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Research Article

Some Properties of a Sequence Similar to Generalized Euler Numbers

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We introduce the sequence $\{U_n^{(x)}\}$ given by generating function $(1/(e^t+e^{-t}-1))^x = \sum_{n=0}^{\infty} U_n^{(x)}(t^n/n!)$ $(|t| < (1/3)\pi, 1^x := 1)$ and establish some explicit formulas for the sequence $\{U_n^{(x)}\}$. Several identities involving the sequence $\{U_n^{(x)}\}$, Stirling numbers, Euler polynomials, and the central factorial numbers are also presented.

1. Introduction and Definitions

For a real or complex parameter α, the generalized Euler polynomials $E_n^{(\alpha)}(x)$ are defined by the following generating function (see [1–4])

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!} \quad \left(|t| < \pi, 1^{\alpha} := 1\right). \tag{1}$$

Obviously, we have

$$E_n^{(1)}(x) = E_n(x) \quad \left(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\right), \tag{2}$$

in terms of the classical Euler polynomials $E_n(x)$, $\mathbb{N}$ being the set of positive integers. The classical Euler numbers E_n are given by the following:

$$E_n = 2^n E_n\left(\frac{1}{2}\right) \quad (n \in \mathbb{N}_0). \tag{3}$$

The so-called the generalized Euler numbers $E_{2n}^{(x)}$ are defined by (see [3, 5])

$$\left(\frac{2}{e^t+e^{-t}}\right)^x = \sum_{n=0}^{\infty} E_{2n}^{(x)}\frac{t^{2n}}{(2n)!} \quad \left(|t| < \frac{\pi}{2}, 1^x := 1\right). \tag{4}$$

In fact, $E_{2n}^{(k)}$ $(k \in \mathbb{Z})$ are the Euler numbers of order k, $\mathbb{Z}$ being the set of integers. The numbers $E_{2n}^{(1)} = E_{2n}$ are the ordinary Euler numbers.

Zhi-Hong Sun introduces the sequence $\{U_n\}$ similar to Euler numbers as follows (see [6, 7]):

$$U_0 = 1, \qquad U_n = -2\sum_{k=1}^{[n/2]}\binom{n}{2k}U_{n-2k}, \quad (n \geq 1), \tag{5}$$

where (and in what follows) $[x]$ is the greatest integer not exceeding x.

Clearly, $U_{2n-1} = 0$ for $n \geq 1$. The first few values of U_{2n} are shown below

$$\begin{gathered} U_2 = -2, \quad U_4 = 22, \quad U_6 = -602, \quad U_8 = 30742, \\ U_{10} = -2523002, \quad U_{12} = 303692662. \end{gathered} \tag{6}$$

The sequence $\{U_n\}$ is related to the classical Bernoulli polynomials $B_n(x)$ (see [8–11]) and the classical Euler polynomials $E_n(x)$. Zhi-Hong Sun gets the generating function of

$\{U_n\}$ and deduces many identities involving $\{U_n\}$. As example, (see [6]),

$$\begin{aligned}\frac{1}{e^t+e^{-t}-1} &= \sum_{n=0}^{\infty} U_n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} U_{2n} \frac{t^{2n}}{(2n)!} \quad \left(|t| < \frac{1}{3}\pi\right),\end{aligned} \tag{7}$$

$$\frac{1}{2\cos t - 1} = \sum_{n=0}^{\infty} (-1)^n U_{2n} \frac{t^{2n}}{(2n)!} \quad \left(|t| < \frac{1}{3}\pi\right), \tag{8}$$

$$U_{2n} = 3^{2n} E_{2n}\left(\frac{1}{3}\right). \tag{9}$$

Similarly, we can define the generalized sequence $\{U_n^{(x)}\}$. For a real or complex parameter x, the generalized sequence $\{U_n^{(x)}\}$ is defined by the following generating function:

$$\left(\frac{1}{e^t+e^{-t}-1}\right)^x = \sum_{n=0}^{\infty} U_n^{(x)} \frac{t^n}{n!} \quad \left(|t| < \frac{1}{3}\pi, 1^x := 1\right). \tag{10}$$

Obviously,

$$U_0^{(x)} = 1, \qquad U_n^{(1)} = U_n \quad (n \in \mathbb{N}). \tag{11}$$

By using (10), we can obtain

$$U_n^{(k)} = n! \sum_{v_1,\ldots,v_k \in \mathbb{N}_0}^{(v_1+\cdots+v_k=n)} \frac{U_{v_1}\cdots U_{v_k}}{v_1!\cdots v_k!} \quad (k \in \mathbb{N}). \tag{12}$$

We now return to the Stirling numbers $s(n,k)$ of the first kind, which are usually defined by (see [2, 5, 8, 11, 12])

$$x(x-1)(x-2)\cdots(x-n+1) = \sum_{k=0}^{n} s(n,k)\, x^k \tag{13}$$

or by the following generating function:

$$(\log(1+x))^k = k! \sum_{n=k}^{\infty} s(n,k) \frac{x^n}{n!}. \tag{14}$$

It follows from (13) or (14) that

$$s(n,k) = s(n-1, k-1) - (n-1)\, s(n-1, k) \tag{15}$$

and that

$$\begin{gathered} s(n,0) = 0 \quad (n \in \mathbb{N}), \qquad s(n,n) = 1 \quad (n \in \mathbb{N}_0), \\ s(n,1) = (-1)^{n-1}(n-1)! \quad (n \in \mathbb{N}), \\ s(n,k) = 0 \quad (k > n \text{ or } k < 0). \end{gathered} \tag{16}$$

The central factorial numbers $T(n,k)$ are given by the following expansion formula (see [3, 5, 13]):

$$\begin{aligned} x^n = \sum_{k=0}^{n} T(n,k)\, x\left(x-1^2\right) \\ \times\left(x-2^2\right)\cdots\left(x-(k-1)^2\right) \end{aligned} \tag{17}$$

or by means of the generating function

$$\left(e^x+e^{-x}-2\right)^k = (2k)! \sum_{n=k}^{\infty} T(n,k) \frac{x^{2n}}{(2n)!}. \tag{18}$$

It follows from (17) or (18) that

$$T(n,k) = T(n-1, k-1) + k^2 T(n-1, k), \tag{19}$$

with

$$\begin{gathered} T(0,0) = 1, \qquad T(n,0) = 0 \quad (n \in \mathbb{N}), \\ T(n,1) = 1 \quad (n \in \mathbb{N}). \end{gathered} \tag{20}$$

We also find from (18) that

$$\begin{gathered} T(n,2) = \frac{1}{4}\left(4^{n-1}-1\right), \\ T(n,3) = \frac{9^n}{360} - \frac{4^n}{60} + \frac{1}{24} \quad (n \in \mathbb{N}). \end{gathered} \tag{21}$$

The main purpose of this paper is to prove some formulas for the generalized sequence $\{U_n^{(x)}\}$ and $E_n(x)$. Some identities involving the sequence $\{U_n^{(x)}\}$, Stirling numbers $s(n,k)$, and the central factorial numbers $T(n,k)$ are deduced.

2. Main Results

Theorem 1. *Let $n \geq k$ $(n,k \in \mathbb{N})$ and*

$$q(n,k) = (-1)^k \sum_{j=k}^{n} \frac{(2j)!}{j!} T(n,j)\, s(j,k). \tag{22}$$

Then,

$$U_{2n}^{(x)} = \sum_{k=1}^{n} q(n,k)\, x^k. \tag{23}$$

Remark 2. By (15), (19), (20), and Theorem 1, we know that $U_{2n}^{(x)}$ is a polynomial of x with integral coefficients. For example, by setting $n = 1, 2, 3, 4$ in Theorem 1, we get

$$\begin{gathered} U_2^{(x)} = -2x, \qquad U_4^{(x)} = 10x + 12x^2, \\ U_6^{(x)} = -182x - 300x^2 - 120x^3, \\ U_8^{(x)} = 6970x + 13692x^2 + 8400x^3 + 1680x^4. \end{gathered} \tag{24}$$

Taking $x = 1$ in Theorem 1, we can obtain the following.

Corollary 3. *Let $n \in \mathbb{N}$. Then,*

$$U_{2n} = \sum_{j=0}^{n} (-1)^j (2j)! T(n,j). \tag{25}$$

From Corollary 3, we may immediately deduce the following results.

Corollary 4. *Let $n \in \mathbb{N}$. Then,*

$$\begin{aligned} U_{2n} &\equiv -2 \pmod{24}, \\ U_{2n} &\equiv -2 + 24T(n,2) \pmod{720}, \\ U_{2n} &\equiv -2 + 24T(n,2) - 720T(n,3) \pmod{40320}. \end{aligned} \tag{26}$$

Theorem 5. *Let $n \geq k$ $(n, k \in \mathbb{N})$. Then,*

$$\begin{aligned} U_{2n} &= \sum_{k=1}^{n} q(n,k), \\ U_{2n} &= 2\sum_{k=1}^{[n/2]} q(n,2k) - 2 \\ &= 2\sum_{k=1}^{[(n-1)/2]} q(n,2k+1) + 2. \end{aligned} \tag{27}$$

Theorem 6. *Let $n \geq k$ $(n, k \in \mathbb{N})$. Suppose also that $q(n,k)$ is defined by* (22). *Then,*

$$\begin{aligned} k!q(n,k) = {} & (2n)!3^{2n-k} \\ & \times \sum_{v_1,\ldots,v_k \in \mathbb{N}}^{(v_1+\cdots+v_k=n)} \left(E_{2v_1-1}(0) - E_{2v_1-1}\left(\frac{2}{3}\right)\right) \\ & \cdots \left(E_{2v_k-1}(0) - E_{2v_k-1}\left(\frac{2}{3}\right)\right) \\ & \times \left((2v_1)!\cdots(2v_k)!\right)^{-1}. \end{aligned} \tag{28}$$

Theorem 7. *Let $n \in \mathbb{N}$. Then,*

$$-2\sum_{k=0}^{n-1}\binom{2n-1}{2k}U_{2k} = 3^{2n-1}\left(E_{2n-1}(0) - E_{2n-1}\left(\frac{2}{3}\right)\right). \tag{29}$$

Theorem 8. *Let $n \in \mathbb{N}$. Then,*

$$U_{n+1} = \sum_{k=0}^{n-1}\binom{n}{k}\left(\left(1-2^{n-k}\right)U_{k+1} - 2^{n-k}U_k\right). \tag{30}$$

Theorem 9. *Let $n \in \mathbb{N}_0$. Then,*

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)!}U_n = \frac{1}{\sqrt{3}}\log\frac{2e-1-\sqrt{3}}{2\left(2-\sqrt{3}\right)e-5+3\sqrt{3}}. \tag{31}$$

3. Proofs of Theorems

Proof of Theorem 1. By (10), (13), and (18), we have

$$\begin{aligned} \sum_{n=0}^{\infty}U_{2n}^{(x)}\frac{t^{2n}}{(2n)!} &= \left(\frac{1}{e^t+e^{-t}-1}\right)^x \\ &= \left(\frac{1}{1+(e^t+e^{-t}-2)}\right)^x \\ &= \sum_{j=0}^{\infty}(-1)^j\binom{x+j-1}{j}\left(e^t+e^{-t}-2\right)^j \\ &= \sum_{j=0}^{\infty}(-1)^j\binom{x+j-1}{j}(2j)!\sum_{n=j}^{\infty}T(n,j)\frac{t^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}\sum_{j=0}^{n}(-1)^j(2j)!\binom{x+j-1}{j}T(n,j), \end{aligned} \tag{32}$$

which readily yields

$$\begin{aligned} U_{2n}^{(x)} &= \sum_{j=0}^{n}(-1)^j(2j)!\binom{x+j-1}{j}T(n,j) \\ &= \sum_{j=0}^{n}(-1)^j(2j)!T(n,j)\frac{1}{j!}x(x+1)\cdots(x+j-1) \\ &= \sum_{j=0}^{n}\frac{(2j)!}{j!}T(n,j)\sum_{k=1}^{j}(-1)^k s(j,k)x^k \\ &= \sum_{k=1}^{n}(-1)^k\sum_{j=k}^{n}\frac{(2j)!}{j!}T(n,j)s(j,k)x^k \\ &= \sum_{k=1}^{n}q(n,k)x^k. \end{aligned} \tag{33}$$

This completes the proof of Theorem 1. □

Proof of Theorem 5. By (10), we have

$$\sum_{n=0}^{\infty}U_{2n}^{(-1)}\frac{t^{2n}}{(2n)!} = e^t+e^{-t}-1 = 2\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!} - 1, \tag{34}$$

and $U_0^{(x)} = 1$, thus

$$\sum_{n=1}^{\infty}U_{2n}^{(-1)}\frac{t^{2n}}{(2n)!} = e^t+e^{-t}-1 = 2\sum_{n=1}^{\infty}\frac{t^{2n}}{(2n)!}. \tag{35}$$

By Theorem 1 and comparing the coefficient of $t^{2n}/(2n)!$ on both sides of (35), we get

$$\sum_{k=1}^{n}q(n,k)(-1)^k = U_{2n}^{(-1)} = 2. \tag{36}$$

Again, by taking $x = 1$ in Theorem 1, we have

$$\sum_{k=1}^{n} q(n,k) = U_{2n}. \tag{37}$$

By (36) and (37), we immediately obtain (27). This completes the proof of Theorem 5. □

Proof of Theorem 6. By applying Theorem 1, we have

$$k!q(n,k) = \frac{d^k}{dx^k}\left\{U_n^{(x)}\right\}\Big|_{x=0}. \tag{38}$$

On the other hand, it follows from (10) that

$$\sum_{n=k}^{\infty} \frac{d^k}{dx^k}\left\{U_n^{(x)}\right\}\Big|_{x=0} \frac{t^{2n}}{(2n)!} = \left(\log\left(\frac{1}{e^t+e^{-t}-1}\right)\right)^k. \tag{39}$$

By using (38) and (39), we find that

$$k!\sum_{n=k}^{\infty} q(n,k)\frac{t^{2n}}{(2n)!} = \left(\log\left(\frac{1}{e^t+e^{-t}-1}\right)\right)^k. \tag{40}$$

We now note that

$$\begin{aligned}
&\frac{d}{dt}\left\{\log\left(\frac{1}{e^t+e^{-t}-1}\right)\right\} \\
&\quad= \frac{e^{-t}-e^t}{e^t+e^{-t}-1} \\
&\quad= \frac{e^{-t}-e^t}{2}\left(\frac{2e^t}{e^{3t}+1}+\frac{2e^{-t}}{e^{-3t}+1}\right) \\
&\quad= \frac{1}{2}\left(\left(\frac{2}{e^{3t}+1}-\frac{2}{e^{-3t}+1}\right)-\left(\frac{2e^{2t}}{e^{3t}+1}-\frac{2e^{-2t}}{e^{-3t}+1}\right)\right) \\
&\quad= \frac{1}{2}\left(\sum_{n=0}^{\infty}E_n(0)\frac{(3t)^n}{n!}-\sum_{n=0}^{\infty}E_n(0)\frac{(-3t)^n}{n!}\right) \\
&\quad- \frac{1}{2}\left(\sum_{n=0}^{\infty}E_n\left(\frac{2}{3}\right)\frac{(3t)^n}{n!}-\sum_{n=0}^{\infty}E_n\left(\frac{2}{3}\right)\frac{(-3t)^n}{n!}\right) \\
&\quad= \sum_{n=0}^{\infty}3^{2n+1}\left(E_{2n+1}(0)-E_{2n+1}\left(\frac{2}{3}\right)\right)\frac{t^{2n+1}}{(2n+1)!}.
\end{aligned} \tag{41}$$

Hence,

$$\begin{aligned}
\log\frac{1}{e^t+e^{-t}-1} &= \sum_{n=0}^{\infty}3^{2n+1}\left(E_{2n+1}(0)-E_{2n+1}\left(\frac{2}{3}\right)\right)\frac{t^{2n+2}}{(2n+2)!} \\
&= \sum_{n=1}^{\infty}3^{2n-1}\left(E_{2n-1}(0)-E_{2n-1}\left(\frac{2}{3}\right)\right)\frac{t^{2n}}{(2n)!}
\end{aligned} \tag{42}$$

yields

$$\begin{aligned}
&k!\sum_{n=k}^{\infty}q(n,k)\frac{t^{2n}}{(2n)!} \\
&\quad= \left(\sum_{n=1}^{\infty}3^{2n-1}\left(E_{2n-1}(0)-E_{2n-1}\left(\frac{2}{3}\right)\right)\frac{t^{2n}}{(2n)!}\right)^k \\
&\quad= \sum_{n=k}^{\infty}\frac{t^{2n}}{(2n)!}(2n)!3^{2n-k} \\
&\qquad\times\sum_{v_1,\dots,v_k\in\mathbb{N}}^{(v_1+\cdots+v_k=n)}\left(E_{2v_1-1}(0)-E_{2v_1-1}\left(\frac{2}{3}\right)\right) \\
&\qquad\cdots\left(E_{2v_k-1}(0)-E_{2v_k-1}\left(\frac{2}{3}\right)\right) \\
&\qquad\times\left((2v_1)!\cdots(2v_k)!\right)^{-1}.
\end{aligned} \tag{43}$$

Comparing the coefficient of $t^{2n}/(2n)!$ on both sides of (43), we immediately get (28). This completes the proof of Theorem 6. □

Proof of Theorem 7. Consider

$$\begin{aligned}
\frac{d}{dt}\left\{\log\left(\frac{1}{e^t+e^{-t}-1}\right)\right\} &= \frac{e^{-t}-e^t}{e^t+e^{-t}-1} \\
&= \sum_{n=0}^{\infty}U_{2n}\frac{t^{2n}}{(2n)!}\left(-2\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\right) \\
&= -2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2n+1}{2k}U_{2k}\frac{t^{2n+1}}{(2n+1)!}.
\end{aligned} \tag{44}$$

Thus,

$$\log\frac{1}{e^t+e^{-t}-1} = -2\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\binom{2n-1}{2k}U_{2k}\frac{t^{2n}}{(2n)!}. \tag{45}$$

By (42) and (45) we obtain (29). This completes the proof of Theorem 7. □

Proof of Theorem 8. By using (7), we have

$$\sum_{n=1}^{\infty}U_n\frac{t^{n-1}}{(n-1)!} = \frac{e^{-t}-e^t}{(e^t+e^{-t}-1)^2}. \tag{46}$$

Thus

$$\begin{aligned}
&\left(e^{2t}-e^t+1\right)\sum_{n=1}^{\infty}U_n\frac{t^{n-1}}{(n-1)!} = \left(1-e^{2t}\right)\sum_{n=0}^{\infty}U_n\frac{t^n}{n!}, \\
&\sum_{n=0}^{\infty}\left(2^n-1\right)\frac{t^n}{n!}\sum_{n=0}^{\infty}U_{n+1}\frac{t^n}{n!}+\sum_{n=0}^{\infty}U_{n+1}\frac{t^n}{n!} \\
&\quad= \sum_{n=0}^{\infty}U_n\frac{t^n}{n!}-\sum_{n=0}^{\infty}2^n\frac{t^n}{n!}\sum_{n=0}^{\infty}U_n\frac{t^n}{n!}.
\end{aligned} \tag{47}$$

That is,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left(2^{n-k}-1\right)U_{k+1}\frac{t^n}{n!}+\sum_{n=0}^{\infty}U_{n+1}\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}U_n\frac{t^n}{n!}-\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}2^{n-k}U_k\frac{t^n}{n!}. \tag{48}$$

Comparing the coefficient of $t^n/n!$ on both sides of (48), we get the following:

$$U_{n+1}-U_n=\sum_{k=0}^{n}\binom{n}{k}\left(\left(1-2^{n-k}\right)U_{k+1}-2^{n-k}U_k\right). \tag{49}$$

By (49) we immediately obtain (30). This completes the proof of Theorem 8. □

Proof of Theorem 9. By integrating (7) with respect to t from 0 to 1, we have

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)!}U_n=\int_0^1\frac{1}{e^t+e^{-t}-1}dt$$
$$=\int_0^1\frac{1}{e^{2t}-e^t+1}de^t=\int_1^e\frac{1}{x^2-x+1}dx. \tag{50}$$

By (50) and $\int(1/(ax^2+bx+c))dx=(1/\sqrt{b^2-4ac})\log|(2ax+b-\sqrt{b^2-4ac})/(2ax+b+\sqrt{b^2-4ac})|+c$ (c is constant), we have (31). This completes the proof of Theorem 9. □

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