Research Article

On the Mazur-Ulam Theorem in Non-Archimedean Fuzzy \( n \)-Normed Spaces

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The motivation of this paper is to present a new notion of non-Archimedean fuzzy \( n \)-normed space over a field with valuation. We obtain a Mazur-Ulam theorem for fuzzy \( n \)-isometric mappings in the strictly convex non-Archimedean fuzzy \( n \)-normed spaces. We also prove that the interior preserving mapping carries the barycenter of a triangle to the barycenter point of the corresponding triangle. And then, using this result, we get a Mazur-Ulam theorem for the interior preserving fuzzy \( n \)-isometry mappings in non-Archimedean fuzzy \( n \)-normed spaces over a linear ordered non-Archimedean field.

1. Introduction

Let \( \mathbb{K} \) be a field. A valuation mapping on \( \mathbb{K} \) is a function \( | \cdot | : \mathbb{K} \to \mathbb{R} \) such that for any \( r, s \in \mathbb{K} \) the following conditions are satisfied: (i) \( |r| \geq 0 \) and equality holds if and only if \( r = 0 \); (ii) \( |rs| = |r| \cdot |s| \); (iii) \( |r + s| \leq |r| + |s| \).

A field endowed with a valuation mapping will be called a valued field. The usual absolute values of \( \mathbb{R} \) and \( \mathbb{C} \) are examples of valuations. A trivial example of a non-Archimedean valuation is the function \( | \cdot | \) taking everything except for 0 into 1 and \( |0| = 0 \). In the following, we will assume that \( | \cdot | \) is nontrivial; that is, there is an \( r_0 \in \mathbb{K} \) such that \( |r_0| \neq 0, 1 \).

Throughout this paper, we assume that \( \mathbb{K} \) is a valued field and \( n \geq 2 \) is a positive integer. We denote the set of all elements of \( \mathbb{K} \) whose norms are 1 by \( \mathbb{C} \); that is, \( \mathbb{C} = \{ r \in \mathbb{K} \mid |r| = 1 \} \). Moreover, \( \mathbb{N} \) stands for the set of all positive integers and \( \mathbb{R} \) (resp., \( \mathbb{C} \)) denotes the set of all real numbers (resp., complex numbers).

If condition (iii) in the definition of a valuation mapping is replaced with a strong triangle inequality (ultrametric), \( |r + s| \leq \max\{|r|, |s|\} \), then the valuation \( | \cdot | \) is said to be non-Archimedean. In any non-Archimedean field, we have \( |1| = |−1| = 1 \) and \( |k| \leq 1 \) for all \( k \in \mathbb{N} \).

Let \( \mathcal{X} \) and \( \mathcal{Y} \) be metric spaces. A map \( f : \mathcal{X} \to \mathcal{Y} \) is called a distance preserving mapping (isometry) if \( d(f(x), f(y)) = d(x, y) \) for any \( x, y \in \mathcal{X} \). Automatically, an isometry is injective. Two metric spaces \( \mathcal{X} \) and \( \mathcal{Y} \) are called isometric if there is an isometry from \( \mathcal{X} \) to \( \mathcal{Y} \).

The classical result of Mazur and Ulam states that if \( \mathcal{X}, \mathcal{Y} \) are real normed linear spaces and \( f : \mathcal{X} \to \mathcal{Y} \) is a surjective isometry, then \( f \) is affine; that is, \( f \) is a linear mapping up to translation. Numerous generalizations of this fact were presented by many authors (see, e.g., [1–13] and references therein). Unfortunately, the Mazur-Ulam theorem is not true for normed complex vector space. It was a natural step to ask if the theorem holds without the onto assumption. In fact, the onto assumption is essential. Without this assumption, Baker [14] proved that every isometry, not necessary surjective, \( f : \mathcal{X} \to \mathcal{Y} \), between real normed linear spaces is affine if \( \mathcal{Y} \) is strictly convex. Moslehian and Sadeghi presented a non-Archimedean version of this result [11]; they also noted that a Mazur-Ulam theorem generally fails in a non-Archimedean case. Choi et al. [3] proved the Mazur-Ulam theorem for the interior preserving mappings in linear 2-normed spaces; they also proved the theorem on non-Archimedean 2-normed spaces over a linear ordered non-Archimedean field without the strict convexity assumption. Chu et al. [4] studied the Mazur-Ulam theorem in linear \( n \)-normed spaces. Alaca [1] introduced the concepts of 2-isometry, collinearity, and 2-Lipschitz mapping in 2-fuzzy 2-normed linear spaces. Also, he gave a new generalization of the Mazur-Ulam theorem when \( \mathcal{X} \) is a 2-fuzzy 2-normed linear space or \( \mathcal{X}(X) \) is a fuzzy 2-normed linear space. Kubzdela [10] gave some new results...
for isometries, Mazur-Ulam theorem, and Aleksandrov problem in the framework of non-Archimedean normed spaces. Kang et al. [9] proved that the Mazur-Ulam theorem holds under some conditions in non-Archimedean fuzzy normed space.

The motivation of this paper is to introduce the notion of non-Archimedean fuzzy n-normed space over a field with valuation as a generalization of n-normed space [2, 15, 16], non-Archimedean 2-normed space [3], fuzzy n-normed space [17], and non-Archimedean fuzzy normed space [9, 18]. We will prove that the Mazur-Ulam theorem holds in the strictly convex non-Archimedean fuzzy n-normed spaces.

2. Preliminaries

In 1897, Hensel discovered the p-adic numbers as a number-theoretical analogue of power series in complex analysis. Let p be a prime number. For any nonzero rational number a, there exists a unique integer r such that a = \( p^r m/k \), where m and k are integers not divisible by p. Then \( |a|_p := p^{-r} \) defines a non-Archimedean norm on \( \mathbb{Q} \). The completion of \( \mathbb{Q} \) with respect to the metric \( d(a, b) = |a-b|_p \), denoted by \( \mathbb{Q}_p \), is called the p-adic number field. Note that if \( p > 2 \), then \( |2^k|_p = 1 \) for each integer \( k \) but \( |2|_2 < 1 \).

During the last three decades, p-adic numbers have gained the interest of physicists for their research, in particular, in problems derived from quantum physics, p-adic strings, and superstrings (see, e.g., [19]).

Definition 1. Let \( \mathcal{X} \) be a linear space over a field \( \mathbb{K} \) with a non-Archimedean valuation \( |\cdot| \). A function \( \|\cdot\|: \mathcal{X} \rightarrow [0, \infty) \) is said to be a non-Archimedean norm if it satisfies the following conditions:

(i) \( \|x\| = 0 \) if and only if \( x = 0 \),

(ii) \( \|rx\| = |r|\|x\|, r \in \mathbb{K}, x \in \mathcal{X} \),

(iii) the strong triangle inequality: \( \|x + y\| \leq \max\{|\|x\||, |\|y\||\} (x, y \in \mathcal{X}) \).

Then \( (\mathcal{X}, \|\cdot\|) \) is called a non-Archimedean normed space. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent.

Definition 2. Let \( \mathcal{X} \) be a linear space over a valued field \( \mathbb{K} \). A function \( N: \mathcal{X}^n \times \mathbb{R} \rightarrow [0, 1] \) is called a non-Archimedean fuzzy n-norm on \( \mathcal{X} \) if the following conditions hold for all \( x, y, x_1, \ldots, x_n \in \mathcal{X} \) and all \( s, t \in \mathbb{R} \):

(\( nN_1 \)) \( N(x_1, \ldots, x_n, t) = 0 \) if \( t \leq 0 \),

(\( nN_2 \)) for all \( t > 0 \), \( N(x_1, \ldots, x_n, t) = 1 \) if and only if \( x_1, \ldots, x_n \) are linearly dependent,

(\( nN_3 \)) \( N(x_1, \ldots, x_n, t) \) is invariant under any permutation of \( x_1, \ldots, x_n \),

(\( nN_4 \)) \( N(cx_1, \ldots, cx_n, t) = N(x_1, \ldots, x_n, ct) \) for all \( t > 0 \) if \( c \in \mathbb{K}, c \neq 0 \),

(\( nN_5 \)) \( N(x + y, x_2, \ldots, x_n, \max\{s, t\}) \geq \min\{N(x, x_2, \ldots, x_n, s), N(y, x_2, \ldots, x_n, t)\} \) for all \( s, t > 0 \),

(\( nN_6 \)) \( \lim_{t \to \infty} N(x_1, \ldots, x_n, t) = 1 \).

If \( N \) is a non-Archimedean fuzzy n-norm on \( \mathcal{X} \), then \( (\mathcal{X}, N) \) is called a non-Archimedean fuzzy n-normed space. It should be noticed that from the condition \((nN_5)\) it follows that

\[
N(x_1, \ldots, x_n, t) \geq \min\{N(0, \ldots, 0, 0), N(x_1, \ldots, x_n, s)\} = N(x_1, \ldots, x_n, s)
\]

for every \( t > s > 0 \) and \( x_1, \ldots, x_n \in \mathcal{X} \); that is, \( N(x_1, \ldots, x_n, t) \) is nondecreasing for every \( x_1, \ldots, x_n \in \mathcal{X} \). This implies that

\[
N(x_1, \ldots, x_n, s + t) \geq N(x_1, \ldots, x_n, \max\{s, t\}) , \quad s, t > 0.
\]

If \((nN_5)\) holds, then so is

(\( nN_7 \)) \( N(x, y, x_2, \ldots, x_n, s + t) \geq \min\{N(x, x_2, \ldots, x_n, s), N(y, x_2, \ldots, x_n, t)\}\).

Example 3. Let \((\mathcal{X}, [1, \ldots, n])\) be an n-normed space (see [2]). For each \( k \in \mathbb{N} \), consider

\[
N_k(x_1, \ldots, x_n, t) = \begin{cases} \frac{t}{t + k\|x_1, \ldots, x_n\|}, & t > 0, \\ 0, & t \leq 0. \end{cases}
\]

Then \((\mathcal{X}, N_k)\) is a non-Archimedean fuzzy n-normed space.

Definition 4. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be non-Archimedean fuzzy n-normed spaces, and let \( f: \mathcal{X} \rightarrow \mathcal{Y} \) be a mapping. We call \( f \) a fuzzy n-isometry if

\[
N_k(x_1 - x_0, \ldots, x_n - x_0, t) = N(f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0), t)
\]

for all \( x_0, x_1, \ldots, x_n \in \mathcal{X} \) and all \( t > 0 \).

For given points \( x, y, \) and \( z \) in \( \mathcal{X} \), \( \Delta xyz \) denotes the triangle determined by \( x, y, \) and \( z \). A point \((x + y + z)/3\) is called a barycenter of \( \Delta xyz \). If \( p \) is a point of a set \( \{t_1x + t_2y + t_3z \mid t_1 + t_2 + t_3 = 1, t_i \in \mathbb{K}, t_i > 0, i = 1, 2, 3\} \), then \( p \) is called an interior point of \( \Delta xyz \). Define a mapping \( f \) between linear n-normed spaces to be an interior preserving mapping of the triangle if \( f \) carries an interior point in a triangle to an interior point in the corresponding triangle.

Remark 5. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be non-Archimedean fuzzy n-normed spaces and \( f: \mathcal{X} \rightarrow \mathcal{Y} \) be a mapping. Then \( f \) is a fuzzy n-isometry if and only if \( f \) satisfies the following property:

\[
\left| N(x_1 - x_0, \ldots, x_n - x_0, t) - N(x'_1 - x'_0, \ldots, x'_n - x'_0, t) \right| = \left| N(f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0), t) - N(f(x'_1) - f(x'_0), \ldots, f(x'_n) - f(x'_0), t) \right|
\]

for all \( x_0, x_1, \ldots, x_n, x'_0, x'_1, \ldots, x'_n \in \mathcal{X} \) and all \( t > 0 \).
Definition 6. Let $X$ and $Y$ be non-Archimedean fuzzy $n$-normed spaces and $f : X \to Y$ be a mapping. Then $f$ is called a weak fuzzy $n$-isometry if for every $e > 0$, there exists positive real number $\delta$ such that

$$|N(x_1 - x_0, \ldots, x_n - x_0, t) - N(x'_1 - x'_0, \ldots, x'_n - x'_0, t)| < \delta$$

implies

$$|N(f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0), t)| < e$$

for all $x_0, x_1, \ldots, x_n, x'_0, x'_1, \ldots, x'_n \in X$ and all $t > 0$.

Definition 7. Let $X$ be a non-Archimedean fuzzy $n$-normed space. The points $x_0, x_1, \ldots, x_n$ are said to be $n$-collinear if for every $i \in \{0, 1, \ldots, n\}$, $|x_j - x_i| \leq e$ implies that $x_j = x_i$ for all $i \neq j$. Then $x_0, x_1, \ldots, x_n$ are said to be 2-collinear if and only if $x_2 - x_0 = t(x_1 - x_0)$ for some $t \in \mathbb{R}^n$.

Now we define the concept of $n$-Lipschitz mapping.

Definition 8. Let $X$ and $Y$ be non-Archimedean fuzzy $n$-normed spaces, and let $f : X \to Y$ be a mapping. Then $f$ is called a fuzzy $n$-Lipschitz mapping if there is a $L \geq 0$ such that

$$N(f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0), Lt) \geq N(x_1 - x_0, \ldots, x_n - x_0, t)$$

for all $x_0, x_1, \ldots, x_n \in X$ and all $t > 0$. The smallest such $L$ is called the $n$-Lipschitz constant.

Definition 9. A non-Archimedean fuzzy $n$-normed space $X$ over a valued field $\mathbb{K}$ is called strictly convex, if for each $x_1, \ldots, x_m, w_2, \ldots, w_n \in X$ and $s_1, \ldots, s_m > 0$,

$$N\left(\sum_{j=1}^{m} s_j w_2, \ldots, w_n \max\{s_1, \ldots, s_m\}\right) = \min\{N(x_1, w_2, \ldots, w_n, s_1), \ldots, N(x_m, w_2, \ldots, w_n, s_m)\}$$

implies that $x_1 = \cdots = x_m$ and $s_1 = \cdots = s_m$.

3. On the Mazur-Ulam Problem

Lemma II. Let $X$ be a non-Archimedean fuzzy $n$-normed space over a valued field $\mathbb{K}$, $x_1, \ldots, x_n \in X$ and all $t > 0$. Then

$$N(x_1, \ldots, x_n, t) = N(x_1, \ldots, x_j + \alpha x_j, x_{j+1}, \ldots, x_n, t)$$

for all $\alpha \in \mathbb{K}$.

Proof. Let $x_1, \ldots, x_n \in X$, $t > 0$, and $\alpha \in \mathbb{K}$, then

$$N(x_1, \ldots, x_j + \alpha x_j, x_{j+1}, \ldots, x_n, t) \geq \min\{N(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n, t), N\left(\frac{x_1 + y}{2}, \ldots, \frac{x_j + y}{2}, \ldots, \frac{x_n - y}{2}, t\right)\}$$

implies $x_1 = \cdots = x_n$ and $s_1 = \cdots = s_m$.
Hence, the existence holds. For the uniqueness of \( u \), we may assume that there is another \( v \in \mathcal{X} \), collinear with \( x, y \) such that

\[
N \left( x - v, x_2 - v, \ldots, x_n - v, t \right) = N \left( y - v, x_2 - v, \ldots, x_n - v, t \right) = N \left( x - y, x_2 - x, \ldots, x_n - x, t \right).
\] (13)

Since \( x, y \), and \( v \) are collinear, \( v = sx + (1-s)y \) for some \( s \in \mathbb{R} \).

We may assume \( s \neq 0 \) and \( s \neq 1 \). Then

\[
N \left( x - y, x_2 - x, \ldots, x_n - x, t \right) = N \left( x - v, x_2 - v, \ldots, x_n - v, t \right) = N \left( y - v, x_2 - v, \ldots, x_n - v, t \right).
\]

(14)

Then

\[
N \left( x - y + x - y, x_2 - x - y, \ldots, x_n - x, \max \left\{ \frac{t}{|s|}, \frac{t}{1 - |s|} \right\} \right) = N \left( x - y, x_2 - x, \ldots, x_n - x, \frac{1}{2} \max \left\{ \frac{t}{|s|}, \frac{t}{1 - |s|} \right\} \right) = N \left( x - y, x_2 - x, \ldots, x_n - x, \frac{t}{|s|} \right) = N \left( x - y, x_2 - x, \ldots, x_n - x, \frac{t}{|s|} \right).
\]

(15)

By the strict convexity of \( \mathcal{X} \), we have \( |1 - s| = |s| = 1 \). Then there exist two integers \( k_1, k_2 \) such that \( 1 - s = 2^k_1 \) and \( s = 2^k_2 \). Since \( 2^k_1 + 2^k_2 = 1 \), we know that \( k_1 < 0, k_2 < 0 \). Without loss of generality, we let \( 1 - s = 2^{-n_1} \) and \( s = 2^{-n_2} \) with \( n_1 \geq n_2 \). If \( n_1 > n_2 \), then \( 1 = 2^{-n_1} + 2^{-n_2} = 2^{-n_1} \left( 1 + 2^{n_1-n_2} \right) \); that is, \( 2^{-n_1} = 2^{n_2-n_2} \). This is a contradiction. Thus, \( n_1 = n_2 \); that is, \( s = 1/2 \). This completes the proof.

\[ \square \]

**Lemma 13.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be non-Archimedean fuzzy \( n \)-normed spaces over a valued field \( \mathbb{K} \). If \( f : \mathcal{X} \to \mathcal{Y} \) is a fuzzy \( n \)-isometry and \( x_0, x_1, \) and \( x_2 \) are 2-collinear, then \( f(x_0), f(x_1), \) and \( f(x_2) \) are 2-collinear.

**Proof.** Since \( \dim \mathcal{X} \geq n \), for any \( x_0 \in \mathcal{X} \), there exist \( y_1, \ldots, y_n \in \mathcal{X} \) such that \( y_1 - x_0, \ldots, y_n - x_0 \) are linearly independent. Then

\[
N \left( y_1 - x_0, \ldots, y_n - x_0, t \right) = N \left( f(y_1) - f(x_0), \ldots, f(y_n) - f(x_0), t \right) \neq 1,
\]

(16)

and hence, the set \( \{ f(x) - f(x_0) : x \in \mathcal{X} \} \) contains \( n \) linearly independent vectors. Assume that \( x_0, x_1, \) and \( x_2 \) are 2-collinear. Then, for any \( x_3, \ldots, x_n \in \mathcal{X} \),

\[
N \left( x_1 - x_0, \ldots, x_n - x_0, t \right) = N \left( f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0), t \right) = 1.
\]

(17)

By \( (nN_2) \), it follows that \( f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0) \) are linearly dependent. If there exist \( x_3, \ldots, x_{n-1} \) such that \( f(x_1) - f(x_0), \ldots, f(x_n-1) - f(x_0) \) are linearly independent, then

\[
\{ f(x) - f(x_0) : x \in \mathcal{X} \} \subset \operatorname{span} \{ f(x_1) - f(x_0), \ldots, f(x_{n-1}) - f(x_0) \}
\]

which contradicts the fact that \( \{ f(x) - f(x_0) : x \in \mathcal{X} \} \) contains \( n \) linearly independent vectors. Hence, for any \( x_3, \ldots, x_{n-1} \in \mathcal{X} \), \( f(x_1) - f(x_0), \ldots, f(x_{n-1}) - f(x_0) \) are linearly dependent. If there exist \( x_3, \ldots, x_{n-2} \) such that \( f(x_1) - f(x_0), \ldots, f(x_{n-2}) - f(x_0) \) are linearly dependent, then

\[
\{ f(x) - f(x_0) : x \in \mathcal{X} \} \subset \operatorname{span} \{ f(x_1) - f(x_0), \ldots, f(x_{n-2}) - f(x_0) \}
\]

which contradicts the fact that \( \{ f(x) - f(x_0) : x \in \mathcal{X} \} \) contains \( n \) linearly independent vectors. And thus, \( f(x_1) - f(x_0), f(x_2) - f(x_0) \) are linearly dependent. Therefore, \( f(x_0), f(x_1), \) and \( f(x_2) \) are 2-collinear. \( \square \)
Theorem 14. Let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean fuzzy $n$-normed spaces over a linear ordered non-Archimedean field $K$ with \( \mathcal{C} = \{3^n \mid n \in \mathbb{Z}\} \) such that $\mathcal{Y}$ is strictly convex. If $f: \mathcal{X} \to \mathcal{Y}$ is a fuzzy isometry, then $f(x) - f(0)$ is additive.

Proof. Let $g(x) = f(x) - f(0)$ for $x \in \mathcal{X}$. Then $g$ is a fuzzy $n$-isometry and $g(0) = 0$. For each $x, y, x_2, \ldots, x_n \in \mathcal{X}$. Since $g$ is a fuzzy $n$-isometry, we have

\[
N\left(g(x) - g\left(\frac{x + y}{2}\right), g(x_2) - g\left(\frac{x + y}{2}\right)\right) = N\left(\frac{x + y}{2}, y - x, x_2 - x, \ldots, x_n - x, t\right)
\]

\[
= N\left(\frac{x + y}{2}, y - x, x_2 - x, \ldots, x_n - x, t\right)
\]

\[
= N\left(\frac{x + y}{2}, y - x, x_2 - x, \ldots, x_n - x, t\right)
\]

\[
= N\left(\frac{x + y}{2}, y - x, x_2 - x, \ldots, x_n - x, t\right)
\]

Since $(x + y)/2, x,$ and $y$ are collinear, by Lemma 13, $g((x + y)/2), g(x),$ and $g(y)$ are also collinear. It follows from Lemma 12 that

\[
g\left(\frac{x + y}{2}\right) = \frac{g(x) + g(y)}{2},
\]

for all $x, y \in \mathcal{X}$. Hence, $g(x) = f(x) - f(0)$ is additive since $g(0) = 0$.

In the following, we prove that the interior preserving mapping carries the barycenter of a triangle to the barycenter point of the corresponding triangle. And then, using this result, we get a Mazur-Ulam theorem on non-Archimedean fuzzy $n$-normed spaces over a linear ordered non-Archimedean field $K$ with $\mathcal{C} = \{3^n \mid n \in \mathbb{Z}\}$.

Lemma 15. Let $\mathcal{X}$ be a strictly convex non-Archimedean fuzzy $n$-normed space over a linear ordered non-Archimedean field $K$.

Hence, the existence holds. For the uniqueness of $u$, we may assume that there is another $v \in \mathcal{X}$ satisfying (22). Since $v \in \{t_1 x + t_2 y + t_3 z \mid t_1 + t_2 + t_3 = 1, t_i \in [0, 1], i = 1, 2, 3\}$, $v = s_1 x + s_2 y + s_3 z$, $s_1 + s_2 + s_3 = 1$ for some $s_i \in [0, 1], i = 1, 2, 3$. Then

\[
N(x - u, y - u, x_3 - u, \ldots, x_n - u, t)
\]

\[
= N(y - u, z - u, x_3 - u, \ldots, x_n - u, t)
\]

\[
= N(x - u, z - u, x_3 - u, \ldots, x_n - u, t)
\]

\[
= N(x - y, x - z, x_3 - x, \ldots, x_n - x, t)
\]

for all $x_1, \ldots, x_n \in \mathcal{X}$ and $u \not\in \{t_1 x + t_2 y + t_3 z \mid t_1 + t_2 + t_3 = 1, t_i \in [0, 1], i = 1, 2, 3\}$. 

Proof. Let $u := (x + y + z)/3$ is the unique element of $\mathcal{X}$ and $t > 0$. By Lemma II, we have

\[
N(x - u, y - u, x_3 - u, \ldots, x_n - u, t)
\]

\[
= N(y - u, z - u, x_3 - u, \ldots, x_n - u, t)
\]

\[
= N(x - u, z - u, x_3 - u, \ldots, x_n - u, t)
\]

\[
= N(x - y, x - z, x_3 - x, \ldots, x_n - x, t)
\]

Similarly, we have

\[
N(y - u, z - u, x_3 - u, \ldots, x_n - u, t)
\]

\[
= N(x - y, x - z, x_3 - x, \ldots, x_n - x, t)
\]

\[
= N(x - y, x - z, x_3 - x, \ldots, x_n - x, t)
\]

Hence, the existence holds. For the uniqueness of $u$, we may assume that there is another $v \in \mathcal{X}$ satisfying (22). Since $v \in \{t_1 x + t_2 y + t_3 z \mid t_1 + t_2 + t_3 = 1, t_i \in [0, 1], i = 1, 2, 3\}$, $v = s_1 x + s_2 y + s_3 z$, $s_1 + s_2 + s_3 = 1$ for some $s_i \in [0, 1], i = 1, 2, 3$. Then

\[
N(x - u, y - u, x_3 - u, \ldots, x_n - u, t)
\]

\[
= N(y - u, z - u, x_3 - u, \ldots, x_n - u, t)
\]

\[
= N(x - u, z - u, x_3 - u, \ldots, x_n - u, t)
\]

\[
= N(x - y, x - z, x_3 - x, \ldots, x_n - x, t)
\]

\[
= N(x - y, x - z, x_3 - x, \ldots, x_n - x, t)
\]

\[
= N(x - y, x - z, x_3 - x, \ldots, x_n - x, t)
\]

\[
= N(x - y, x - z, x_3 - x, \ldots, x_n - x, t)
\]

\[
= N(x - y, x - z, x_3 - x, \ldots, x_n - x, t)
\]
By the strict convexity, we have $|s_1| = |s_2| = |s_3| = 1$. Then there exist integers $k_1, k_2$, and $k_3$ such that $s_1 = 3^k_1$, $s_2 = 3^k_2$, and $s_3 = 3^{k_3}$. Since $3^2 + 3^2 + 3^2 = 1$, we know that $k_1 < 0$, $k_2 < 0$, $k_3 < 0$. Without loss of generality, we let $s_1 = 3^{-n_1}$, $s_2 = 3^{-n_2}$, and $s_3 = 3^{-n_3}$ with $n_1 \geq n_2 \geq n_3$. Assume that the above one of the inequalities holds. Then $1 = s_1 + s_2 + s_3 = 3^{n_1}(1 + 3^{n_2-n_1} + 3^{n_3-n_1})$. So $3^{n_1} = 1 + 3^{n_2-n_1} + 3^{n_3-n_1}$. This is a contradiction. Thus, $s_1 = s_2 = s_3 = 1/3$ which means $u = v$. This completes the proof. 

**Theorem 16.** Let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean fuzzy n-normed spaces over a linear ordered non-Archimedean field $\mathbb{K}$ with $\mathbb{C} = \{3^n \mid n \in \mathbb{Z}\}$ such that $\mathcal{Y}$ is strictly convex. If $f : \mathcal{X} \to \mathcal{Y}$ is an interior preserving fuzzy n-isometry, then $f(x) - f(0)$ is additive.

**Proof.** Let $g(x) = f(x) - f(0)$ for $x \in \mathcal{X}$. Then $g$ is a fuzzy n-isometry and $g(0) = 0$. For $a, b, c \in \mathcal{X}$, let $\Delta abc$ be a triangle determined by the points $a, b, c,$ and $x$, an interior point of $\Delta abc$. Since $f$ is an interior preserving map, there exist $s_i \in K, s_i > 0$ $(i = 1, 2, 3)$ with $s_1 + s_2 + s_3 = 1$ such that $f(x) = s_1 f(a) + s_2 f(b) + s_3 f(c)$. Then

$$g(x) = s_1 f(a) + s_2 f(b) + s_3 f(c) - f(0)$$

$$= s_1 (f(a) - f(0)) + s_2 (f(b) - f(0)) + s_3 (f(c) - f(0))$$

$$= s_1 g(a) + s_2 g(b) + s_3 g(c),$$

and hence, $g(x)$ is an interior point of $\Delta g(a)g(b)g(c)$. Therefore, $g$, is also an interior preserving mapping.
Now let $x, y, z, x_3, \ldots, x_n \in X$. Since $g$ is a fuzzy $n$-isometry, we have

\[ N\left( g(x) - g\left( \frac{x + y + z}{3} \right) \right), g(y) - g\left( \frac{x + y + z}{3} \right), g(x_3) - g\left( \frac{x + y + z}{3} \right), \ldots, g(x_n) - g\left( \frac{x + y + z}{3} \right), \]
\[ -g\left( \frac{x + y + z}{3} \right), t \]
\[ = N\left( x - x + y + z \right) , y - \frac{x + y + z}{3}, x_3 - \frac{x + y + z}{3} , \ldots, x_n - \frac{x + y + z}{3}, t \right) \]
\[ = N\left( x - y, x - \frac{x + y + z}{3}, x_3 - x, \ldots, x_n - x, t \right) \]
\[ = N\left( x - y, 2x - y - z, x_3 - x, \ldots, x_n - x, t \right) \]
\[ = N\left( g(x) - g(y), g(x) - g(z), g(x_3) - g(x) \right), \]
\[ g(x_3) - g(x_3), \ldots, g(x_n) - g(x), t \right), \]

and similarly, we can obtain

\[ N\left( g(y) - g\left( \frac{x + y + z}{3} \right) \right), g(z) - g\left( \frac{x + y + z}{3} \right), g(x_3) - g\left( \frac{x + y + z}{3} \right), \ldots, g(x_n) - g\left( \frac{x + y + z}{3} \right), t \right) \]
\[ = N\left( g(x) - g(y), g(x) - g(z), g(x_3) - g(x) \right), \]
\[ g(x_3) - g(x), \ldots, g(x_n) - g(x), t \right), \]
\[ = N\left( g(x) - g(y), g(x) - g(z), g(x_3) - g(x_3), \ldots, g(x_n) - g(x), t \right), \]
\[ = N\left( g(x) - g(y), g(x) - g(z), g(x_3) - g(x), \ldots, g(x_n) - g(x), t \right), \]

Since $(x + y + z)/3$ is an interior point of the triangle $\triangle xyz$ and $g$ is an interior preserving mapping, $g((x + y + z)/3) \in \{ t_1 g(x) + t_2 g(y) + t_3 g(z) \ | \ t_1 + t_2 + t_3 = 1, t_i \in [0, 1], t_i > 0, i = 1, 2, 3 \}$. By Lemma 15,

\[ g\left( \frac{x + y + z}{3} \right) = \frac{g(x) + g(y) + g(z)}{3}. \]

Hence, $g(x) = f(x) - f(0)$ is additive since $g(0) = 0$. This completes the proof.

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**References**


