Hindawi Publishing Corporation
ISRN Mathematical Analysis
Volume 2013, Article ID 814067, 7 pages
http://dx.doi.org/10.1155/2013/814067

*Research Article*

# On the Mazur-Ulam Theorem in Non-Archimedean Fuzzy $n$-Normed Spaces

**Tian Zhou Xu**

*School of Mathematics, Beijing Institute of Technology, Beijing 100081, China*

Correspondence should be addressed to Tian Zhou Xu; xutianzhou@bit.edu.cn

Received 11 June 2013; Accepted 4 August 2013

Academic Editors: M. Tang and C. Zhu



The motivation of this paper is to present a new notion of non-Archimedean fuzzy $n$-normed space over a field with valuation. We obtain a Mazur-Ulam theorem for fuzzy $n$-isometry mappings in the strictly convex non-Archimedean fuzzy $n$-normed spaces. We also prove that the interior preserving mapping carries the barycenter of a triangle to the barycenter point of the corresponding triangle. And then, using this result, we get a Mazur-Ulam theorem for the interior preserving fuzzy $n$-isometry mappings in non-Archimedean fuzzy $n$-normed spaces over a linear ordered non-Archimedean field.

## 1. Introduction

Let $\mathbb{K}$ be a field. A valuation mapping on $\mathbb{K}$ is a function $|\cdot| : \mathbb{K} \to \mathbb{R}$ such that for any $r, s \in \mathbb{K}$ the following conditions are satisfied: (i) $|r| \geq 0$ and equality holds if and only if $r = 0$; (ii) $|rs| = |r| \cdot |s|$; (iii) $|r + s| \leq |r| + |s|$.

A field endowed with a valuation mapping will be called a valued field. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$. In the following, we will assume that $|\cdot|$ is nontrivial; that is, there is an $r_0 \in \mathbb{K}$ such that $|r_0| \neq 0, 1$.

Throughout this paper, we assume that $\mathbb{K}$ is a valued field and $n \geq 2$ is a positive integer. We denote the set of all elements of $\mathbb{K}$ whose norms are 1 by $\mathscr{C}$; that is, $\mathscr{C} = \{r \in \mathbb{K} \mid |r| = 1\}$. Moreover, $\mathbb{N}$ stands for the set of all positive integers and $\mathbb{R}$ (resp., $\mathbb{C}$) denotes the set of all real numbers (resp., complex numbers).

If condition (iii) in the definition of a valuation mapping is replaced with a strong triangle inequality (ultrametric), $|r + s| \leq \max\{|r|, |s|\}$, then the valuation $|\cdot|$ is said to be non-Archimedean. In any non-Archimedean field, we have $|1| = |-1| = 1$ and $|k| \leq 1$ for all $k \in \mathbb{N}$.

Let $\mathscr{X}$ and $\mathscr{Y}$ be metric spaces. A map $f : \mathscr{X} \to \mathscr{Y}$ is called a distance preserving mapping (isometry) if $d(f(x), f(y)) = d(x, y)$ for any $x, y \in \mathscr{X}$. Automatically, an isometry is injective. Two metric spaces $\mathscr{X}$ and $\mathscr{Y}$ are called isometric if there is an isometry from $\mathscr{X}$ to $\mathscr{Y}$.

The classical result of Mazur and Ulam states that if $\mathscr{X}, \mathscr{Y}$ are real normed linear spaces and $f : \mathscr{X} \to \mathscr{Y}$ is a surjective isometry, then $f$ is affine; that is, $f$ is a linear mapping up to translation. Numerous generalizations of this fact were presented by many authors (see, e.g., [1–13] and references therein). Unfortunately, the Mazur-Ulam theorem is not true for normed complex vector space. It was a natural step to ask if the theorem holds without the onto assumption. In fact, the onto assumption is essential. Without this assumption, Baker [14] proved that every isometry, not necessary surjective, $f : \mathscr{X} \to \mathscr{Y}$, between real normed linear spaces is affine if $\mathscr{Y}$ is strictly convex. Moslehian and Sadeghi presented a non-Archimedean version of this result [11]; they also noted that a Mazur-Ulam theorem generally fails in a non-Archimedean case. Choy et al. [3] proved the Mazur-Ulam theorem for the interior preserving mappings in linear 2-normed spaces; they also proved the theorem on non-Archimedean 2-normed spaces over a linear ordered non-Archimedean field without the strict convexity assumption. Chu et al. [4] studied the Mazur-Ulam theorem in linear $n$-normed spaces. Alaca [1] introduced the concepts of 2-isometry, collinearity, and 2-Lipschitz mapping in 2-fuzzy 2-normed linear spaces. Also, he gave a new generalization of the Mazur-Ulam theorem when $\mathscr{X}$ is a 2-fuzzy 2-normed linear space or $\mathfrak{I}(X)$ is a fuzzy 2-normed linear space. Kubzdela [10] gave some new results

for isometries, Mazur-Ulam theorem, and Aleksandrov problem in the framework of non-Archimedean normed spaces. Kang et al. [9] proved that the Mazur-Ulam theorem holds under some conditions in non-Archimedean fuzzy normed space.

The motivation of this paper is to introduce the notion of non-Archimedean fuzzy $n$-normed space over a field with valuation as a generalization of $n$-normed space [2, 15, 16], non-Archimedean 2-normed space [3], fuzzy $n$-normed space [17], and non-Archimedean fuzzy normed space [9, 18]. We will prove that the Mazur-Ulam theorem holds in the strictly convex non-Archimedean fuzzy $n$-normed spaces.

## 2. Preliminaries

In 1897, Hensel discovered the $p$-adic numbers as a number-theoretical analogue of power series in complex analysis. Let $p$ be a prime number. For any nonzero rational number $a$, there exists a unique integer $r$ such that $a = p^r m/k$, where $m$ and $k$ are integers not divisible by $p$. Then $|a|_p := p^{-r}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(a,b) = |a-b|_p$ is denoted by $\mathbb{Q}_p$ which is called the $p$-adic number field. Note that if $p > 2$, then $|2^k|_p = 1$ for each integer $k$ but $|2|_2 < 1$.

During the last three decades, $p$-adic numbers have gained the interest of physicists for their research, in particular, in problems derived from quantum physics, $p$-adic strings, and superstrings (see, e.g., [19]).

*Definition 1.* Let $\mathcal{X}$ be a linear space over a field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\|: \mathcal{X} \to [0,\infty)$ is said to be a non-Archimedean norm if it satisfies the following conditions:

(i) $\|x\| = 0$ if and only if $x = 0$,

(ii) $\|rx\| = |r|\|x\|$, $r \in \mathbb{K}$, $x \in \mathcal{X}$,

(iii) the strong triangle inequality: $\|x + y\| \le \max\{\|x\|, \|y\|\}$ $(x, y \in \mathcal{X})$.

Then $(\mathcal{X}, \|\cdot\|)$ is called a non-Archimedean normed space. By a complete non-Archimedean normed space, we mean one in which every Cauchy sequence is convergent.

*Definition 2.* Let $\mathcal{X}$ be a linear space over a valued field $\mathbb{K}$. A function $N : \mathcal{X}^n \times \mathbb{R} \to [0,1]$ is called a non-Archimedean fuzzy $n$-norm on $\mathcal{X}$ if the following conditions hold for all $x, y, x_1, \ldots, x_n \in \mathcal{X}$ and all $s, t \in \mathbb{R}$:

$(nN_1)$ $N(x_1, \ldots, x_n, t) = 0$ if $t \le 0$,

$(nN_2)$ for all $t > 0$, $N(x_1, \ldots, x_n, t) = 1$ if and only if $x_1, \ldots, x_n$ are linearly dependent,

$(nN_3)$ $N(x_1, \ldots, x_n, t)$ is invariant under any permutation of $x_1, \ldots, x_n$,

$(nN_4)$ $N(cx_1, \ldots, x_n, t) = N(x_1, \ldots, x_n, t/|c|)$ for all $t > 0$ if $c \in \mathbb{K}$, $c \ne 0$,

$(nN_5)$ $N(x + y, x_2, \ldots, x_n, \max\{s,t\}) \ge \min\{N(x, x_2, \ldots, x_n, s), N(y, x_2, \ldots, x_n, t)\}$ for all $s, t > 0$,

$(nN_6)$ $\lim_{t\to\infty} N(x_1, \ldots, x_n, t) = 1$.

If $N$ is a non-Archimedean fuzzy $n$-norm on $\mathcal{X}$, then $(\mathcal{X}, N)$ is called a non-Archimedean fuzzy $n$-normed space. It should be noticed that from the condition $(nN_5)$ it follows that

$$\begin{aligned} N(x_1, \ldots, x_n, t) &\ge \min\{N(0, \ldots, 0, t), N(x_1, \ldots, x_n, s)\} \\ &= N(x_1, \ldots, x_n, s) \end{aligned} \tag{1}$$

for every $t > s > 0$ and $x_1, \ldots, x_n \in \mathcal{X}$; that is, $N(x_1, \ldots, x_n, \cdot)$ is nondecreasing for every $x_1, \ldots, x_n \in \mathcal{X}$. This implies that

$$N(x_1, \ldots, x_n, s + t) \ge N(x_1, \ldots, x_n, \max\{s,t\}), \quad s, t > 0. \tag{2}$$

If $(nN_5)$ holds, then so is

$(nN_7)$ $N(x+y, x_2, \ldots, x_n, s+t)$
$\ge \min\{N(x, x_2, \ldots, x_n, s), N(y, x_2, \ldots, x_n, t)\}$.

*Example 3.* Let $(\mathcal{X}, \|\cdot, \ldots, \cdot\|)$ be an $n$-normed space (see [2]). For each $k \in \mathbb{N}$, consider

$$N_k(x_1, \ldots, x_n, t) = \begin{cases} \dfrac{t}{t + k\|x_1, \ldots, x_n\|}, & t > 0, \\ 0, & t \le 0. \end{cases} \tag{3}$$

Then $(\mathcal{X}, N_k)$ is a non-Archimedean fuzzy $n$-normed space.

*Definition 4.* Let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean fuzzy $n$-normed spaces, and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. We call $f$ a fuzzy $n$-isometry if

$$\begin{aligned} &N(x_1 - x_0, \ldots, x_n - x_0, t) \\ &\quad = N(f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0), t) \end{aligned} \tag{4}$$

for all $x_0, x_1, \ldots, x_n \in \mathcal{X}$ and all $t > 0$.

For given points $x$, $y$, and $z$ in $\mathcal{X}$, $\Delta xyz$ denotes the triangle determined by $x$, $y$, and $z$. A point $(x + y + z)/3$ is called a barycenter of $\Delta xyz$. If $p$ is a point of a set $\{t_1 x + t_2 y + t_3 z \mid t_1 + t_2 + t_3 = 1,\ t_i \in \mathbb{K},\ t_i > 0,\ i = 1, 2, 3\}$, then $p$ is called an interior point of $\Delta xyz$. Define a mapping $f$ between linear $n$-normed spaces to be an interior preserving mapping of the triangle if $f$ carries an interior point in a triangle to an interior point in the corresponding triangle.

*Remark 5.* Let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean fuzzy $n$-normed spaces and $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. Then $f$ is a fuzzy $n$-isometry if and only if $f$ satisfies the following property:

$$\begin{aligned} &\left|N(x_1 - x_0, \ldots, x_n - x_0, t) - N(x_1' - x_0', \ldots, x_n' - x_0', t)\right| \\ &\quad = \left|N(f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0), t)\right. \\ &\qquad \left. - N(f(x_1') - f(x_0'), \ldots, f(x_n') - f(x_0'), t)\right| \end{aligned} \tag{5}$$

for all $x_0, x_1, \ldots, x_n, x_0', x_1', \ldots, x_n' \in \mathcal{X}$ and all $t > 0$.

*Definition 6.* Let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean fuzzy $n$-normed spaces and $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. Then $f$ is called a weak fuzzy $n$-isometry if for every $\epsilon > 0$, there exists positive real number $\delta$ such that

$$\begin{aligned}&\left|N(x_1 - x_0, \ldots, x_n - x_0, t) - N\left(x_1' - x_0', \ldots, x_n' - x_0', t\right)\right| \\ &\quad < \delta \text{ implies } \left|N(f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0), t) \right. \\ &\qquad - N\left(f\left(x_1'\right) - f\left(x_0'\right), \ldots, f\left(x_n'\right)\right. \\ &\qquad \left.\left. - f\left(x_0'\right), t\right)\right| < \epsilon\end{aligned} \tag{6}$$

for all $x_0, x_1, \ldots, x_n, x_0', x_1', \ldots, x_n' \in \mathcal{X}$ and all $t > 0$.

*Definition 7.* Let $\mathcal{X}$ be a non-Archimedean fuzzy $n$-normed space. The points $x_0, x_1, \ldots, x_n$ are said to be $n$-collinear if for every $i \in \{0, 1, \ldots, n\}$, $\{x_j - x_i \mid 0 \le j \ne i \le n\}$ is linearly dependent.

*Remark 8.* Let $\mathcal{X}$ be a non-Archimedean fuzzy $n$-normed space over a valued field $\mathbb{K}$ and $x_0, x_1$, $x_2$ mutually disjoint elements of $\mathcal{X}$. Then $x_0, x_1$, and $x_2$ are said to be 2-collinear if and only if $x_2 - x_0 = t(x_1 - x_0)$ for some $t \in \mathbb{K}$.

Now we define the concept of $n$-Lipschitz mapping.

*Definition 9.* Let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean fuzzy $n$-normed spaces, and let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping. Then $f$ is called a fuzzy $n$-Lipschitz mapping if there is a $L \ge 0$ such that

$$\begin{aligned}&N(f(x_1) - f(x_0), \ldots, f(x_n) - f(x_0), Lt) \\ &\quad \ge N(x_1 - x_0, \ldots, x_n - x_0, t)\end{aligned} \tag{7}$$

for all $x_0, x_1, \ldots, x_n \in \mathcal{X}$ and all $t > 0$. The smallest such $L$ is called the $n$-Lipschitz constant.

*Definition 10.* A non-Archimedean fuzzy $n$-normed space $\mathcal{X}$ over a valued field $\mathbb{K}$ is called strictly convex, if for each $x_1, \ldots, x_m, w_2, \ldots, w_n \in \mathcal{X}$ and $s_1, \ldots, s_m > 0$,

$$\begin{aligned}&N\left(\sum_{i=1}^{m} x_i, w_2, \ldots, w_n, \max\{s_1, \ldots, s_m\}\right) \\ &= \min\{N(x_1, w_2, \ldots, w_n, s_1), \ldots, N(x_m, w_2, \ldots, w_n, s_m)\}, \\ &\quad N(x_1, w_2, \ldots, w_n, s_1) = \cdots = N(x_m, w_2, \ldots, w_n, s_m)\end{aligned} \tag{8}$$

implies that $x_1 = \cdots = x_m$ and $s_1 = \cdots = s_m$.

## 3. On the Mazur-Ulam Problem

**Lemma 11.** *Let $\mathcal{X}$ be a non-Archimedean fuzzy $n$-normed space over a valued field $\mathbb{K}$, $x_1, \ldots, x_n \in \mathcal{X}$ and all $t > 0$. Then*

$$\begin{aligned}&N\left(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n, t\right) \\ &\quad = N\left(x_1, \ldots, x_i, \ldots, x_{j-1}, x_j + \alpha x_i, x_{j+1}, \ldots, x_n, t\right)\end{aligned} \tag{9}$$

*for all $\alpha \in \mathbb{K}$.*

*Proof.* Let $x_1, \ldots, x_n \in \mathcal{X}$, $t > 0$, and $\alpha \in \mathbb{K}$, then

$$\begin{aligned}&N\left(x_1, \ldots, x_i, \ldots, x_{j-1}, x_j + \alpha x_i, x_{j+1}, \ldots, x_n, t\right) \\ &\quad \ge \min\left\{N\left(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n, t\right),\right. \\ &\qquad \left. N\left(\frac{x_1, \ldots, x_i, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n, t}{|\alpha|}\right)\right\} \\ &\quad \ge N\left(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n, t\right), \\ &N\left(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n, t\right) \\ &\quad = N\left(x_1, \ldots, x_i, \ldots, x_{j-1}, x_j + \alpha x_i - \alpha x_i, x_{j+1}, \ldots, x_n, t\right) \\ &\quad \ge \min\left\{N\left(x_1, \ldots, x_i, \ldots, x_{j-1}, x_j + \alpha x_i, x_{j+1}, \ldots, x_n, t\right),\right. \\ &\qquad \left. N\left(x_1, \ldots, x_i, \ldots, x_{j-1}, -\alpha x_i, x_{j+1}, \ldots, x_n, t\right)\right\} \\ &\quad = N\left(x_1, \ldots, x_i, \ldots, x_{j-1}, x_j + \alpha x_i, x_{j+1}, \ldots, x_n, t\right).\end{aligned} \tag{10}$$

It follows that (9) holds. □

**Lemma 12.** *Let $\mathcal{X}$ be a strictly convex non-Archimedean fuzzy $n$-normed space over a linear ordered non-Archimedean field $\mathbb{K}$ with $\mathcal{C} = \{2^n \mid n \in \mathbb{Z}\}$, and let $x, y \in \mathcal{X}$ and $t > 0$. Then $u = (x + y)/2$ is the unique element of $\mathcal{X}$ satisfying*

$$\begin{aligned}&N(x - u, x_2 - u, \ldots, x_n - u, t) \\ &\quad = N(y - u, x_2 - u, \ldots, x_n - u, t) \\ &\quad = N(x - y, x_2 - x, \ldots, x_n - x, t)\end{aligned} \tag{11}$$

*for all $x_2, \ldots, x_n \in \mathcal{X}$ and $x$, $y$, and $u$ are 2-collinear.*

*Proof.* Let $u := (x + y)/2, x_2, \ldots, x_n \in \mathcal{X}$ and $t > 0$. By Lemma 11, we have

$$\begin{aligned}&N(x - u, x_2 - u, \ldots, x_n - u, t) \\ &\quad = N\left(x - \frac{x+y}{2}, x_2 - \frac{x+y}{2}, \ldots, x_n - \frac{x+y}{2}, t\right) \\ &\quad = N\left(x - \frac{x+y}{2}, x_2 - x, \ldots, x_n - x, t\right) \\ &\quad = N(y - x, x_2 - x, \ldots, x_n - x, |2|t), \\ &N(y - u, x_2 - u, \ldots, x_n - u, t) \\ &\quad = N\left(y - \frac{x+y}{2}, x_2 - \frac{x+y}{2}, \ldots, x_n - \frac{x+y}{2}, t\right) \\ &\quad = N\left(y - \frac{x+y}{2}, x_2 - y, \ldots, x_n - y, t\right) \\ &\quad = N(y - x, x_2 - y, \ldots, x_n - y, |2|t) \\ &\quad = N(y - x, x_2 - x, \ldots, x_n - x, |2|t).\end{aligned} \tag{12}$$

Hence, the existence holds. For the uniqueness of $u$, we may assume that there is another $v \in \mathcal{X}$, collinear with $x, y$ such that

$$\begin{aligned} &N(x-v, x_2-v, \ldots, x_n-v, t) \\ &\quad = N(y-v, x_2-v, \ldots, x_n-v, t) \\ &\quad = N(x-y, x_2-x, \ldots, x_n-x, t). \end{aligned} \tag{13}$$

Since $x$, $y$, and $v$ are collinear, $v = sx+(1-s)y$ for some $s \in \mathbb{K}$. We may assume $s \neq 0$ and $s \neq 1$. Then

$$\begin{aligned} &N(x-y, x_2-x, \ldots, x_n-x, t) \\ &\quad = N(x-v, x_2-v, \ldots, x_n-v, t) \\ &\quad = N(x-v, x_2-x, \ldots, x_n-x, t) \\ &\quad = N(x-sx-(1-s)y, x_2-x, \ldots, x_n-x, t) \\ &\quad = N\left(x-y, x_2-x, \ldots, x_n-x, \frac{t}{|1-s|}\right), \\ &N(x-y, x_2-x, \ldots, x_n-x, t) \\ &\quad = N(y-v, x_2-v, \ldots, x_n-v, t) \\ &\quad = N(y-v, x_2-y, \ldots, x_n-y, t) \\ &\quad = N(y-sx-(1-s)y, x_2-y, \ldots, x_n-y, t) \\ &\quad = N\left(x-y, x_2-y, \ldots, x_n-y, \frac{t}{|s|}\right) \\ &\quad = N\left(x-y, x_2-x, \ldots, x_n-x, \frac{t}{|s|}\right). \end{aligned} \tag{14}$$

Then

$$\begin{aligned} &N\left(x-y+x-y, x_2-x, \ldots, x_n-x, \max\left\{\frac{t}{|s|}, \frac{t}{|1-s|}\right\}\right) \\ &\quad = N\left(x-y, x_2-x, \ldots, x_n-x, \frac{1}{|2|}\max\left\{\frac{t}{|s|}, \frac{t}{|1-s|}\right\}\right) \\ &\quad = N(x-y, x_2-x, \ldots, x_n-x, t) \\ &\quad = \min\left\{N\left(x-y, x_2-x, \ldots, x_n-x, \frac{t}{|s|}\right),\right. \\ &\qquad\qquad \left. N\left(x-y, x_2-x, \ldots, x_n-x, \frac{t}{|1-s|}\right)\right\}, \\ &N\left(x-y+x-y, x_2-x, \ldots, x_n-x, \max\left\{t, \frac{t}{|s|}\right\}\right) \\ &\quad = N\left(x-y, x_2-x, \ldots, x_n-x, \frac{1}{|2|}\max\left\{t, \frac{t}{|s|}\right\}\right) \\ &\quad = N(x-y, x_2-x, \ldots, x_n-x, t) \\ &\quad = \min\left\{N(x-y, x_2-x, \ldots, x_n-x, t),\right. \\ &\qquad\qquad \left. N\left(x-y, x_2-x, \ldots, x_n-x, \frac{t}{|s|}\right)\right\}. \end{aligned} \tag{15}$$

By the strict convexity of $\mathcal{X}$, we have $|1-s| = |s| = 1$. Then there exist two integers $k_1, k_2$ such that $1-s = 2^{k_1}$ and $s = 2^{k_2}$. Since $2^{k_1}+2^{k_2} = 1$, we know that $k_1 < 0$, $k_2 < 0$. Without loss of generality, we let $1-s = 2^{-n_1}$ and $s = 2^{-n_2}$ with $n_1 \geq n_2$. If $n_1 > n_2$, then $1 = 2^{-n_1}+2^{-n_2} = 2^{-n_1}(1+2^{n_1-n_2})$; that is, $2^{n_1} = 1+2^{n_1-n_2}$. This is a contradiction. Thus, $n_1 = n_2$; that is, $s = 1/2$. This completes the proof. □

**Lemma 13.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean fuzzy $n$-normed spaces over a valued field $\mathbb{K}$. If $f : \mathcal{X} \to \mathcal{Y}$ is a fuzzy $n$-isometry and $x_0, x_1$, and $x_2$ are 2-collinear, then $f(x_0)$, $f(x_1)$, and $f(x_2)$ are 2-collinear.*

*Proof.* Since $\dim \mathcal{X} \geq n$, for any $x_0 \in \mathcal{X}$, there exist $y_1, \ldots, y_n \in \mathcal{X}$ such that $y_1-x_0, \ldots, y_n-x_0$ are linearly independent. Then

$$\begin{aligned} &N(y_1-x_0, \ldots, y_n-x_0, t) \\ &\quad = N(f(y_1)-f(x_0), \ldots, f(y_n)-f(x_0), t) \neq 1, \end{aligned} \tag{16}$$

and hence, the set $\{f(x)-f(x_0) : x \in \mathcal{X}\}$ contains $n$ linearly independent vectors. Assume that $x_0, x_1$, and $x_2$ are 2-collinear. Then, for any $x_3, \ldots, x_n \in \mathcal{X}$,

$$\begin{aligned} &N(x_1-x_0, \ldots, x_n-x_0, t) \\ &\quad = N(f(x_1)-f(x_0), \ldots, f(x_n)-f(x_0), t) \\ &\quad = 1. \end{aligned} \tag{17}$$

By $(nN_2)$, it follows that $f(x_1)-f(x_0), \ldots, f(x_n)-f(x_0)$ are linearly dependent. If there exist $x_3, \ldots, x_{n-1}$ such that $f(x_1)-f(x_0), \ldots, f(x_{n-1})-f(x_0)$ are linearly independent, then

$$\begin{aligned} &\{f(x)-f(x_0) : x \in \mathcal{X}\} \\ &\quad \subset \operatorname{span}\{f(x_1)-f(x_0), \ldots, f(x_{n-1})-f(x_0)\} \end{aligned} \tag{18}$$

which contradicts the fact that $\{f(x)-f(x_0) : x \in \mathcal{X}\}$ contains $n$ linearly independent vectors. Hence, for any $x_3, \ldots, x_{n-1} \in \mathcal{X}$, $f(x_1)-f(x_0), \ldots, f(x_{n-1})-f(x_0)$ are linearly dependent. If there exist $x_3, \ldots, x_{n-2}$ such that $f(x_1)-f(x_0), \ldots, f(x_{n-2})-f(x_0)$ are linearly independent, then

$$\begin{aligned} &\{f(x)-f(x_0) : x \in \mathcal{X}\} \\ &\quad \subset \operatorname{span}\{f(x_1)-f(x_0), \ldots, f(x_{n-2})-f(x_0)\} \end{aligned} \tag{19}$$

which contradicts the fact that $\{f(x)-f(x_0) : x \in \mathcal{X}\}$ contains $n$ linearly independent vectors. And thus, $f(x_1)-f(x_0)$, $f(x_2)-f(x_0)$ are linearly dependent. Therefore, $f(x_0)$, $f(x_1)$, and $f(x_2)$ are 2-collinear. □

**Theorem 14.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean fuzzy $n$-normed spaces over a linear ordered non-Archimedean field $\mathbb{K}$ with $\mathcal{C} = \{2^n \mid n \in \mathbb{Z}\}$ such that $\mathcal{Y}$ is strictly convex. If $f : \mathcal{X} \to \mathcal{Y}$ is a fuzzy $n$-isometry, then $f(x) - f(0)$ is additive.*

*Proof.* Let $g(x) = f(x) - f(0)$ for $x \in \mathcal{X}$. Then $g$ is a fuzzy $n$-isometry and $g(0) = 0$. For each $x, y, x_2, \ldots, x_n \in \mathcal{X}$. Since $g$ is a fuzzy $n$-isometry, we have

$$\begin{aligned}
&N\left(g(x) - g\left(\frac{x+y}{2}\right), g(x_2)\right.\\
&\quad \left. - g\left(\frac{x+y}{2}\right), \ldots, g(x_n) - g\left(\frac{x+y}{2}\right), t\right)\\
&\quad = N\left(x - \frac{x+y}{2}, x_2 - \frac{x+y}{2}, \ldots, x_n - \frac{x+y}{2}, t\right)\\
&\quad = N\left(x - \frac{x+y}{2}, x_2 - x, \ldots, x_n - x, t\right)\\
&\quad = N(x - y, x_2 - x, \ldots, x_n - x, t)\\
&\quad = N(g(x) - g(y), g(x_2) - g(x), \ldots, g(x_n) - g(x), t),\\
&N\left(g(y) - g\left(\frac{x+y}{2}\right), g(x_2)\right.\\
&\quad \left. - g\left(\frac{x+y}{2}\right), \ldots, g(x_n) - g\left(\frac{x+y}{2}\right), t\right)\\
&\quad = N\left(y - \frac{x+y}{2}, x_2 - \frac{x+y}{2}, \ldots, x_n - \frac{x+y}{2}, t\right)\\
&\quad = N\left(y - \frac{x+y}{2}, x_2 - y, \ldots, x_n - y, t\right)\\
&\quad = N(x - y, x_2 - y, \ldots, x_n - y, t)\\
&\quad = N(x - y, x_2 - x, \ldots, x_n - x, t)\\
&\quad = N(g(x) - g(y), g(x_2) - g(x), \ldots, g(x_n) - g(x), t).
\end{aligned} \tag{20}$$

Since $(x + y)/2$, $x$, and $y$ are collinear, by Lemma 13, $g((x + y)/2)$, $g(x)$, and $g(y)$ are also collinear. It follows from Lemma 12 that

$$g\left(\frac{x+y}{2}\right) = \frac{g(x) + g(y)}{2}, \tag{21}$$

for all $x, y \in \mathcal{X}$. Hence, $g(x) = f(x) - f(0)$ is additive since $g(0) = 0$. □

In the following, we prove that the interior preserving mapping carries the barycenter of a triangle to the barycenter point of the corresponding triangle. And then, using this result, we get a Mazur-Ulam theorem on non-Archimedean fuzzy $n$-normed spaces over a linear ordered non-Archimedean field $\mathbb{K}$ with $\mathcal{C} = \{3^n \mid n \in \mathbb{Z}\}$.

**Lemma 15.** *Let $\mathcal{X}$ be a strictly convex non-Archimedean fuzzy $n$-normed space over a linear ordered non-Archimedean field $\mathbb{K}$ with $\mathcal{C} = \{3^n \mid n \in \mathbb{Z}\}$, and let $x, y, z \in \mathcal{X}$ and $t > 0$. Then $u := (x + y + z)/3$ is the unique element of $\mathcal{X}$ satisfying*

$$\begin{aligned}
&N(x - u, y - u, x_3 - u, \ldots, x_n - u, t)\\
&\quad = N(y - u, z - u, x_3 - u, \ldots, x_n - u, t)\\
&\quad = N(x - u, z - u, x_3 - u, \ldots, x_n - u, t)\\
&\quad = N(x - y, x - z, x_3 - x, \ldots, x_n - x, t)
\end{aligned} \tag{22}$$

*for all $x_3, \ldots, x_n \in \mathcal{X}$ and $u \in \{t_1 x + t_2 y + t_3 z \mid t_1 + t_2 + t_3 = 1,\ t_i \in \mathbb{K}, t_i > 0,\ i = 1, 2, 3\}$.*

*Proof.* Let $u := (x + y + z)/3, x_3, \ldots, x_n \in \mathcal{X}$ and $t > 0$. By Lemma 11, we have

$$\begin{aligned}
&N(x - u, y - u, x_3 - u, \ldots, x_n - u, t)\\
&\quad = N(x - u, y - x, x_3 - x, \ldots, x_n - x, t)\\
&\quad = N\left(x - \frac{x+y+z}{3}, y - x, x_3 - x, \ldots, x_n - x, t\right)\\
&\quad = N(2x - y - z, y - x, x_3 - x, \ldots, x_n - x, |3|\, t)\\
&\quad = N(y - z, y - x, x_3 - x, \ldots, x_n - x, t)\\
&\quad = N(y - x, z - x, x_3 - x, \ldots, x_n - x, t)\\
&\quad = N(x - y, x - z, x_3 - x, \ldots, x_n - x, t).
\end{aligned} \tag{23}$$

Similarly, we have

$$\begin{aligned}
&N(y - u, z - u, x_3 - u, \ldots, x_n - u, t)\\
&\quad = N(x - y, x - z, x_3 - x, \ldots, x_n - x, t),\\
&N(x - u, z - u, x_3 - u, \ldots, x_n - u, t)\\
&\quad = N(x - y, x - z, x_3 - x, \ldots, x_n - x, t).
\end{aligned} \tag{24}$$

Hence, the existence holds. For the uniqueness of $u$, we may assume that there is another $v \in \mathcal{X}$ satisfying (22). Since $v \in \{t_1 x + t_2 y + t_3 z \mid t_1 + t_2 + t_3 = 1,\ t_i \in \mathbb{K},\ t_i > 0,\ i = 1, 2, 3\}$, $v = s_1 x + s_2 y + s_3 z$, $s_1 + s_2 + s_3 = 1$ for some $s_i \in \mathbb{K}$, $s_i > 0$, $i = 1, 2, 3$. Then

$$\begin{aligned}
&N(x - y, z - x, x_3 - x, \ldots, x_n - x, t)\\
&\quad = N(x - v, y - v, x_3 - v, \ldots, x_n - v, t)\\
&\quad = N(x - v, y - x, x_3 - x, \ldots, x_n - x, t)\\
&\quad = N((1 - s_1)x - s_2 y - (1 - s_1 - s_2)z,\\
&\qquad y - x, x_3 - x, \ldots, x_n - x, t)\\
&\quad = N((1 - s_1 - s_2)x - (1 - s_1 - s_2)z,\\
&\qquad y - x, x_3 - x, \ldots, x_n - x, t)
\end{aligned}$$

$$
\begin{aligned}
&= N\left(x-z, y-x, x_3-x, \ldots, x_n-x, \frac{t}{|1-s_1-s_2|}\right) \\
&= N\left(x-y, z-x, x_3-x, \ldots, x_n-x, \frac{t}{|1-s_1-s_2|}\right), \\
N&(x-y, z-x, x_3-x, \ldots, x_n-x, t) \\
&= N(x-v, z-v, x_3-v, \ldots, x_n-v, t) \\
&= N(x-v, z-x, x_3-x, \ldots, x_n-x, t) \\
&= N((1-s_1)x-s_2y-(1-s_1-s_2)z, \\
&\qquad z-x, x_3-x, \ldots, x_n-x, t) \\
&= N(s_2x-s_2y, z-x, x_3-x, \ldots, x_n-x, t) \\
&= N\left(x-z, y-x, x_3-x, \ldots, x_n-x, \frac{t}{|s_2|}\right), \\
N&(x-y, z-x, x_3-x, \ldots, x_n-x, t) \\
&= N(y-v, z-v, x_3-v, \ldots, x_n-v, t) \\
&= N(y-v, z-y, x_3-y, \ldots, x_n-y, t) \\
&= N((1-s_2)y-s_1x-(1-s_1-s_2)z, \\
&\qquad z-y, x_3-y, \ldots, x_n-y, t) \\
&= N(-s_1x+s_1z, z-y, x_3-y, \ldots, x_n-y, t) \\
&= N\left(x-z, z-y, x_3-y, \ldots, x_n-y, \frac{t}{|s_1|}\right) \\
&= N\left(x-y, z-x, x_3-x, \ldots, x_n-x, \frac{t}{|s_1|}\right).
\end{aligned}
\tag{25}
$$

Hence,

$$
\begin{aligned}
N&\left(x-y+x-y+x-y, z-x, x_3-x, \ldots, x_n-x, \right. \\
&\qquad \left. \max\left\{\frac{t}{|s_1|}, \frac{t}{|s_2|}, \frac{t}{|1-s_1-s_2|}\right\}\right) \\
&= N\left(x-y, z-x, x_3-x, \ldots, x_n-x, \right. \\
&\qquad \left. \frac{1}{|3|}\max\left\{\frac{t}{|s_1|}, \frac{t}{|s_2|}, \frac{t}{|1-s_1-s_2|}\right\}\right) \\
&= N(x-y, z-x, x_3-x, \ldots, x_n-x, t) \\
&= \min\left\{N\left(x-y, z-x, x_3-x, \ldots, x_n-x, \frac{t}{|s_1|}\right), \right. \\
&\qquad N\left(x-y, z-x, x_3-x, \ldots, x_n-x, \frac{t}{|s_2|}\right), \\
&\qquad \left. N\left(x-y, z-x, x_3-x, \ldots, x_n-x, \frac{t}{|1-s_1-s_2|}\right)\right\}, \\
N&\left(x-y+x-y+x-y, z-x, x_3-x, \ldots, x_n-x, \right. \\
&\qquad \left. \max\left\{t, \frac{t}{|s_1|}, \frac{t}{|s_2|}\right\}\right) \\
&= N\left(x-y, z-x, x_3-x, \ldots, x_n-x, \right. \\
&\qquad \left. \frac{1}{|3|}\max\left\{t, \frac{t}{|s_1|}, \frac{t}{|s_2|}\right\}\right) \\
&= N(x-y, z-x, x_3-x, \ldots, x_n-x, t) \\
&= \min\left\{N(x-y, z-x, x_3-x, \ldots, x_n-x, t), \right. \\
&\qquad N\left(x-y, z-x, x_3-x, \ldots, x_n-x, \frac{t}{|s_1|}\right), \\
&\qquad \left. N\left(x-y, z-x, x_3-x, \ldots, x_n-x, \frac{t}{|s_2|}\right)\right\}.
\end{aligned}
\tag{26}
$$

By the strict convexity, we have $|s_1| = |s_2| = |s_3| = 1$. Then there exist integers $k_1$, $k_2$, and $k_3$ such that $s_1 = 3^{k_1}$, $s_2 = 3^{k_2}$, and $s_3 = 3^{k_3}$. Since $3^{k_1} + 3^{k_2} + 3^{k_3} = 1$, we know that $k_1 < 0$, $k_2 < 0$, $k_3 < 0$. Without loss of generality, we let $s_1 = 3^{-n_1}$, $s_2 = 3^{-n_2}$, and $s_3 = 3^{-n_3}$ with $n_1 \geq n_2 \geq n_3$. Assume that the above one of the inequalities holds. Then $1 = s_1 + s_2 + s_3 = 3^{-n_1}(1 + 3^{n_1-n_2} + 3^{n_1-n_3})$. So $3^{n_1} = 1 + 3^{n_1-n_2} + 3^{n_1-n_3}$. This is a contradiction. Thus, $s_1 = s_2 = s_3 = 1/3$ which means $u = v$. This completes the proof. □

**Theorem 16.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be non-Archimedean fuzzy $n$-normed spaces over a linear ordered non-Archimedean field $\mathbb{K}$ with $\mathcal{C} = \{3^n \mid n \in \mathbb{Z}\}$ such that $\mathcal{Y}$ is strictly convex. If $f : \mathcal{X} \to \mathcal{Y}$ is an interior preserving fuzzy $n$-isometry, then $f(x) - f(0)$ is additive.*

*Proof.* Let $g(x) = f(x) - f(0)$ for $x \in \mathcal{X}$. Then $g$ is a fuzzy $n$-isometry and $g(0) = 0$. For $a, b, c \in \mathcal{X}$, let $\Delta abc$ be a triangle determined by the points $a$, $b$, $c$, and $x$, an interior point of $\Delta abc$. Since $f$ is an interior preserving map, there exist $s_i \in \mathbb{K}$, $s_i > 0$ $(i = 1, 2, 3)$ with $s_1 + s_2 + s_3 = 1$ such that $f(x) = s_1 f(a) + s_2 f(b) + s_3 f(c)$. Then

$$
\begin{aligned}
g(x) &= s_1 f(a) + s_2 f(b) + s_3 f(c) - f(0) \\
&= s_1 (f(a) - f(0)) + s_2 (f(b) - f(0)) \\
&\quad + s_3 (f(c) - f(0)) \\
&= s_1 g(a) + s_2 g(b) + s_3 g(c),
\end{aligned}
\tag{27}
$$

and hence, $g(x)$ is an interior point of $\Delta g(a)g(b)g(c)$. Therefore, $g$ is also an interior preserving mapping.

Now let $x, y, z, x_3, \ldots, x_n \in \mathcal{X}$. Since $g$ is a fuzzy $n$-isometry, we have

$$\begin{aligned}
&N\left(g(x)-g\left(\frac{x+y+z}{3}\right), g(y)-g\left(\frac{x+y+z}{3}\right),\right.\\
&\quad g(x_3)-g\left(\frac{x+y+z}{3}\right), \ldots, g(x_n)\\
&\quad \left.-g\left(\frac{x+y+z}{3}\right), t\right)\\
&= N\left(x-\frac{x+y+z}{3}, y-\frac{x+y+z}{3},\right.\\
&\qquad \left. x_3-\frac{x+y+z}{3}, \ldots, x_n-\frac{x+y+z}{3}, t\right) \qquad (28)\\
&= N\left(x-y, x-\frac{x+y+z}{3}, x_3-x, \ldots, x_n-x, t\right)\\
&= N(x-y, 2x-y-z, x_3-x, \ldots, x_n-x, t)\\
&= N(x-y, x-z, x_3-x, \ldots, x_n-x, t)\\
&= N(g(x)-g(y), g(x)-g(z),\\
&\qquad g(x_3)-g(x), \ldots, g(x_n)-g(x), t),
\end{aligned}$$

and similarly, we can obtain

$$\begin{aligned}
&N\left(g(y)-g\left(\frac{x+y+z}{3}\right), g(z)-g\left(\frac{x+y+z}{3}\right),\right.\\
&\quad \left. g(x_3)-g\left(\frac{x+y+z}{3}\right), \ldots, g(x_n)-g\left(\frac{x+y+z}{3}\right), t\right)\\
&= N(g(x)-g(y), g(x)-g(z), g(x_3)\\
&\qquad -g(x), \ldots, g(x_n)-g(x), t), \qquad (29)
\end{aligned}$$

$$\begin{aligned}
&N\left(g(x)-g\left(\frac{x+y+z}{3}\right), g(z)-g\left(\frac{x+y+z}{3}\right),\right.\\
&\quad \left. g(x_3)-g\left(\frac{x+y+z}{3}\right), \ldots, g(x_n)-g\left(\frac{x+y+z}{3}\right), t\right)\\
&= N(g(x)-g(y), g(x)-g(z),\\
&\qquad g(x_3)-g(x), \ldots, g(x_n)-g(x), t). \qquad (30)
\end{aligned}$$

Since $(x+y+z)/3$ is an interior point of the triangle $\Delta xyz$ and $g$ is an interior preserving mapping, $g((x+y+z)/3) \in \{t_1 g(x) + t_2 g(y) + t_3 g(z) \mid t_1 + t_2 + t_3 = 1, t_i \in \mathbb{K}, t_i > 0, i = 1, 2, 3\}$. By Lemma 15,

$$g\left(\frac{x+y+z}{3}\right) = \frac{g(x)+g(y)+g(z)}{3}. \qquad (31)$$

Hence, $g(x) = f(x) - f(0)$ is additive since $g(0) = 0$. This completes the proof. □

## Acknowledgment

This work was supported by the National Natural Science Foundation of China (NNSFC) (Grant no. 11171022).

## References

[1] C. Alaca, "A new perspective to the Mazur-Ulam problem in 2-fuzzy 2-normed linear spaces," *Iranian Journal of Fuzzy Systems*, vol. 7, no. 2, pp. 109–119, 2010.

[2] X. Y. Chen and M. M. Song, "Characterizations on isometries in linear $n$-normed spaces," *Nonlinear Analysis*, vol. 72, no. 3-4, pp. 1895–1901, 2010.

[3] J. Choy, H. Y. Chu, and S. H. Ku, "Characterizations on Mazur-Ulam theorem," *Nonlinear Analysis*, vol. 72, no. 3-4, pp. 1291–1297, 2010.

[4] H. Y. Chu, S. K. Choi, and D. S. Kang, "Mappings of conservative distances in linear $n$-normed spaces," *Nonlinear Analysis*, vol. 70, no. 3, pp. 1168–1174, 2009.

[5] H. Y. Chu, K. Lee, and C. G. Park, "On the Aleksandrov problem in linear $n$-normed spaces," *Nonlinear Analysis*, vol. 59, no. 7, pp. 1001–1011, 2004.

[6] H. Y. Chu, C. G. Park, and W. G. Park, "The Aleksandrov problem in linear 2-normed spaces," *Journal of Mathematical Analysis and Applications*, vol. 289, no. 2, pp. 666–672, 2004.

[7] H. Y. Chu, "On the Mazur-Ulam problem in linear 2-normed spaces," *Journal of Mathematical Analysis and Applications*, vol. 327, no. 2, pp. 1041–1045, 2007.

[8] S. Cobzaş, "A Mazur-Ulam theorem for probabilistic normed spaces," *Aequationes Mathematicae*, vol. 77, no. 1-2, pp. 197–205, 2009.

[9] D. Kang, H. Koh, and I. G. Cho, "On the Mazur-Ulam theorem in non-Archimedean fuzzy normed spaces," *Applied Mathematics Letters*, vol. 25, no. 3, pp. 301–304, 2012.

[10] A. Kubzdela, "Isometries, Mazur-Ulam theorem and Aleksandrov problem for non-Archimedean normed spaces," *Nonlinear Analysis*, vol. 75, no. 4, pp. 2060–2068, 2012.

[11] M. S. Moslehian and G. Sadeghi, "A Mazur-Ulam theorem in non-Archimedean normed spaces," *Nonlinear Analysis*, vol. 69, no. 10, pp. 3405–3408, 2008.

[12] C. Park and C. Alaca, "An introduction to 2-fuzzy $n$-normed linear spaces and a new perspective to the Mazur-Ulam problem," *Journal of Inequalities and Applications*, vol. 2012, article 14, 2012.

[13] T. M. Rassias, "Properties of isometric mappings," *Journal of Mathematical Analysis and Applications*, vol. 235, no. 1, pp. 108–121, 1999.

[14] J. A. Baker, "Isometries in normed spaces," *The American Mathematical Monthly*, vol. 78, pp. 655–658, 1971.

[15] Y. J. Cho, P. C. S. Lin, S. S. Kim, and A. Misiak, *Theory of 2-Inner Product Spaces*, Nova Science Publishers, Huntington, NY, USA, 2001.

[16] T. Z. Xu and J. M. Rassias, "On the Hyers-Ulam stability of a general mixed additive and cubic functional equation in $n$-Banach spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 926390, 23 pages, 2012.

[17] Al. Narayanan and S. Vijayabalaji, "Fuzzy $n$-normed linear space," *International Journal of Mathematics and Mathematical Sciences*, vol. 2005, no. 24, pp. 3963–3977, 2005.

[18] T. Z. Xu, J. M. Rassias, and W. X. Xu, "Stability of a general mixed additive-cubic functional equation in non-Archimedean fuzzy normed spaces," *Journal of Mathematical Physics*, vol. 51, no. 9, Article ID 093508, 19 pages, 2010.

[19] A. Khrennikov, *Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1997.