldots, v_{(2k)+(2k)j}, \\ & e_{1+(2k)j}, e_{2+(2k)j}, \ldots e_{k+(2k)j}, \\ & e_{(k+1)+(2k)j}, \ldots e_{(2k)+(2k)j} \}, \\ N_k(v_n) = \{ & v_{n-1}, v_{n-2}, \ldots, v_{n-k}, e_{n-1}, e_{n-2}, \ldots, e_{n-k} \}. \end{aligned} \tag{7}$$

This implies that $N_k[D] = V(T(P_n))$ for $n \equiv 1, 2, \ldots, 2k \pmod{(2k)}$. Now from the nature of $T(P_n)$, one can observe that every vertex d of D is at a distance at least $k+1$ apart from every other vertex of D in $T(P_n)$.

Thus by Proposition 1, above defined D is a minimal distance k-dominating set of $T(P_n)$. Hence, from (2), for $n > 2k$,

$$\gamma_k(T(P_n)) = \begin{cases} \left\lfloor \dfrac{n}{2k} \right\rfloor & \text{for } n \equiv 0, 1 \pmod{2k} \\ \left\lfloor \dfrac{n}{2k} \right\rfloor + 1 & \text{for } n \equiv 2, 3, \ldots, 2k-1 \pmod{2k}. \end{cases} \tag{8}$$

□

Theorem 4. *For $n \le 2k+1$, $\gamma_k(D_2(P_n)) = 1$.*

Proof. Consider two copies of path P_n. Let $v_1, v_2, \ldots, v_n$ be the vertices of first copy of path P_n and $u_1, u_2, \ldots, u_n$ be the vertices of second copy of path P_n. Then $D = \{v_{\lfloor n/2 \rfloor}\}$ is distance k-dominating set as $n \le 2k+1$. The set D being a singleton, set it is obviously a minimal distance k-dominating set of $D_2(P_n)$. □

Theorem 5. *For $n > 2k+1$,*

$$\gamma_k(D_2(P_n)) = \begin{cases} \left\lfloor \dfrac{n}{2k+1} \right\rfloor + 1 & \text{for } n \equiv 1, 2, \ldots, 2k \pmod{(2k+1)} \\ \dfrac{n}{2k+1} & \text{for } n \equiv 0 \pmod{(2k+1)}. \end{cases} \tag{9}$$

Proof. Consider two copies of path P_n. Let $v_1, v_2, \ldots, v_n$ be the vertices of first copy of path P_n and $u_1, u_2, \ldots, u_n$ be the vertices of second copy of path P_n. Now every vertex from $v_{k+1}, v_{k+2}, \ldots, v_{n-k}$ dominates $2k+1$ vertices of v_i's and $2k+1$ vertices of u_i's at a distance k while every vertex from $u_{k+1}, u_{k+2}, \ldots, u_{n-k}$ dominates $2k+1$ vertices of v_i's and $2k+1$ vertices of u_i's at a distance k. Therefore at least one vertex from $\{v_{1+(2k+1)j}, v_{2+(2k+1)j}, \ldots, v_{(k+1)+(2k+1)j}, v_{(k+2)+(2k+1)j}, \ldots, v_{(2k+1)+(2k+1)j}\}$ must belongs to any distance k-dominating set D of $D_2(P_n)$.

Hence

$$\gamma_k(D_2(P_n)) \ge \left\lfloor \frac{n}{2k+1} \right\rfloor. \tag{10}$$

Now depending upon the number of vertices of P_n, consider the following subsets.

For $n \equiv 1, 2, \ldots, k \,(\mathrm{mod}(2k+1))$,

$$D = \left\{ v_{(k+1)+(2k+1)j}, v_n \mid 0 \le j < \left\lfloor \frac{n}{2k+1} \right\rfloor \right\}, \quad |D| = \left\lfloor \frac{n}{2k+1} \right\rfloor + 1, \tag{11}$$

for $n \equiv k+1, k+2, \ldots, 2k \,(\mathrm{mod}(2k+1))$,

$$D = \left\{ v_{(k+1)+(2k+1)j} \mid 0 \le j \le \left\lfloor \frac{n}{2k+1} \right\rfloor \right\}, \quad |D| = \left\lfloor \frac{n}{2k+1} \right\rfloor + 1, \tag{12}$$

for $n \equiv 0 \,(\mathrm{mod}(2k+1))$,

$$D = \left\{ v_{(k+1)+(2k+1)j} \mid 0 \le j < \frac{n}{2k+1} \right\}, \quad |D| = \frac{n}{2k+1}. \tag{13}$$

We claim that each D is a distance k-dominating set because

$$d\left(v_{(k+1)+(2k+1)j}, v_{i+(2k+1)j}\right) \le k, \quad d\left(v_{(k+1)+(2k+1)j}, u_{i+(2k+1)j}\right) \le k, \tag{14}$$

where $1 \le i \le 2k+1$, $d(v_n, v_{n-l}) \le k$ and $d(v_n, u_{n-l}) \le k$, where $1 \le l \le k$.

Therefore

$$\begin{aligned} N_k\left(v_{(k+1)+(2k+1)j}\right) &= \{ v_{1+(2k+1)j}, v_{2+(2k+1)j}, \ldots, v_{k+(2k+1)j}, \\ &\quad v_{(k+2)+(2k+1)j}, \ldots, v_{(2k+1)+(2k+1)j}, \\ &\quad u_{1+(2k+1)j}, u_{2+(2k+1)j}, \ldots u_{k+(2k+1)j}, \\ &\quad u_{(k+1)+(2k+1)j}, \ldots u_{(2k+1)+(2k+1)j} \}, \\ N_k(v_n) &= \{ v_{n-1}, v_{n-2}, \ldots, v_{n-k}, u_n, u_{n-1}, \ldots, u_{n-k} \}, \end{aligned} \tag{15}$$

which implies that $N_k[D] = V(D_2(P_n))$ for any n. Now from the nature of $D_2(P_n)$, one can observe that every vertex d of D is at a distance at least $k+1$ apart from every other vertex of D in $D_2(P_n)$.

Thus by Proposition 1, above defined D is a minimal distance k-dominating set of $D_2(P_n)$. Hence from (10), for $n > 2k+1$,

$$\gamma_k(D_2(P_n)) = \begin{cases} \left\lfloor \dfrac{n}{2k+1} \right\rfloor + 1 & \text{for } n \equiv 1, 2, \ldots, 2k \,(\mathrm{mod}\,(2k+1)) \\ \dfrac{n}{2k+1} & \text{for } n \equiv 0 \,(\mathrm{mod}\,(2k+1)). \end{cases} \tag{16}$$

□

Theorem 6. *For $n \le 2k$, $\gamma_k(M(P_n)) = 1$.*

Proof. Let $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots e_{n-1}$ be the vertices and the edges of P_n, respectively. Then $v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}$ will be the vertices of $M(P_n)$. Then $D = \{v_{\lfloor n/2 \rfloor}\}$ is distance k-dominating set of $M(P_n)$ as $n \le 2k$. The set D being a singleton set, it is obviously a minimal distance k-dominating set of $M(P_n)$. □

Theorem 7. *For $n > 2k$,*

$$\gamma_k(M(P_n)) = \begin{cases} \left\lfloor \dfrac{n}{2k} \right\rfloor + 1 & \text{for } n \equiv 1, 2, \ldots, 2k-1 \,(\mathrm{mod}\,(2k)) \\ \dfrac{n}{2k} & \text{for } n \equiv 0 \,(\mathrm{mod}\,(2k)). \end{cases} \tag{17}$$

Proof. Let $v_1, v_2, \ldots, v_n$ and $e_1, e_2, \ldots e_{n-1}$ be the vertices and the edges of P_n respectively. Then $v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}$ will be the vertices of $M(P_n)$. Now every vertex from $v_k, v_{k+1}, \ldots, v_{n-k}$ dominates $2k-1$ vertices of v_i's and $2k$ vertices of e_i's at a distance k while every vertex from $e_k, e_{k+1}, \ldots, e_{n-k}$ dominates $2k$ vertices of v_i's and $2k+1$ vertices of e_i's at a distance k. Therefore at least one vertex from $\{e_{1+(2k)j}, e_{2+(2k)j}, \ldots, e_{k+(2k)j}, e_{k+1+(2k)j}, \ldots, e_{2k+(2k)j}\}$ must belong to any distance k-dominating set D of $M(P_n)$.

Hence,

$$\gamma_k(M(P_n)) \ge \left\lfloor \frac{n}{2k} \right\rfloor. \tag{18}$$

Now depending upon the number of vertices of P_n, consider the following subsets.

For $n \equiv 1, 2, \ldots, k \,(\mathrm{mod}(2k))$,

$$D = \left\{ e_{k+(2k)j}, e_{n-1} \mid 0 \le j < \left\lfloor \frac{n}{2k} \right\rfloor \right\}, \quad |D| = \left\lfloor \frac{n}{2k} \right\rfloor + 1, \tag{19}$$

for $n \equiv k+1, k+2, \ldots, 2k-1 \,(\mathrm{mod}(2k))$,

$$D = \left\{ e_{k+(2k)j} \mid 0 \le j \le \left\lfloor \frac{n}{2k} \right\rfloor \right\}, \quad |D| = \left\lfloor \frac{n}{2k} \right\rfloor + 1, \tag{20}$$

for $n \equiv 0 \,(\mathrm{mod}(2k))$,

$$D = \left\{ e_{k+(2k)j} \mid 0 \le j < \frac{n}{2k} \right\}, \quad |D| = \frac{n}{2k}. \tag{21}$$

We claim that each D is a distance k-dominating set because

$$d\left(e_{k+(2k)j}, e_{i+(2k)j}\right) \le k, \quad d\left(e_{k+(2k)j}, v_{i+(2k)j}\right) \le k, \tag{22}$$

where $1 \le i \le 2k$, $d(e_{n-1}, e_{n-l}) \le k$, and $d(e_{n-l}, v_{n-l}) \le k$, where $1 \le l \le k$.

Therefore

$$\begin{aligned} N_k\left(e_{k+(2k)j}\right) &= \left\{e_{1+(2k)j}, e_{2+(2k)j}, \ldots, e_{(k-1)+(2k)j},\right. \\ &\quad e_{(k+1)+(2k)j}, \ldots, e_{(2k)+(2k)j}, \\ &\quad v_{1+(2k)j}, v_{2+(2k)j}, \ldots v_{k+(2k)j}, \\ &\quad \left. v_{(k+1)+(2k)j}, \ldots v_{(2k)+(2k)j}\right\}, \\ N_k\left(e_{n-1}\right) &= \left\{e_{n-2}, e_{n-3}, \ldots, e_{n-k}, v_{n-1}, v_{n-2}, \ldots, v_{n-k}\right\}. \end{aligned} \tag{23}$$

This implies that $N_k[D] = V(M(P_n))$ for $n > 2k$. Now from the nature of $M(P_n)$, one can observe that every vertex d of D is at a distance at least $k+1$ apart from every other vertex of D in $M(P_n)$.

Thus by Proposition 1, above defined D is a minimal distance k-dominating set of $M(P_n)$. Hence from (18), for $n > 2k$,

$$\gamma_k\left(M\left(P_n\right)\right) = \begin{cases} \left\lfloor \dfrac{n}{2k} \right\rfloor + 1 & \text{for } n \equiv 1, 2, \ldots, 2k-1 \,(\mathrm{mod}\,(2k)) \\ \dfrac{n}{2k} & \text{for } n \equiv 0 \,(\mathrm{mod}\,(2k)). \end{cases} \tag{24}$$

□

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