Research Article

Domination Integrity of Splitting Graph of Path and Cycle

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If S is a dominating set of a connected graph G then the domination integrity is the minimum of the sum of two parameters, the number of elements in S and the order of the maximum component of $G - S$. We investigate domination integrity of splitting graph of path P_n and cycle C_n . This work is an effort to relate network expansion and vulnerability parameter.

1. Introduction

The stability of a communication network is of prime importance for any network designer. A shrinking network eventually loses links or nodes, its effectiveness is continuously decreasing and network becomes vulnerable. Many graph theoretic parameters have been introduced for the measurement of vulnerability. Some of them are connectivity, toughness, integrity, and rupture degree. The integrity of a graph is one of the well explored concepts which was introduced by Barefoot et al. [1].

Definition 1. The integrity of a graph G is denoted by $I(G)$ and defined by $I(G) = \min\{|S| + m(G - S) : S \subset V(G)\}$, where $m(G - S)$ is the order of a maximum component of $G - S$.

Definition 2. An I -set of G is any (proper) subset S of $V(G)$ for which $I(G) = |S| + m(G - S)$.

The integrity of the complete graph K_n , path P_n , cycle C_n , star $K_{1,n}$, complete bipartite graph $K_{m,n}$, and power graph of cycle were discussed by Barefoot et al. [1, 2] while Goddard and Swart [3, 4] have investigated the bounds for integrity of graphs and its complement. They have also investigated the integrity of graphs in the context of some graph operations. The integrity of middle graphs is discussed by Mamut and

Vumar [5] while integrity of total graphs is discussed by Dündar and Aytaç [6].

Definition 3. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called *dominating set* if every vertex $v \in V$ is either an element of S or is adjacent to an element of S .

Definition 4. The *domination number* $\gamma(G)$ of a graph G equals the minimum cardinality of minimal dominating set of graph G .

If D is any minimal dominating set and if the order of the largest component of $G - D$ is small, then the removal of D will crash the communication network. Considering this aspect, the concept of domination integrity was introduced by Sundareswaran and Swaminathan [7].

Definition 5. The *domination integrity* of a connected graph G is denoted as $DI(G) = \min\{|S| + m(G - S) : S \text{ is a dominating set}\}$, where $m(G - S)$ is the order of a maximum component of $G - S$.

Sundareswaran and Swaminathan [8] have investigated domination integrity of middle graph of some graphs while Vaidya and Kothari [9] have discussed domination integrity of a graph obtained by duplication of an edge by a vertex and

duplication of vertex by an edge in C_n and P_n . In the present work we investigate domination integrity of splitting graphs of path and cycle.

Definition 6. For a graph G the *splitting* graph $S'(G)$ of graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$ where $N(v)$ and $N(v')$ are the neighborhood sets of v and v' , respectively.

For standard graph theoretic notation we refer to West [10] while for terminology related to domination in Graphs we refer to Haynes et al. [11].

2. Main Results

Theorem 7. For all $n \geq 2$,

$$\gamma(S'(P_n)) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{4} \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod{4}. \end{cases} \quad (1)$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of path P_n and $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S'(P_n)$. As $N(u_1) = \{v_2\}$ and $N(u_n) = \{v_{n-1}\}$, at least one vertex from each pair from u_1, v_2 and u_n, v_{n-1} must belong to any dominating set of $S'(P_n)$. Also at least one vertex from v_{i-1} and v_{i+1} must belong to any dominating set as $N(u_i) = \{v_{i-1}, v_{i+1}\}$ and $N(v_i) = \{v_{i-1}, v_{i+1}, u_{i-1}, u_{i+1}\}$, where $2 \leq i \leq n-1$. Consequently

$$|S| \geq \frac{n}{2} \quad \text{for any dominating set } S. \quad (2)$$

Now depending upon the number of vertices in P_n we consider following subsets for $0 \leq j < \lfloor n/4 \rfloor$:

$$\begin{aligned} S &= \{v_{2+4j}, v_{3+4j}\}, & |S| &= \frac{n}{2} \\ &\text{for } n \equiv 0 \pmod{4}, \\ S &= \{v_{2+4j}, v_{3+4j}, v_{n-1}\}, & |S| &= \frac{n+1}{2} \\ &\text{for } n \equiv 1 \pmod{4}, \\ S &= \{v_{2+4j}, v_{3+4j}, v_{n-2}, v_{n-1}\}, & |S| &= \frac{n+2}{2} \\ &\text{for } n \equiv 2 \pmod{4}, \\ S &= \{v_{2+4j}, v_{3+4j}, v_{n-2}, v_{n-1}\}, & |S| &= \frac{n+1}{2} \\ &\text{for } n \equiv 3 \pmod{4}. \end{aligned} \quad (3)$$

We claim that each S is dominating set because $N(v_{2+4j}) = \{v_{1+4j}, v_{3+4j}, u_{1+4j}, u_{3+4j}\}$ and $N(v_{3+4j}) = \{v_{2+4j}, v_{4+4j}, u_{2+4j}, u_{4+4j}\}$.

Also each S is minimal because the vertex u_{1+4j} will not be dominated by any of the vertices when the vertex v_{2+4j} is removed.

Thus above defined S is minimal dominating set of $S'(P_n)$, hence from (2) we get

$$\gamma(S'(P_n)) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{4} \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod{4}. \end{cases} \quad (4)$$

□

Observation 1. If S is any dominating set of $S'(P_n)$ with $m(S'(P_n) - S) = 2$ then $|S| \geq n - 2$.

Observation 2. If S is any dominating set of $S'(P_n)$ with $m(S'(P_n) - S) = 1$ then $|S| \geq n$.

Theorem 8.

$$DI(S'(P_n)) = \begin{cases} 3 & \text{if } n = 2 \\ n & \text{if } 3 \leq n \leq 9 \\ n-1 & \text{if } 10 \leq n \leq 12 \\ n-2 & \text{if } 13 \leq n \leq 15 \\ n-3 & \text{if } 16 \leq n \leq 18 \\ 15 & \text{if } n = 19. \end{cases} \quad (5)$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of path P_n and $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S'(P_n)$.

Case 1 ($n = 2$). From Theorem 7 $\gamma(S'(P_2)) = 2$ and $D = \{v_1, v_2\}$ is a γ -set of $S'(P_2)$. Then $m(S'(P_2) - D) = 1$. This implies that $DI(S'(P_2)) = \gamma(S'(P_2)) + m(S'(P_2) - D) = 2 + 1 = 3$ since $\gamma(S'(P_2)) \leq |S|$ and $m(S'(P_2) - D) \leq m(S'(P_2) - S)$ for any dominating set S of $S'(P_2)$. Consequently $\gamma(S'(P_2)) + m(S'(P_2) - D) \leq |S| + m(S'(P_2) - S)$ for any dominating set S of $S'(P_2)$. Hence $DI(S'(P_2)) = 3$.

Case 2 ($3 \leq n \leq 9$). Consider the following subcases.

Subcase (i) ($n = 3$). From Theorem 7 $\gamma(S'(P_3)) = 2$ and $D = \{v_2, u_2\}$ is a γ -set of $S'(P_3)$. Then $m(S'(P_3) - D) = 1$. This implies that $DI(S'(P_3)) = \gamma(S'(P_3)) + m(S'(P_3) - D) = 2 + 1 = 3$ since $\gamma(S'(P_3)) \leq |S|$ and $m(S'(P_3) - D) \leq m(S'(P_3) - S)$ for any dominating set S of $S'(P_3)$. Consequently $\gamma(S'(P_3)) + m(S'(P_3) - D) \leq |S| + m(S'(P_3) - S)$ for any dominating set S of $S'(P_3)$. Hence $DI(S'(P_3)) = 3$.

Subcase (ii) ($4 \leq n \leq 6$). From Theorem 7 $\gamma(S'(P_n)) = n - 2$ and $D = \{v_2, v_3, \dots, v_{n-1}\}$ is a γ -set of $S'(P_n)$. Then $m(S'(P_n) - D) = 2$.

Therefore

$$\begin{aligned} DI(S'(P_n)) &\leq \gamma(S'(P_n)) + m(S'(P_n) - D) \\ &= n - 2 + 2 = n. \end{aligned} \quad (6)$$

If S is any dominating set other than D of $S'(P_n)$ with $m(S'(P_n) - S) = 1$ then $|S| \geq n$. This implies that

$$|S| + m(S'(P_n) - S) \geq n + 1 > n. \quad (7)$$

Hence from (6) and (7) $DI(S'(P_n)) = n$.

Subcase (iii) ($n = 7$). From Theorem 7 $\gamma(S'(P_7)) = 4$ and $D = \{v_2, v_3, v_5, v_6\}$ is a γ -set of $S'(P_7)$. Then $m(S'(P_7) - D) = 3$.

Therefore

$$DI(S'(P_7)) \leq \gamma(S'(P_7)) + m(S'(P_7) - D) = 4 + 3 = 7. \quad (8)$$

Now if S_1 is any dominating set other than D of $S'(P_7)$ and $m(S'(P_7) - S_1) = 2$ then $|S_1| \geq 5$. This implies that

$$|S_1| + m(S'(P_7) - S_1) \geq 5 + 2 = 7. \quad (9)$$

Also if S_2 is any dominating set other than D and S_1 of $S'(P_7)$ with $m(S'(P_7) - S_2) = 1$ then $|S_2| \geq 7$.

This implies that

$$|S_2| + m(S'(P_7) - S_2) \geq 7 + 1 = 8. \quad (10)$$

Hence from (8), (9), and (10) $DI(S'(P_7)) = 7$.

Case 3 ($n = 10$). From Theorem 7 $\gamma(S'(P_{10})) = 6$ and $D = \{v_2, v_3, v_5, v_6, v_8, v_9\}$ is a γ -set of $S'(P_{10})$. Then $m(S'(P_{10}) - D) = 3$.

Therefore

$$\begin{aligned} DI(S'(P_{10})) &\leq \gamma(S'(P_{10})) + m(S'(P_{10}) - D) \\ &= 6 + 3 = 9. \end{aligned} \quad (11)$$

Now if S_1 is any dominating set other than D of $S'(P_{10})$ and $m(S'(P_{10}) - S_1) = 2$ then $|S_1| \geq 8$.

This implies that

$$|S_1| + m(S'(P_{10}) - S_1) \geq 8 + 2 = 10. \quad (12)$$

Also if S_2 is any dominating set other than D and S_1 of $S'(P_{10})$ with $m(S'(P_{10}) - S_2) = 1$ then $|S_2| \geq 10$.

This implies that

$$|S_2| + m(S'(P_{10}) - S_2) \geq 10 + 1 = 11. \quad (13)$$

Hence from (11), (12) and (13) $DI(S'(P_{10})) = 9$.

Case 4 ($n = 8, 9$ and $11 \leq n \leq 19$). We know that

$$DI(S'(P_n)) \leq \gamma(S'(P_n)) + m(S'(P_n) - D). \quad (14)$$

The domination number with $\gamma(S'(C_n)) + m(S'(P_n) - D)$ and set S with $|S| + m(S'(P_n) - S)$ are shown in Table 1.

From Table 1

$$|S| + m(S'(P_n) - S) \leq \gamma(S'(P_n)) + m(S'(P_n) - D). \quad (15)$$

If S_1 is any dominating set of $S'(P_n)$ other than D and S with $m(S'(P_n) - S_1) = 4$ or 5 then $|S_1| \leq |S_1|$.

This implies that

$$|S| + m(S'(P_n) - S) \leq |S_1| + m(S'(P_n) - S_1). \quad (16)$$

If S_2 is any dominating set other than D , S_1 , and S with $m(S'(P_n) - S_2) = 2$ then $|S_2| \geq n - 2$.

This implies that

$$|S_2| + m(S'(P_n) - S_2) \geq (n - 2) + 2 = n. \quad (17)$$

If S_3 is any dominating set other than D , S_1 , S_2 , and S with $m(S'(P_n) - S_3) = 1$ then $|S_3| \geq n$.

This implies that

$$|S_3| + m(S'(P_n) - S_3) \geq n + 1. \quad (18)$$

Hence from Table 1 and the results (14) to (18) we conclude that

$$\begin{aligned} DI(S'(P_n)) &= |S| + m(S'(P_n) - S) \\ &= \begin{cases} 3 & \text{if } n = 2 \\ n & \text{if } 3 \leq n \leq 9 \\ n - 1 & \text{if } 10 \leq n \leq 12 \\ n - 2 & \text{if } 13 \leq n \leq 15 \\ n - 3 & \text{if } 16 \leq n \leq 18 \\ 15 & \text{if } n = 19. \end{cases} \end{aligned} \quad (19)$$

Hence the result. $\square$

Theorem 9.

$$DI(S'(P_n)) = \gamma(S'(P_n)) + 6, \quad \forall n \geq 20. \quad (20)$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of path P_n and $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S'(P_n)$. Then from Theorem 7

$$\gamma(S'(P_n)) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{4} \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod{4}, \end{cases} \quad (21)$$

TABLE 1

n	$\gamma(S'(P_n))$	$m(S'(P_n) - D)$	$\gamma(S'(P_n)) + m(S'(P_n) - D)$	S	$m(S'(P_n) - S)$	$ S + m(S'(P_n) - S)$
8	4	6	10	$\{v_2, v_3, v_4, v_6, v_7\}$	3	8
9	5	6	11	$\{v_2, v_3, v_5, v_6, v_7, v_8\}$	3	9
11	6	6	12	$\{v_2, v_3, v_5, v_6, v_7, v_9, v_{10}\}$	3	10
12	6	6	12	$\{v_2, v_3, v_5, v_6, v_7, v_8, v_{10}, v_{11}\}$	3	11
13	7	6	13	$\{v_2, v_3, v_5, v_6, v_8, v_9, v_{11}, v_{12}\}$	3	11
14	8	6	14	$\{v_2, v_3, v_4, v_6, v_7, v_9, v_{10}, v_{12}, v_{13}\}$	3	12
15	8	6	14	$\{v_2, v_3, v_4, v_6, v_7, v_8, v_{10}, v_{11}, v_{13}, v_{14}\}$	3	13
16	8	6	14	$\{v_2, v_3, v_5, v_6, v_8, v_9, v_{11}, v_{12}, v_{14}, v_{15}\}$	3	13
17	9	6	15	$\{v_2, v_3, v_5, v_6, v_8, v_9, v_{11}, v_{12}, v_{14}, v_{15}, v_{16}\}$	3	14
18	10	6	16	$\{v_2, v_3, v_5, v_6, v_8, v_9, v_{11}, v_{12}, v_{14}, v_{15}, v_{16}, v_{17}\}$	3	15
19	10	6	16	$\{v_2, v_3, v_5, v_6, v_8, v_9, v_{11}, v_{12}, v_{14}, v_{15}, v_{17}, v_{18}\}$	3	15

$D = \{v_{4j+2}, v_{4j+3}\}$ where $0 \leq j < \lfloor n/4 \rfloor$ for $n \equiv 0 \pmod{4}$,
 $D = \{v_{4j+2}, v_{4j+3}, v_{n-1}\}$ where $0 \leq j < \lfloor n/4 \rfloor - 1$ for $n \equiv 1 \pmod{4}$,
 $D = \{v_{4j+2}, v_{4j+3}, v_{n-2}, v_{n-1}\}$ where $0 \leq j < \lfloor n/4 \rfloor$
for $n \equiv 2, 3 \pmod{4}$ are γ -sets of $S'(P_n)$. Then $m(S'(P_n) - D) = 6$.

Therefore

$$DI(S'(P_n)) \leq \gamma(S'(P_n)) + 6. \quad (22)$$

Now we claim that if $m(G - S) \neq 6$ for any dominating set S other than D then

$$|S| + m(S'(P_n) - S) \geq \gamma(S'(P_n)) + 6. \quad (23)$$

Case 1. If S_1 is any dominating set other than D and $m(S'(P_n) - S_1) > 4$ or 5 then $|S_1| \geq \gamma(S'(P_n)) + 2$. Consequently $|S_1| + m(S'(P_n) - S_1) \geq \gamma(S'(P_n)) + 6$.

Case 2. If S_2 is any dominating set other than D and S_1 of $S'(P_n)$ with $m(S'(P_n) - S_2) = 3$ then $|S_2| \geq (n - \lfloor n/3 \rfloor)$. Consequently $|S_2| + m(S'(P_n) - S_2) \geq (n - \lfloor n/3 \rfloor) + 3 > (\lfloor n/2 \rfloor + 1) + 6 = \gamma(S'(P_n)) + 6$.

Case 3. If S_3 is any dominating set other than D, S_1 and S_2 of $S'(P_n)$ with $m(S'(P_n) - S_3) = 2$ then $|S_3| \geq n - 2$. Consequently $|S_3| + m(S'(P_n) - S_3) \geq (n - 2) + 2 = n > (\lfloor n/2 \rfloor + 1) + 6 = \gamma(S'(P_n)) + 6$.

Case 4. If S_4 is any dominating set other than D, S_1, S_2 and S_3 of $S'(P_n)$ with $m(S'(P_n) - S_4) = 1$ then $|S_4| \geq n$. Consequently $|S_4| + m(S'(P_n) - S_4) \geq n + 1 > (\lfloor n/2 \rfloor + 1) + 6 = \gamma(S'(P_n)) + 6$.

Thus from (22) and (23) we have $DI(S'(P_n)) = \gamma(S'(P_n)) + 6$. $\square$

Theorem 10. For all $n \geq 3$

$$\gamma(S'(C_n)) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{4} \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod{4}. \end{cases} \quad (24)$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of cycle C_n and $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S'(C_n)$. Now either of the vertex from v_{i-1} and v_{i+1} must belong to any dominating set S as $N(u_i) = \{v_{i-1}, v_{i+1}\}$.

Consequently

$$|S| \geq \frac{n}{2} \text{ for any dominating set } S. \quad (25)$$

Now depending upon the different possibilities of n we choose S as follows.

For $n \equiv 0 \pmod{4}$, $S = \{v_1, v_{4j}, v_{4j+1}, v_n\}$ with $|S| = n/2$ (where $1 \leq j < \lfloor n/4 \rfloor$),

for $n \equiv 1 \pmod{4}$, $S = \{v_1, v_{4j}, v_{4j+1}\}$ with $|S| = (n+1)/2$ (where $1 \leq j \leq \lfloor n/4 \rfloor$),

for $n \equiv 2 \pmod{4}$, $S = \{v_1, v_{4j}, v_{4j+1}, v_n\}$ with $|S| = (n+2)/2$ (where $1 \leq j \leq \lfloor n/4 \rfloor$),

for $n \equiv 3 \pmod{4}$, $S = \{v_1, v_{4j}, v_{4j+1}, v_n\}$ with $|S| = (n+1)/2$ (where $1 \leq j \leq \lfloor n/4 \rfloor$).

We claim that each S is a minimal dominating set of $S'(C_n)$ since $N(v_1) = \{v_2, v_n, u_2, u_n\}$, $N(v_{4j}) = \{v_{4j-1}, v_{4j+1}, u_{4j-1}, u_{4j+1}\}$, $N(v_{4j+1}) = \{v_{4j}, v_{4j+2}, u_{4j}, u_{4j+2}\}$ and $N(v_n) = \{v_1, v_{n-1}, u_1, u_{n-1}\}$ and removal of v_{4j} , a vertex u_{4j+1} will not be dominated by any vertex hence from (25):

$$\gamma(S'(C_n)) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{4} \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod{4}. \end{cases} \quad (26)$$

$\square$

Theorem 11.

$$DI(S'(C_n)) = \begin{cases} n+1; & 3 \leq n \leq 6 \\ n - \left\lfloor \frac{n}{3} \right\rfloor + 3; & 7 \leq n \leq 18. \end{cases} \quad (27)$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of cycle C_n and $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S'(C_n)$.

Case 1 ($n = 3$). From Theorem 10 $\gamma(S'(C_3)) = 2$ and $D = \{v_1, v_2\}$ is a γ -set of $S'(C_3)$ then $m(S'(C_3) - D) = 3$. Therefore

$$\begin{aligned} DI(S'(C_3)) &\leq \gamma(S'(C_3)) + m(S'(C_3) - D) \\ &= 2 + 3 = 5. \end{aligned} \quad (28)$$

If S_1 is any dominating set other than D of $S'(C_3)$ with $m(S'(C_3) - S_1) = 2$ then $|S_1| \geq 3$. This implies that

$$|S_1| + m(S'(C_3) - S_1) \geq 2 + 3 = 5. \quad (29)$$

Let $S_2 = \{v_1, v_2, v_3\}$ be dominating set of $S'(C_3)$ and $m(S'(C_3) - S_2) = 1$. This implies that

$$|S_2| + m(S'(C_3) - S_2) = 3 + 1 = 4. \quad (30)$$

Hence from (28), (29), and (30) $DI(S'(C_3)) = 4$.

Case 2 ($n = 4$). From Theorem 10 $\gamma(S'(C_4)) = 2$ and $D = \{v_1, v_2\}$ is a γ -set of $S'(C_4)$ then $m(S'(C_4) - D) = 3$.

Therefore

$$\begin{aligned} DI(S'(C_4)) &\leq \gamma(S'(C_4)) + m(S'(C_4) - D) \\ &= 2 + 3 = 5. \end{aligned} \quad (31)$$

If S_1 is any dominating set other than D of $S'(C_4)$ and $m(S'(C_4) - S_1) = 2$ or 1 then $|S_1| \geq 4$. Therefore

$$|S_1| + m(S'(C_4) - S_1) \geq 4 + 2 = 6. \quad (32)$$

Hence from (31) and (32) $DI(S'(C_4)) = 5$.

Case 3 ($n = 5$). From Theorem 10 $\gamma(S'(C_5)) = 3$ and $D = \{v_2, v_3, v_4\}$ is a γ -set of $S'(C_5)$ then $m(S'(C_5) - D) = 6$.

Therefore

$$\begin{aligned} DI(S'(C_5)) &\leq \gamma(S'(C_5)) + m(S'(C_5) - D) \\ &= 3 + 6 = 9. \end{aligned} \quad (33)$$

If S_1 is any dominating set other than D of $S'(C_5)$ and $m(S'(C_5) - S_1) = 4$ or 5 then $|S_1| \geq \gamma(S'(C_5)) = 3$.

This implies

$$\begin{aligned} |S_1| + m(S'(C_5) - S_1) &\geq \gamma(S'(C_5)) + 4 \\ &= 3 + 4 = 7. \end{aligned} \quad (34)$$

If S_2 is any dominating set other than D and S_1 with $m(S'(C_5) - S_2) = 2$ or 3 then $|S_2| \geq 5$. This implies that

$$|S_2| + m(S'(C_5) - S_2) \geq 5 + 2 = 7. \quad (35)$$

Let $S_3 = \{v_1, v_2, v_3, v_4, v_5\}$ be dominating set of $S'(C_5)$ then $m(S'(C_5) - S_3) = 1$. This implies that

$$|S_3| + m(S'(C_5) - S_3) = 5 + 1 = 6. \quad (36)$$

Hence from (33), (34), (35), and (36) $DI(S'(C_5)) = 6$.

Case 4 ($n = 6$). From Theorem 10 $\gamma(S'(C_6)) = 4$ and $D = \{v_1, v_2, v_4, v_5\}$ is a γ -set of $S'(C_6)$ then $m(S'(C_6) - D) = 3$.

Therefore

$$\begin{aligned} DI(S'(C_6)) &\leq \gamma(S'(C_6)) + m(S'(C_6) - D) \\ &= 4 + 3 = 7. \end{aligned} \quad (37)$$

If S_1 is any dominating set other than D of $S'(C_6)$ and $m(S'(C_6) - S_1) = 2$ then $|S_1| \geq 6$. This implies that

$$|S_1| + m(S'(C_6) - S_1) \geq 6 + 2 = 8. \quad (38)$$

Let $S_2 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be dominating set of $S'(C_6)$ then $m(S'(C_6) - S_2) = 1$.

Therefore

$$|S_2| + m(S'(C_6) - S_2) = 6 + 1 = 7. \quad (39)$$

Hence from (37), (38), and (39) $DI(S'(C_6)) = 7$.

Case 5 ($7 \leq n \leq 18$). We know that

$$DI(S'(C_n)) \leq \gamma(S'(C_n)) + m(S'(C_n) - D). \quad (40)$$

The domination number with $\gamma(S'(C_n)) + m(S'(C_n) - D)$ and set S with $|S| + m(S'(C_n) - S)$ are shown in Table 2.

From Table 2 for any dominating set S other than D of $S'(C_n)$, we have

$$|S| + m(S'(C_n) - S) \leq \gamma(S'(C_n)) + m(S'(C_n) - D). \quad (41)$$

If S_1 is any dominating set of $S'(C_n)$ other than D and S with $m(S'(C_n) - S_1) = 4$ or 5 then $|S| \leq |S_1|$.

This implies that

$$|S| + m(S'(C_n) - S) \leq |S_1| + m(S'(C_n) - S_1). \quad (42)$$

If S_2 is any dominating set of $S'(C_n)$ other than D , S , and S_1 with $m(S'(C_n) - S_2) = 2$ or 1 then $|S_2| \geq n$.

TABLE 2

n	$\gamma(S'(C_n))$	$m(S'(C_n) - D)$	$\gamma(S'(C_n)) + m(S'(C_n) - D)$	S	$m(S'(C_n) - S)$	$ S + m(S'(C_n) - S)$
7	4	6	10	$\{v_1, v_2, v_4, v_5, v_7\}$	3	8
8	4	6	10	$\{v_1, v_2, v_3, v_5, v_6, v_8\}$	3	9
9	5	6	11	$\{v_1, v_2, v_4, v_5, v_7, v_8\}$	3	9
10	6	6	12	$\{v_1, v_2, v_4, v_5, v_7, v_8, v_{10}\}$	3	10
11	6	6	12	$\{v_1, v_2, v_4, v_5, v_7, v_8, v_{10}, v_{11}\}$	3	11
12	6	6	12	$\{v_1, v_2, v_4, v_5, v_7, v_8, v_{10}, v_{12}\}$	3	11
13	7	6	13	$\{v_1, v_2, v_4, v_5, v_7, v_8, v_{10}, v_{11}, v_{13}\}$	3	12
14	8	6	14	$\{v_1, v_2, v_4, v_5, v_7, v_8, v_{10}, v_{11}, v_{13}, v_{14}\}$	3	13
15	8	6	14	$\{v_1, v_2, v_4, v_5, v_7, v_8, v_{10}, v_{11}, v_{13}, v_{14}\}$	3	13
16	8	6	14	$\{v_1, v_2, v_4, v_5, v_7, v_8, v_{10}, v_{11}, v_{13}, v_{14}, v_{16}\}$	3	14

This implies that

$$|S_2| + m(S'(C_n) - S_2) \geq n + 1. \quad (43)$$

Hence from Table 2 and the results (40) to (43) we conclude that

$$\begin{aligned} DI(S'(C_n)) &= |S| + m(S'(C_n) - S) \\ &= n - \left\lfloor \frac{n}{3} \right\rfloor + 3. \end{aligned} \quad (44)$$

Hence the result. $\square$

Theorem 12. For all $n \geq 17$ $DI(S'(C_n)) = \gamma(S'(C_n)) + 6$.

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of cycle C_n and $u_1, u_2, \dots, u_n$ be the vertices corresponding to $v_1, v_2, \dots, v_n$ which are added to obtain $S'(C_n)$. Then from Theorem 10

$$\gamma(S'(C_n)) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{4} \\ \frac{n+2}{2} & \text{if } n \equiv 2 \pmod{4}, \end{cases} \quad (45)$$

$D = \{v_1, v_{4j}, v_{4j+1}, v_n\}$ (where $1 \leq j < \lfloor n/4 \rfloor$) for $n \equiv 0 \pmod{4}$, $D = \{v_1, v_{4j}, v_{4j+1}\}$ (where $1 \leq j \leq \lfloor n/4 \rfloor$) for $n \equiv 1 \pmod{4}$, $D = \{v_1, v_{4j}, v_{4j+1}, v_n\}$ (where $1 \leq j \leq \lfloor n/4 \rfloor$) for $n \equiv 2 \pmod{4}$ or $n \equiv 3 \pmod{4}$ are γ -sets of $S'(C_n)$. Then $m(S'(C_n) - D) = 6$.

Therefore

$$DI(S'(C_n)) \leq \gamma(S'(C_n)) + 6. \quad (46)$$

Now we claim that if $m(S'(C_n)) \neq 6$ for any dominating set S other than D then

$$|S| + m(S'(C_n) - S) \geq \gamma(S'(C_n)) + 6. \quad (47)$$

Case 1. If S_1 is any dominating set other than D and $m(S'(C_n) - S_1) = 4$ or 5 then $|S_1| \geq \gamma(S'(C_n)) + 2$. Consequently $|S_1| + m(S'(C_n) - S_1) \geq \gamma(S'(C_n)) + 6$.

Case 2. If S_2 is any dominating set other than D and S_1 of $S'(C_n)$ with $m(S'(C_n) - S_2) = 3$ then $|S_2| \geq (n - \lfloor n/3 \rfloor)$. Consequently $|S_2| + m(S'(C_n) - S_2) \geq (n - \lfloor n/3 \rfloor) + 3 > \gamma(S'(C_n)) + 6$.

Case 3. If S_3 is any dominating set other than D , S_1 , and S_2 of $S'(C_n)$ with $m(S'(C_n) - S_3) = 2$ or 1 then $|S_3| \geq n$. Consequently $|S_3| + m(S'(C_n) - S_3) \geq n + 1 > \gamma(S'(C_n)) + 6$.

Thus from (46) and (47) we have $DI(S'(C_n))$. $\square$

3. Concluding Remarks

The domination and vulnerability of network are two important concepts for the network security. We have studied an important measure of vulnerability known as domination integrity and investigate domination integrity of splitting graphs of path and cycle. The results reported here throw some light in the direction to find the domination integrity of larger graph obtained from the given graph.

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