Research Article

Finite Difference Solution of Elastic-Plastic Thin Rotating Annular Disk with Exponentially Variable Thickness and Exponentially Variable Density

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Elastic-plastic stresses, strains, and displacements have been obtained for a thin rotating annular disk with exponentially variable thickness and exponentially variable density with nonlinear strain hardening material by finite difference method using Von-Mises' yield criterion. Results have been computed numerically and depicted graphically. From the numerical results, it can be concluded that disk whose thickness decreases radially and density increases radially is on the safer side of design as compared to the disk with exponentially varying thickness and exponentially varying density as well as to flat disk.

1. Introduction

Due to wide applications of rotating disk, circular disk, spherical shells, cylinders, and shafts in engineering, elastic-plastic analysis of rotating disk, becoming more and more active topic in the field of solid mechanics. The research on them is always an important topic, and their benefits have been included in some books [1–3]. Sharma et al. [4] uses the concept of transition theory to evaluate the stresses for disk with different parameters like variable thickness and variable densities. The elastic-plastic strain hardening problems of annular disks with constant thickness under external pressure were firstly studied by Gamer [5] using linear stress-plastic strain relation. Later on, the work was extended to annular disk with some especially thickness functions by Güven [6]. The elastic-plastic linear strain hardening problems of rotating annular disks subjected to angular velocity are easier to deal with as compared to nonlinear strain hardening problems of annular disks. Therefore, main focus of research now these days is on rotating annular disk made of strain hardening material. In contrast, very few researchers are there who investigated elastic-plastic deformation and stresses for rotating disks with nonlinear strain hardening.

The obvious advantage using a linear strain hardening stress-plastic strain is that a closed form solution can be obtained for annular disks with constant thickness and some especially variable thickness functions. However, most of the materials exhibit nonlinear strain hardening behavior; thus, this nonlinearity is obvious in the transition region from elastic to plastic parts of stress-strain curve. Due to the previously reason, a polynomial stress-strain relation of nonlinear strain hardening material is proposed in the papers of You et al. [7–9]. Numerical method such as finite difference method is an effective technique to the stresses and strains for these rotating disks. However, for scientific research and engineering analysis, analytical methods and numerical methods are still very active. Therefore, Sterner et al. [10] proposed truncated Taylor's series numerical method to solve elastic problems of rotating disk with arbitrary variable thickness. You and Zhang [8] examine elastic-plastic stresses in rotating disks, using Runge-Kutta's method.

In this paper, we proposed a more straightforward and most effective numerical method such as finite difference method which is most celebrated method to solve boundary value problems. The proposed method is used to analyze the stresses, strains, and displacements for annular disk having

FIGURE 1: Radial stress (T_{rr}) and circumferential stress ($T_{\theta\theta}$) for annular disk with exponentially variable thickness and exponentially variable density with $n = 0.7, 0.2, 0.5$ and $m = 0.7, 0.1, 0.5$ under (a) 500 rad/s and (b) 780 rad/s for both approaches.

exponential variable thickness and exponential variable density with nonlinear strain hardening material behavior.

2. Mathematical Formulation

Assuming that the stresses vary over the thickness of the disk, the theory of the disks of variable thickness can give good result as that of the disks of constant thickness as long as they meet the assumption of plane stress. For disk profile, it is assumed that disk is symmetric with respect to the mid plane. This profile is defined by the thickness function $h(r)$ and density function $\rho(r)$

$$h(r) = h_0 e^{-k(r/b)^n}; \quad \rho(r) = \rho_0 e^{t(r/b)^m}, \quad (1)$$

where n, m, k and t are geometric parameters, b is the radius of the disk, h_0 is the thickness at the axis of the disk, and ρ_0 is the density of the material.

2.1. Basic Equations. For rotating disk with variable thickness and variable density, the governing equilibrium equation is

$$\frac{d}{dr} (hrT_{rr}) - hT_{\theta\theta} + h\rho\omega^2 r^2 = 0, \quad (2)$$

where r is the radial coordinate, T_{rr} and $T_{\theta\theta}$ are the radial and circumferential stresses, and ω is the angular velocity of the disk.

The strains and radial displacement are

$$e_r = \frac{du}{dr}; \quad e_\theta = \frac{u}{r}, \quad (3)$$

where u is the displacement component in the radial direction, e_r is the radial strain, and e_θ is the circumferential strain.

The equation of compatibility can be derived from (3) as follows:

$$r \frac{de_\theta}{dr} + e_\theta - e_r = 0. \quad (4)$$

The relations between stresses and elastic strains for plane stress problems can be determined according to theory of elasticity

$$e_r^e = \frac{1}{2\mu(3\lambda + 2\mu)} (2(\lambda + \mu)T_{rr} - \lambda T_{\theta\theta}); \quad (5)$$

$$e_\theta^e = \frac{1}{2\mu(3\lambda + 2\mu)} (2(\lambda + \mu)T_{\theta\theta} - \lambda T_{rr}),$$

where e_r^e, e_θ^e are the elastic radial and circumferential strains, and λ and μ are Lamé's constants.

For plastic deformation, the relation between the stresses and plastic strains can be determined according to the deformation theory of plasticity [3]

$$e_r^p = \frac{e_e^p}{T_{ee}} \left(T_{rr} - \frac{1}{2} T_{\theta\theta} \right); \quad (6)$$

$$e_\theta^p = \frac{e_e^p}{T_{ee}} \left(T_{\theta\theta} - \frac{1}{2} T_{rr} \right),$$

where e_r^p and e_θ^p are the plastic radial and circumferential strains, e_e^p is the equivalent plastic strain, and T_{ee} is the equivalent stress.

The Von-Mises yield criterion is given by $T_{ee} = \sqrt{T_{rr}^2 - T_{rr}T_{\theta\theta} + T_{\theta\theta}^2}$.

The total strains are the sum of elastic and plastic strain

$$e_r = e_r^e + e_r^p; \quad e_\theta = e_\theta^e + e_\theta^p. \quad (7)$$

FIGURE 2: Radial stress (T_{rr}) and circumferential stress ($T_{\theta\theta}$) for annular disk with exponentially variable thickness and exponentially variable density with (a) $n = 0.7, 0.2, 0.5$ and $m = 0.1$, (b) $n = 0.7, 0.2, 0.5$, and $m = 0.5$ (c) $n = 0.7, 0.2, 0.5$ and $m = 0.7$, under 500 rad/s for both approaches.

Let us now introduce a stress function ϕ and assume that the relations between stresses and the stress function are

$$T_{rr} = \frac{1}{hr}\phi; \quad T_{\theta\theta} = \frac{1}{h} \frac{d\phi}{dr} + \rho\omega^2 r^2. \quad (8)$$

Substituting (8) into (5), and further substitution into (7), one obtains

$$e_r = \frac{1}{2\mu(3\lambda + 2\mu)} \left(2(\lambda + \mu) \frac{\phi}{hr} - \lambda \frac{\phi'}{h} - \lambda \rho \omega^2 r^2 \right) + e_r^p,$$

$$e_\theta = \frac{1}{2\mu(3\lambda + 2\mu)} \left(2(\lambda + \mu) \left(\frac{\phi'}{h} + \rho \omega^2 r^2 \right) - \lambda \frac{\phi}{hr} \right) + e_\theta^p, \quad (9)$$

where $\phi' = d\phi/dr$.

By the substitution of (9) into (1), we have

$$2(\lambda + \mu) r^2 \phi'' + 2(\lambda + \mu) \left(1 + kn \left(\frac{r}{b} \right)^n \right) r \phi' - \left(2(\lambda + \mu) + \lambda kn \left(\frac{r}{b} \right)^n \right) \phi$$

FIGURE 3: Radial stress (T_{rr}) and circumferential stress ($T_{\theta\theta}$) for annular disk with exponentially variable thickness and exponentially variable density with (a) $m = 0.1, 0.5, 0.7$ and $n = 0.2$, (b) $m = 0.1, 0.5, 0.7$ and $n = 0.5$, and (c) $m = 0.1, 0.5, 0.7$ and $n = 0.7$, 500 rad/s for both approaches.

$$\begin{aligned}
 & + 2\mu(3\lambda + 2\mu)h_0e^{-k(r/b)^n} \\
 & \times r \left(r \frac{de_\theta^p}{dr} + e_\theta^p - e_r^p \right) \\
 & = -(7\lambda + 6\mu)h_0\rho_0 \times e^{t(r/b)^m - k(r/b)^n} \omega^2 r^3 \\
 & - 2(\lambda + \mu)h_0\rho_0 e^{t(r/b)^m - k(r/b)^n} m t \left(\frac{r}{b} \right)^m \omega^2 r^3,
 \end{aligned} \tag{10}$$

where $\phi'' = d^2\phi/dr^2$, $\phi' = d\phi/dr$.

By the nonlinear strain-hardening material model proposed by You and Zhang [8], the stress-strain relationship can be written as

$$\begin{aligned}
 e_e &= \frac{(\lambda + \mu)T_{ee}}{\mu(3\lambda + 2\mu)}, \quad e_e \leq e_0; \\
 e_e^p &= a_1 T_{ee}^3 + a_2 T_{ee}^5, \quad e_e > e_0,
 \end{aligned} \tag{11}$$

where e_e is the equivalent total strain and e_0 the yield strain and a_1, a_2 are material constants.

FIGURE 4: Radial stress (T_{rr}) and circumferential stress ($T_{\theta\theta}$) for annular disk with variable thickness and variable density with $n = 0.2, 0.5, 0.7$ and $m = 0.1, 0.5, 0.7$ under (a) 500 rad/s and (b) 780 rad/s for both approaches.

Substitution of the second equation of (11) into (6), the plastic strains can be written as

$$\begin{aligned} e_r^p &= \frac{(a_1 T_{ee}^3 + a_2 T_{ee}^5)}{T_{ee}} \left(T_{rr} - \frac{1}{2} T_{\theta\theta} \right); \\ e_{\theta}^p &= \frac{(a_1 T_{ee}^3 + a_2 T_{ee}^5)}{T_{ee}} \left(T_{\theta\theta} - \frac{1}{2} T_{rr} \right). \end{aligned} \quad (12)$$

The governing equation in the plastic region of the rotating disks in terms of stresses and stress function can be obtained by substituting (12) into (10) as

$$\begin{aligned} & \left\{ 2(\lambda + \mu) + 2\mu(3\lambda + 2\mu) \right. \\ & \times \left[\frac{1}{2} (a_1 + 2a_2 T_{ee}^2) (2T_{\theta\theta} - T_{rr})^2 \right. \\ & \left. \left. + (a_1 T_{ee}^2 + a_2 T_{ee}^4) \right] \right\} r^2 \phi'' \\ & = -2(\lambda + \mu) \left(1 + kn \left(\frac{r}{b} \right)^n \right) r \phi' \\ & + \left(2(\lambda + \mu) + \lambda kn \left(\frac{r}{b} \right)^n \right) \phi \\ & - (7\lambda + 6\mu) h_0 \rho_0 e^{t(r/b)^m - k(r/b)^n} \omega^2 r^3 \\ & - 2(\lambda + \mu) h_0 \rho_0 e^{t(r/b)^m - k(r/b)^n} m t \left(\frac{r}{b} \right)^m \omega^2 r^3 \end{aligned}$$

$$\begin{aligned} & - 2\mu(3\lambda + 2\mu) \\ & \times \left\{ \frac{1}{2} (a_1 + 2a_2 T_{ee}^2) (2T_{\theta\theta} - T_{rr}) \right. \\ & \times \left\{ (2T_{rr} - T_{\theta\theta}) \left[r \phi' - \left(1 - kn \left(\frac{r}{b} \right)^n \right) \phi \right] \right. \\ & \left. + (2T_{\theta\theta} - T_{rr}) \right. \\ & \times \left(r kn \left(\frac{r}{b} \right)^n \phi' + 2h_0 \rho_0 e^{t(r/b)^m - k(r/b)^n} \omega^2 r^3 \right. \\ & \left. \left. + h_0 \rho_0 e^{t(r/b)^m - k(r/b)^n} m t \left(\frac{r}{b} \right)^m \omega^2 r^3 \right) \right\} \\ & + (a_1 T_{ee}^2 + a_2 T_{ee}^4) \\ & \times \left\{ - \left(\frac{1}{2} - kn \left(\frac{r}{b} \right)^n \right) r \phi' + \frac{1}{2} \left(1 - kn \left(\frac{r}{b} \right)^n \right) \phi \right. \\ & \left. + 2h_0 \rho_0 e^{t(r/b)^m - k(r/b)^n} \omega^2 r^3 \right. \\ & \left. + h_0 \rho_0 e^{t(r/b)^m - k(r/b)^n} m t \left(\frac{r}{b} \right)^m \omega^2 r^3 \right. \\ & \left. + \frac{3}{2} r h_0 e^{-k(r/b)^n} (T_{\theta\theta} - T_{rr}) \right\}. \end{aligned} \quad (13)$$

The values of the stress function ϕ at the elastic-plastic interface radius are the same, and, therefore, the stress function ϕ is continuous at the interface radius [9]. It can be seen from the continuity of the stress function and (3),

FIGURE 5: Radial stress (T_{rr}) and circumferential stress ($T_{\theta\theta}$) for annular disk with variable thickness and variable density with (a) $n = 0.7, 0.2, 0.5$ and $m = 0.1$, (b) $n = 0.7, 0.2, 0.5$ and $m = 0.5$, and (c) $n = 0.7, 0.2, 0.5$ and $m = 0.7$, under 500 rad/s for both approaches.

(5)–(7), and (8) that the continuity conditions of the stresses and displacements at the elastic-plastic interface radius are satisfied.

The boundary conditions for the rotating annular disks are

$$T_{rr} = 0 \quad \text{at } r = a; \quad T_{rr} = 0 \quad \text{at } r = b, \quad (14)$$

where a and b are the inner and outer radii of the rotating disks, respectively.

3. Finite Difference Algorithm

To determine the elastic-plastic stresses, strains, and displacement in thin rotating disks with a nonlinear strain hardening material, we have to solve the second-order nonlinear differential equation (13) under the given boundary condition (14). The general form of (13) can be written as

$$\phi'' = f(r, \phi, \phi'). \quad (15)$$

3.1. Finite Difference Algorithm Steps. The second-order differential equation (13) with the given boundary conditions can be solved by using finite difference method.

FIGURE 6: Radial stress (T_{rr}) and circumferential stress ($T_{\theta\theta}$) for annular disk with variable thickness and variable density with (a) $m = 0.1, 0.5, 0.7$ and $n = 0.2$, (b) $m = 0.1, 0.5, 0.7$ and $n = 0.5$, and (c) $m = 0.1, 0.5, 0.7$ and $n = 0.7$, under 500 rad/s for both approaches.

- (i) First, partition the domain $[0.1 \ 0.5]$ into p subintervals of length h .
- (ii) To express the differential operators $d\phi/dr$ and $d^2\phi/dr^2$ in a discrete form, we use the finite difference approximations as

$$\frac{d^2\phi}{dr^2} = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2}, \quad \frac{d\phi}{dr} = \frac{\phi_{i+1} - \phi_{i-1}}{2h}. \quad (16)$$

- (iii) With $h = 1/p$, we have $p + 1$ nodal points $\phi_1, \phi_2, \dots, \phi_{p+1}$. The values at the end points are given by the boundary conditions; that is, we are given that $\phi_1 = 0, \phi_{p+1} = 0$. Using the finite

difference approximation, we get the following system of equations:

$$\begin{aligned} & \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2} \\ & = f\left(r, \phi_i, \frac{\phi_{i+1} - \phi_{i-1}}{2h}\right), \quad i = 2, 3, \dots, n. \end{aligned} \quad (17)$$

- (iv) After simplifying and collecting coefficients of ϕ_{i+1}, ϕ_i , and ϕ_{i-1} , in Step 3, the boundary value problem results in system of nonlinear equations.
- (v) The solution of the $(p - 1)$ system of nonlinear equations can be obtained using Newton-Raphson's

FIGURE 7: Radial stress (T_{rr}) and circumferential stress ($T_{\theta\theta}$) for annular disk with constant thickness and constant density under (a) 500 rad/s and (b) 780 rad/s for both approaches.

method where the unknowns are $\phi_2, \phi_3, \dots, \phi_{p-1}$. By this process, we will get the stress function and then stresses, strains, and displacement.

4. Numerical Illustration and Discussion

A two-dimensional plane stress analysis of rotating disk with nonuniform thickness and nonuniform density are carried out using finite difference method. The radius of the rotating disk is taken to be $b = 0.5$, and its materials properties are material density $\rho = 8000 \text{ Kg/m}^3$, Lamé's constant are $\lambda = 79.61 \text{ GPa}$, $\mu = 119.42 \text{ GPa}$, and Poisson's ratio $\nu = 0.3$.

4.1. Rotating Annular Disk with Exponentially Variable Thickness and Exponentially Variable Density. It has been observed from Figures 1(a) and 1(b) that as thickness decreases and density increases radially, circumferential stress is going on decreasing. Also, circumferential stress is maximum at the internal surface. With the increase in angular velocity, circumferential stress is going on increasing. Figures 2(a), 2(b) and 2(c) are sketched for uniform density and we observed that circumferential stresses are maximum at internal surface and these stresses decreases with decrease in thickness radially. As with the increase in density radially, circumferential goes on decreasing which gives appropriate design of the disk. It has been observed from Figures 3(a), 3(b), and 3(c) that for uniform thickness, circumferential stress is maximum at internal surface. Also, circumferential stress for the disk whose density decreases radially is maximum as compared to the disk with high density. Also, decrease in thickness radially yields decrease in circumferential stress, which gives appropriate design of disk.

From all previous analysis, we can conclude that disk whose thickness decreases radially and density increases

radially is on the safer side of the design as compared to other thickness and density parameters, because circumferential stress is less for the disk whose thickness decreases radially and density increases radially as compared to other thickness and density parameters.

4.2. Rotating Annular Disk with Variable Thickness and Variable Density Using Power Law ($h = h_0(r/b)^{-n}$; $\rho = \rho_0(r/b)^m$). It has been observed from Figures 4(a) and 4(b) that circumferential stress is maximum at internal surface. Circumferential stress is maximum for the disk whose density is less and thickness is high as compared to the disk whose thickness is less and density is high. Also, it has been observed from the figure that with the increase in angular speed, circumferential stress goes on increasing, but again circumferential stress is less as compared to other cases. It has been observed from Figures 5(a), 5(b), and 5(c) that for uniform density disk, circumferential stress is maximum at internal surface. With the decrease in thickness radially, circumferential stress goes on decreasing. Also with the increase in density, circumferential stress further decreases which provides an appropriate design of the disk. It has been observed from Figures 6(a), 6(b), and 6(c) that for uniform thickness, circumferential stresses are maximum at internal surface. Also, it has been noted that circumferential stress is less for disk profile with high density as compared to disk profiles with less density. With the decrease in thickness radially, circumferential stress decreases which yields that disk, whose thickness is less and density is high, is appropriate for the design as compared to other disk profiles.

From the previous analysis, we observed that disk with high density and less thickness is on the safer side of the design as compared to other parameters because circumferential stress is less for previous case as compared to other

disk profiles. The results calculated for variable thickness and variable density ($h = h_0(r/b)^{-n}$; $\rho = \rho_0(r/b)^m$) using finite difference method are in very good agreement with Runge-Kutta's method studied by You et. al [9] for all the rotating disks.

4.3. Annular Disk with Constant Thickness and Constant Density. It has been observed from Figures 7(a) and 7(b) that for constant thickness and constant density, circumferential stress is maximum at internal surface. Also, with the increase in angular speed, circumferential stress increases significantly.

5. Conclusion

After analyzing all the three disk profiles, it can be concluded that circumferential stresses are maximum at internal surface. It is also concluded that disk whose thickness decreases radially and density increases radially is on the safer side of design as compared to the disk with exponentially varying thickness and exponentially varying density as well as to flat disk. This is because circumferential stresses for the disks whose thickness decreases and density increases radially are less as compared to the disk with exponentially varying thickness and exponentially varying density as well as to flat disk.

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