Research Article

Some Properties of a Sequence Similar to Generalized Euler Numbers

Haiqing Wang and Guodong Liu

Department of Mathematics, Huizhou University, Huizhou, Guangdong 516007, China

Correspondence should be addressed to Guodong Liu; gdliu@pub.huizhou.gd.cn

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We introduce the sequence \( \{U_n(x)\} \) given by generating function

\[
\frac{1}{(e^t + e^{-t} - 1)} x = \sum_{n=0}^{\infty} U_n(x) \frac{t^n}{n!} \quad (|t| < (1/3)\pi, 1^x := 1)
\]

and establish some explicit formulas for the sequence \( \{U_n(x)\} \). Several identities involving the sequence \( \{U_n(x)\} \), Stirling numbers, Euler polynomials, and the central factorial numbers are also presented.

1. Introduction and Definitions

For a real or complex parameter \( \alpha \), the generalized Euler polynomials \( E_n^{(\alpha)}(x) \) are defined by the following generating function (see [1–4])

\[
\left( \frac{2}{e^t + 1} \right)^\alpha e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x) \frac{t^n}{n!} \quad (|t| < \pi, 1^{\alpha} := 1). \tag{1}
\]

Obviously, we have

\[
E_n^{(1)}(x) = E_n(x) \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{2}
\]

in terms of the classical Euler polynomials \( E_n(x) \), \( \mathbb{N} \) being the set of positive integers. The classical Euler numbers \( E_n \) are given by the following:

\[
E_n = 2^n E_n\left(\frac{1}{2}\right) \quad (n \in \mathbb{N}_0). \tag{3}
\]

The so-called the generalized Euler numbers \( E_n^{(x)}(2n) \) are defined by (see [3, 5])

\[
\left( \frac{2}{e^t + e^{-t}} \right)^x = \sum_{n=0}^{\infty} E_n^{(x)}(2n) \frac{t^{2n}}{(2n)!} \quad \left(|t| < \frac{\pi}{2}, 1^x := 1\right). \tag{4}
\]

In fact, \( E_n^{(x)}(k \in \mathbb{Z}) \) are the Euler numbers of order \( k \), \( \mathbb{Z} \) being the set of integers. The numbers \( E_{2n}^{(1)} = E_{2n} \) are the ordinary Euler numbers.

Zhi-Hong Sun introduces the sequence \( \{U_n\} \) similar to Euler numbers as follows (see [6, 7]):

\[
U_0 = 1, \quad U_n = -2^{\lfloor n/2 \rfloor} \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} U_{n-2k}, \quad (n \geq 1), \tag{5}
\]

where (and in what follows) \( \lfloor x \rfloor \) is the greatest integer not exceeding \( x \).

Clearly, \( U_{2n-1} = 0 \) for \( n \geq 1 \). The first few values of \( U_{2n} \) are shown below

\[
U_2 = -2, \quad U_4 = 22, \quad U_6 = -602, \quad U_8 = 30742, \quad U_{10} = -2523002, \quad U_{12} = 303692662. \tag{6}
\]

The sequence \( \{U_n\} \) is related to the classical Bernoulli polynomials \( B_n(x) \) (see [8–11]) and the classical Euler polynomials \( E_n(x) \). Zhi-Hong Sun gets the generating function of
\{U_n\} and deduces many identities involving \{U_n\}. As example, (see [6]),

\[
\frac{1}{e^t + e^{-t} - 1} = \sum_{n=0}^{\infty} U_n \frac{t^n}{n!}
\]

or by means of the generating function

\[
(e^x + e^{-x} - 2)^k = (2k)! \sum_{n=k}^{\infty} T(n, k) \frac{x^{2n}}{(2n)!}
\]

It follows from (17) or (18) that

\[
T(n, k) = T(n - 1, k - 1) + k^2 T(n - 1, k),
\]

with

\[
T(0, 0) = 1, \quad T(n, 0) = 0 \quad (n \in \mathbb{N}),
\]

\[
T(n, 1) = 1 \quad (n \in \mathbb{N}).
\]

We also find from (18) that

\[
T(n, 2) = \frac{1}{4} \left(4^{n-1} - 1\right),
\]

\[
T(n, 3) = \frac{9n}{360} - \frac{4n}{60} + \frac{1}{24} \quad (n \in \mathbb{N}).
\]

The main purpose of this paper is to prove some formulas for the generalized sequence \{U^{(x)}_n\} and \(E_n(x)\). Some identities involving the sequence \{U^{(x)}_n\}, Stirling numbers \(s(n, k)\), and the central factorial numbers \(T(n, k)\) are deduced.

### 2. Main Results

**Theorem 1.** Let \(n \geq k\) \((n, k \in \mathbb{N})\) and

\[
q(n, k) = (-1)^k \sum_{j=k}^{\infty} \frac{(2j)!}{j!} T(n, j) s(j, k).
\]

Then,

\[
U^{(x)}_{2n} = \sum_{k=1}^{n} q(n, k) x^k.
\]

**Remark 2.** By (15), (19), (20), and Theorem 1, we know that \(U^{(x)}_{2n}\) is a polynomial of \(x\) with integral coefficients. For example, by setting \(n = 1, 2, 3, 4\) in Theorem 1, we get

\[
U^{(x)}_2 = -2x, \quad U^{(x)}_4 = 10x + 12x^2,
\]

\[
U^{(x)}_6 = -182x - 300x^2 - 120x^3,
\]

\[
U^{(x)}_8 = -6970x + 13692x^2 + 8400x^3 + 1680x^4.
\]

Taking \(x = 1\) in Theorem 1, we can obtain the following.

**Corollary 3.** Let \(n \in \mathbb{N}\). Then,

\[
U_{2n} = \sum_{j=0}^{n} (-1)^j (2j)! T(n, j).
\]
\textbf{Corollary 4.} Let \( n \in \mathbb{N} \). Then,
\[
U_{2n} \equiv -2 \pmod{24},
\]
\[
U_{2n} \equiv -2 + 24T(n,2) \pmod{720},
\]
\[
U_{2n} \equiv -2 + 24T(n,2) - 720T(n,3) \pmod{40320}.
\]

\textbf{Theorem 5.} Let \( n \geq k \ (n, k \in \mathbb{N}) \). Then,
\[
U_{2n} = \sum_{k=1}^{n} q(n,k),
\]
\[
U_{2n} = 2 \sum_{k=1}^{\lfloor n/2 \rfloor} q(n,2k) - 2 = 2 \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} q(n,2k+1) + 2.
\]

\textbf{Theorem 6.} Let \( n \geq k \ (n, k \in \mathbb{N}) \). Suppose also that \( q(n,k) \) is defined by (22). Then,
\[
k!q(n,k) = (2n)!3^{2n-k} \times \sum_{v_1,\ldots,v_n \in \mathbb{N}} \left( E_{2v_1-1}(0) - E_{2v_1-1} \left( \frac{2}{3} \right) \right) \times \left( E_{2v_1-1}(0) - E_{2v_1-1} \left( \frac{2}{3} \right) \right) \times \cdots \times ((2v_1)! \cdots (2v_k)!)^{-1}.
\]
\[ (28) \]

\textbf{Theorem 7.} Let \( n \in \mathbb{N} \). Then,
\[
-2 \sum_{k=0}^{n-1} \left( \frac{2n-1}{2k} \right) U_{2k} = 3^{2n-1} \left( E_{2n-1}(0) - E_{2n-1} \left( \frac{2}{3} \right) \right) \times \left( E_{2n-1}(0) - E_{2n-1} \left( \frac{2}{3} \right) \right) \times \cdots \times ((2v_1)! \cdots (2v_k)!)^{-1}. \]
\[ (29) \]

\textbf{Theorem 8.} Let \( n \in \mathbb{N} \). Then,
\[
U_{n+1} = \sum_{k=0}^{n-1} \binom{n}{k} \left( (1 - 2^{n-k}) U_{k+1} - 2^{n-k} U_k \right).
\]
\[ (30) \]

\textbf{Theorem 9.} Let \( n \in \mathbb{N}_0 \). Then,
\[
\sum_{n=0}^{\infty} \frac{1}{(n+1)!} U_n = \frac{1}{\sqrt{3}} \log \frac{2e - 1 - \sqrt{3}}{2(2 - \sqrt{3})e - 5 + 3\sqrt{3}}.
\]
\[
\sum_{k=1}^{\infty} q(n,k)(-1)^k = U_{2n}^{(-1)} = 2.
\]

\textbf{3. Proofs of Theorems}

\textit{Proof of Theorem 1.} By (10), (13), and (18), we have
\[
\sum_{n=0}^{\infty} U_{2n}^{(x)} \frac{t^{2n}}{(2n)!} = \left( \frac{1}{e^t + e^{-t} - 1} \right)^x = \left( \frac{1}{1 + (e^t + e^{-t} - 2)} \right)^x = \sum_{j=0}^{\infty} (-1)^j \binom{x+j-1}{j} (e^t + e^{-t} - 2)^j = \sum_{j=0}^{\infty} (-1)^j \binom{x+j-1}{j} (2j)! \sum_{n=j}^{\infty} T(n,j) \frac{t^{2n}}{(2n)!}
\]
\[
= \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \sum_{j=0}^{n} (-1)^j (2j)! \left( \binom{x+j-1}{j} T(n,j) \right),
\]
\[ (32) \]

which readily yields
\[
U_{2n}^{(x)} = \sum_{j=0}^{n} (-1)^j (2j)! \left( \binom{x+j-1}{j} T(n,j) \right)
\]
\[ (33) \]

This completes the proof of Theorem 1. \( \square \)

\textit{Proof of Theorem 5.} By (10), we have
\[
\sum_{n=0}^{\infty} U_{2n}^{(-1)} \frac{t^{2n}}{(2n)!} = e^t + e^{-t} - 1 = 2 \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} - 1,
\]
\[ (34) \]

and \( U_0^{(-1)} = 1 \), thus
\[
\sum_{n=1}^{\infty} U_{2n}^{(-1)} \frac{t^{2n}}{(2n)!} = e^t + e^{-t} - 1 = 2 \sum_{n=1}^{\infty} \frac{t^{2n}}{(2n)!}.
\]
\[ (35) \]

By Theorem 1 and comparing the coefficient of \( t^{2n} / (2n)! \) on both sides of (35), we get
\[
\sum_{n=k}^{\infty} \frac{1}{(n+1)!} U_n = \frac{1}{\sqrt{3}} \log \frac{2e - 1 - \sqrt{3}}{2(2 - \sqrt{3})e - 5 + 3\sqrt{3}}.
\]
\[ (31) \]

By Theorem 1 and comparing the coefficient of \( t^{2n} / (2n)! \) on both sides of (35), we get
\[
\sum_{k=1}^{\infty} q(n,k)(-1)^k = U_{2n}^{(-1)} = 2.
\]
\[ (36) \]
Again, by taking $x = 1$ in Theorem 1, we have

$$\sum_{k=1}^{n} q(n, k) = U_{2n}. \quad (37)$$

By (36) and (37), we immediately obtain (27). This completes the proof of Theorem 5. \hfill $\square$

**Proof of Theorem 6.** By applying Theorem 1, we have

$$k! q(n, k) = \frac{d^k}{dx^k} \left[ U_n^{(x)} \right] \bigg|_{x=0}. \quad (38)$$

On the other hand, it follows from (10) that

$$\sum_{n=k}^{\infty} \frac{d^k}{dx^k} \left[ U_n^{(x)} \right] \bigg|_{x=0} \frac{t^{2n}}{(2n)!} = \left( \log \frac{1}{e^t + e^{-t} - 1} \right)^k. \quad (39)$$

By using (38) and (39), we find that

$$k! \sum_{n=k}^{\infty} q(n, k) \frac{t^{2n}}{(2n)!} = \left( \log \frac{1}{e^t + e^{-t} - 1} \right)^k. \quad (40)$$

We now note that

$$\frac{d}{dt} \left\{ \log \left( \frac{1}{e^t + e^{-t} - 1} \right) \right\} = \frac{e^{-t} - e^t}{e^t + e^{-t} - 1}$$

$$= \frac{e^{-t} - e^t}{2} \left( \frac{2e^t}{e^{2t} + 1} + \frac{2e^{-t}}{e^{-2t} + 1} \right)$$

$$= \frac{1}{2} \left( 2 \frac{e^t}{e^{2t} + 1} - 2 \frac{e^{-t}}{e^{-2t} + 1} - 2\frac{e^{2t} - 2t}{e^{2t} + 1} \right)$$

$$= \frac{1}{2} \left( \sum_{n=0}^{\infty} E_n (0) \left( \frac{3t}{n!} \right)^n - \sum_{n=0}^{\infty} E_n \left( \frac{-3t}{n!} \right)^n \right)$$

$$- \frac{1}{2} \left( \sum_{n=0}^{\infty} E_n \left( \frac{2}{3} \right) \left( \frac{3t}{n!} \right)^n - \sum_{n=0}^{\infty} E_n \left( \frac{-3t}{n!} \right)^n \right)$$

$$= \sum_{n=0}^{\infty} \frac{2^{2n+1}}{n!} \left( E_{2n+1} (0) - E_{2n+1} \left( \frac{2}{3} \right) \right) \frac{t^{2n+1}}{(2n+1)!}. \quad (41)$$

Hence,

$$\log \frac{1}{e^t + e^{-t} - 1} = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{n!} \left( E_{2n+1} (0) - E_{2n+1} \left( \frac{2}{3} \right) \right) \frac{t^{2n+2}}{(2n+2)!}$$

$$= \sum_{n=0}^{\infty} \frac{2^{2n+1}}{n!} \left( E_{2n} (0) - E_{2n-1} \left( \frac{2}{3} \right) \right) \frac{t^{2n}}{(2n)!}. \quad (42)$$

yields

$$k! \sum_{n=k}^{\infty} q(n, k) \frac{t^{2n}}{(2n)!} = \left( \sum_{n=1}^{\infty} 2^{2n-1} \left( E_{2n-1} (0) - E_{2n-1} \left( \frac{2}{3} \right) \right) \frac{t^{2n}}{(2n)!} \right)^k$$

$$= \sum_{n=k}^{\infty} \frac{2^{2n}}{(2n)!} \left( 2n \sum_{n=0}^{\infty} U_n \frac{t^n}{n!} \right)$$

$$= \sum_{n=k}^{\infty} \frac{2^{2n}}{(2n)!} \left( n! \sum_{n=0}^{\infty} U_n \frac{t^n}{n!} \right)$$

$$= \sum_{n=k}^{\infty} \frac{2^{2n}}{(2n)!} \left( n! \sum_{n=0}^{\infty} U_n \frac{t^n}{n!} \right). \quad (43)$$

Comparing the coefficient of $t^{2n}/(2n)!$ on both sides of (43), we immediately get (28). This completes the proof of Theorem 6. \hfill $\square$

**Proof of Theorem 7.** Consider

$$\frac{d}{dt} \left\{ \log \left( \frac{1}{e^t + e^{-t} - 1} \right) \right\} = \frac{e^{-t} - e^t}{e^t + e^{-t} - 1}$$

$$= \sum_{n=0}^{\infty} U_n \frac{t^{2n}}{(2n)!} \left\{ -2 \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \right\}$$

$$= -2 \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left( \frac{2n+1}{2k} \right) U_{2k} \frac{t^{2n+1}}{(2n+1)!}. \quad (44)$$

Thus,

$$\log \frac{1}{e^t + e^{-t} - 1} = -2 \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \left( \frac{2n+1}{2k} \right) U_{2k} \frac{t^{2n+1}}{(2n+1)!}. \quad (45)$$

By (42) and (45) we obtain (29). This completes the proof of Theorem 7. \hfill $\square$

**Proof of Theorem 8.** By using (7), we have

$$\sum_{n=1}^{\infty} U_n \frac{t^{n-1}}{(n-1)!} = \frac{e^{-t} - e^t}{(e^t + e^{-t} - 1)^2}. \quad (46)$$

Thus

$$\left( e^{2t} - e^t + 1 \right) \sum_{n=1}^{\infty} U_n \frac{t^{n-1}}{(n-1)!} = \left( 1 - e^{2t} \right) \sum_{n=0}^{\infty} U_n \frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} \sum_{n=0}^{\infty} U_{n+1} \frac{t^n}{n!} + \sum_{n=0}^{\infty} U_{n+1} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} U_n \frac{t^n}{n!} - \sum_{n=0}^{\infty} \frac{2^n}{n!} \sum_{n=0}^{\infty} U_n \frac{t^n}{n!}. \quad (47)$$
That is,
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (2^{n-k} - 1) U_{k+1} \frac{t^n}{n!} + \sum_{n=0}^{\infty} U_{n+1} \frac{t^n}{n!} = \sum_{n=0}^{\infty} U_n \frac{t^n}{n!} - \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} U_k \frac{t^n}{n!}.
\] (48)

Comparing the coefficient of $t^n/n!$ on both sides of (48), we get the following:
\[
U_{n+1} - U_n = \sum_{k=0}^{n} \binom{n}{k} (1 - 2^{n-k}) U_{k+1} - 2^{n-k} U_k).
\] (49)

By (49) we immediately obtain (30). This completes the proof of Theorem 8.

Proof of Theorem 9. By integrating (7) with respect to $t$ from 0 to 1, we have
\[
\sum_{n=0}^{\infty} \frac{1}{(n+1)!} U_n = \int_0^1 \frac{1}{e^t + e^{-t} - 1} dt = \int_0^1 \frac{1}{e^{2t} - e^{-2t} + 1} dt = \int_1^e \frac{1}{x^2 - x + 1} dx.
\] (50)

By (50) and \[\int \frac{1}{(ax^2 + bx + c)} dx = \frac{1}{\sqrt{\Delta}} \log |(2ax + b - \sqrt{\Delta})/(2ax + b + \sqrt{\Delta})| + c \] (c is constant), we have (31). This completes the proof of Theorem 9.

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References
