Research Article

Slant Curves in the Unit Tangent Bundles of Surfaces

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Let (M, g) be a surface and let $(U(TM), G)$ be the unit tangent bundle of M endowed with the Sasaki metric. We know that any curve $\Gamma(s)$ in $U(TM)$ consist of a curve $\gamma(s)$ in M and as unit vector field $X(s)$ along $\gamma(s)$. In this paper we study the geometric properties $\gamma(s)$ and $X(s)$ satisfying when $\Gamma(s)$ is a slant geodesic.

1. Introduction

Let (M, g, ϕ, ξ, η) be a 3-dimensional contact metric manifold. The slant curves in M are generalization of Legendrian curves which form a constant angle with the Reeb vector field ξ . Cho et al. [1] studied Lancret type problem for curves in Sasakian 3-manifold. They showed that a curve $\gamma(s) \subset M$ is slant if and only if $(\tau \pm 1)/k_\gamma$ is constant where τ and k_γ are torsion and curvature of $\gamma(s)$, respectively, and they also gave some examples of slant curves. One can find some other papers about slant curves in almost contact metric manifolds. For examples, Călin et al. [2] studied the slant curves in f -Kenmotsu manifolds. In [3], Călin and Crasmăreanu studied slant curves in normal almost contact manifolds.

Let (M, g) be a Riemannian manifold. Sasaki [4, 5] studied the geometries of TM endowed with the Sasaki metric and introduced the almost complex structure J in TM which is compatible with G_s . Tashiro [6] constructed an almost contact metric structure (G', ϕ', ξ', η') in the unit tangent bundle $U(TM)$ of M which is induced from the almost complex structure (TM, G_s, J) in TM . Klingenberg and Sasaki [7] studied geodesics in the unit tangent bundle of 2-sphere S^2 endowed with Sasaki metric and showed that $U(S^2)$ is isometric to $RP^3(1/4)$. Sasaki [8] studied the geodesics on the unit tangent bundles over space forms.

In this paper, we study the slant geodesics in the unit tangent bundle $U(TM)$ of some surface M . For any curve $\gamma(s)$ in M , let $T(s) = |T(s)|E(s)$ be the tangent vector field of $\gamma(s)$ and let $K_M(s)$ be the sectional curvature of M at $\gamma(s)$, we have the following theorems.

Theorem 1. *Let $\Gamma(s) = (\gamma(s), X(s))$ be a Legendrian geodesic parameterized by arc length in $U(TM)$ with domain $s \in [a, b]$. If the set consisting of points $s \in [a, b]$ such that $K_M(s) = 1$ is discrete, then $\gamma(s)$ is a geodesic of velocity 2 and $X(s)$ is the normal direction of γ in M .*

Theorem 2. *Let $\Gamma(s) = (\gamma(s), X(s))$ be a slant geodesic parameterized by arc length in $U(TM)$ which is not Legendrian. Under the assumptions of $K_M(s)$ as in Theorem 1, we have the following.*

- (1) *If $X(s) \neq \pm E(s)$, then $\gamma(s)$ is a geodesic of velocity 2 and X is a parallel vector field along $\gamma(s)$.*
- (2) *If $X(s) = \pm E(s)$, then $\gamma(s)$ is a curve of velocity $2|\cos \theta|$ with constant curvature $k_\gamma = \pm \tan \theta$.*

2. Preliminaries

Firstly, we introduce the (almost) contact metric structure on a Riemannian manifold of odd dimension. With the same notations as in [9]; let M be a real $(2n+1)$ -dimensional $\mathcal{C}^\infty$ manifold and $\mathfrak{X}(M)$ the Lie algebra of $\mathcal{C}^\infty$ vector fields on M . An almost cocomplex structure on M is defined by a $\mathcal{C}^\infty$ (1,1)-tensor ϕ , a $\mathcal{C}^\infty$ vector field ξ and a $\mathcal{C}^\infty$ 1-form η on M such that for any point $x \in M$ we have

$$\phi_x^2 = -I + \eta_x \otimes \xi_x, \quad \eta_x(\xi_x) = 1, \quad (1)$$

where I denotes the identity transformation of the tangent space $T_x M$ at x . Manifolds equipped with an almost

cocomplex structure are called *almost contact manifolds*. A Riemannian manifold with a metric g and an almost contact structure (ϕ, ξ, η) is called *almost contact metric* or *almost co-Hermitian manifold* if g and (ϕ, ξ, η) satisfies

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2)$$

for any vector fields $X, Y \in \mathfrak{X}(M)$.

As in Kahler geometry, we can define the fundamental 2-form Φ on the almost contact metric manifold (M, g, ϕ, ξ, η) as

$$\Phi(X, Y) = g(X, \phi Y), \quad (3)$$

for any vector fields $X, Y \in \mathfrak{X}(M)$. Obviously, this form satisfies $\eta \wedge \Phi^n \neq 0$ which means every almost contact metric manifold is orientable and that (η, Φ) defines an *almost cosymplectic structure* on M . If the fundamental 2-form $\Phi = d\eta$, then we call (M, g, ϕ, ξ, η) a contact metric manifold.

Let (x^i, u^i) be the locally coordinate systems on the tangent bundle TM of M . Sasaki [4, 5] defined a metric G_s on TM which is the natural lifts of the metric g on M as follows:

$$\begin{aligned} G_s(X^h, Y^h) &= g(X, Y), \\ G_s(X^h, Y^v) &= 0, \\ G_s(X^v, Y^v) &= g(X, Y), \end{aligned} \quad (4)$$

where $X, Y \in \mathfrak{X}(M)$ and $X^h = X^i \partial_{x^i} - X^i u^j \Gamma_{ij}^k \partial_{u^k}$ and $X^v = X^i \partial_{u^i}$ are the horizontal and vertical lifts of X at (x, u) with respect to Levi-Civita connection ∇ of g , respectively, and $\{\Gamma_{ij}^k\}$ are the Christoffel symbols of ∇ . The Levi-Civita connection $\bar{\nabla}$ of G_s is defined as

$$\begin{aligned} \bar{\nabla}_{X^h} Y^h &= (\nabla_X Y)^h - \frac{1}{2} (R_{XY}^M u)^v, & \bar{\nabla}_{X^v} Y^h &= \frac{1}{2} (R_{uX}^M Y)^h, \\ \bar{\nabla}_{X^h} Y^v &= (\nabla_X Y)^v + \frac{1}{2} (R_{uY}^M X)^h, & \bar{\nabla}_{X^v} Y^v &= 0, \end{aligned} \quad (5)$$

where $X, Y \in \mathfrak{X}(M)$ and R^M is the curvature tensor on (M, g) . The almost complex structure J on TM which is compatible with G_s is given by

$$JX^h = X^v, \quad JX^v = -X^h, \quad (6)$$

for any vector field $X \in \mathfrak{X}(M)$.

We know that the normal vector field of the unit tangent bundle $U(TM) = \{(x, u) \in TM; g(u, u) = 1\}$ of M at $(x, u) \in U(TM)$ in (TM, G_s) is u^v . Let $\mathfrak{X}(U(TM)) = \{X^h + Y^t; X, Y \in \mathfrak{X}(M)\}$ be the Lie algebra of $\mathcal{E}^\infty$ vector fields on $U(TM)$ where $X^h = X^i \partial_{x^i} - X^i u^j \Gamma_{ij}^k \partial_{u^k}$ and $X^t = (X - g(X, u)u)^v$ are the horizontal and tangential lifts of X at $(x, u) \in U(TM)$, respectively. Let $i : U(TM) \rightarrow TM$ be the including map

from $U(TM)$ to TM and this map induces a metric $G' = i^* G_s$ in $U(TM)$ as follows:

$$\begin{aligned} G'(X^h, Y^h) &= g(X, Y), \\ G'(X^h, Y^t) &= 0, \\ G'(X^t, Y^t) &= g(X, Y) - g(X, u)g(Y, u), \end{aligned} \quad (7)$$

for any vector fields $X, Y \in \mathfrak{X}(M)$.

From Tashiro [6], we know that there is an almost contact metric structure (G', ϕ', ξ', η') in $U(TM)$ which is induced from the almost complex structure J in TM such that

$$J \circ i_* = i_* \circ \phi' + \eta' \otimes u^v, \quad J \circ i_* (\xi') = u^v. \quad (8)$$

This implies that

$$\eta'(\xi') = 1, \quad \phi'(\xi') = 0, \quad \eta' \circ \phi' = 0. \quad (9)$$

From Blair et al. [10], at any point $(x, u) \in U(TM)$ we have

$$\begin{aligned} \xi' &= u^h, \\ \eta'(\bar{X}) &= G'(\bar{X}, u^h), \\ \Phi'(\bar{X}, \bar{Y}) &= G'(\bar{X}, \phi' \bar{Y}) = 2d\eta'(\bar{X}, \bar{Y}), \\ \phi' X^h &= X^t, \quad \phi' X^t = -X^h + \eta'(X^h) \xi', \end{aligned} \quad (10)$$

where $X \in \mathfrak{X}(M)$ and $\bar{X}, \bar{Y} \in \mathfrak{X}(U(TM))$. Hence, we know that there is a contact metric structure (G, ϕ, ξ, η) in $U(TM)$ such that

$$\eta = \frac{1}{2} \eta', \quad \xi = 2\xi', \quad \phi = \phi', \quad G = \frac{1}{4} G'. \quad (11)$$

By (5), we know that the Levi-Civita connection D of $(U(TM), G)$ is determined by the following:

$$\begin{aligned} D_{X^h} Y^h &= (\bar{\nabla}_{X^h} Y^h)^\top = (\nabla_X Y)^h - \frac{1}{2} (R_{XY} u)^t, \\ D_{X^t} Y^h &= (\bar{\nabla}_{X^t} Y^h)^\top = \frac{1}{2} (R_{uX} Y)^h, \\ D_{X^h} Y^t &= (\bar{\nabla}_{X^h} Y^t)^\top = (\nabla_X Y)^t + \frac{1}{2} (R_{uY} X)^h, \\ D_{X^t} Y^t &= (\bar{\nabla}_{X^t} Y^t)^\top = -g(Y, u) X^t, \end{aligned} \quad (12)$$

for any $(x, u) \in U(TM)$ and $X, Y \in \mathfrak{X}(U(TM))$.

3. Geodesic Slant Curves in $U(TM)$

Let (M, g) be a surface and let $\gamma : [a, b] \rightarrow M$ be a curve in M . Let $\Gamma(s) = (\gamma(s), X(s))$ be a curve in $(U(TM), G, \phi, \xi, \eta)$, where the contact metric structure is given by (11).

Definition 3. We say that $\Gamma(s)$ is a slant curve in $U(TM)$ if the angle between the tangent vector field $\Gamma'(s)$ of $\Gamma(s)$ and ξ is constant.

Assume that $\Gamma(s)$ is parameterized by the arc length. We know that

$$\begin{aligned}\Gamma'(s) &= \frac{dx^i}{ds} \partial_{x^i} + \frac{dX^i}{ds} \partial_{u^i} \\ &= \frac{dx^i}{ds} (\partial_{x^i})^h (\Gamma(s)) + \left(\frac{dX^i}{ds} + \frac{dx^j}{ds} X^k \Gamma_{jk}^i \right) \partial_{u^i} (\Gamma(s)) \\ &= (T^h + (\nabla_T X)^v) (\Gamma(s)),\end{aligned}\quad (13)$$

where $T(s) = \gamma'(s)$. Since $G(\Gamma', \Gamma') = 1$ and $G = G'/4$ we have

$$g(T, T) + g(\nabla_T X, \nabla_T X) = 4. \quad (14)$$

Let θ be the angle between $\Gamma'(s)$ and ξ . We have

$$\cos \theta(s) = G(\Gamma'(s), \xi) = \frac{g(T(s), X(s))}{2}. \quad (15)$$

Taking the derivative on both sides of (15) with respect to s , we derive that

$$\begin{aligned}-\sin \theta \cdot \theta' &= \frac{d}{ds} G(\Gamma'(s), \xi) \\ &= G(D_{\Gamma'(s)} \Gamma'(s), \xi) \\ &\quad + G(\Gamma'(s), D_{\Gamma'(s)} \xi) \\ &= k_\Gamma \eta(N_\Gamma) + G(\Gamma'(s), 2(\nabla_T X)^h \\ &\quad + (R_{u\nabla_T X} u)^h - (R_{Tu} u)^t) \\ &= k_\Gamma \eta(N_\Gamma) + 8g(\nabla_T X, T) - 8R(T, X, X, \nabla_T X),\end{aligned}\quad (16)$$

where $\theta' = d\theta/ds$, $\xi(s) = 2X^h(s)$, and k_Γ and N_Γ are the curvature and the direction of the acceleration of $\Gamma(s)$, respectively.

If $\Gamma(s)$ is a slant curve, that is, $\theta \equiv \text{Const}$, from (16) we have

$$k_\Gamma \eta(N_\Gamma) + 8g(\nabla_T X, T) - 8R(T, X, X, \nabla_T X) = 0. \quad (17)$$

Let $\{E, N\}$ be the Frenet frame on $\gamma(s)$. It follows from (15) that we have

$$X = \frac{2 \cos \theta}{\alpha} E + 2\beta N, \quad (18)$$

where $\alpha = |T|$ and β is a $\mathcal{C}^\infty$ function. It follows that

$$\begin{aligned}\nabla_T X &= \left(\frac{2 \cos \theta}{\alpha} \right)' E + 2 \cos \theta \cdot k_\gamma N + 2\beta' N - 2\alpha\beta k_\gamma E \\ &= -2 \left(\frac{\cos \theta}{\alpha^2} \alpha' + \alpha\beta k_\gamma \right) E + 2(k_\gamma \cos \theta + \beta') N.\end{aligned}\quad (19)$$

Substituting (19) into (14), we have

$$\alpha^2 + 4 \left(\frac{\cos \theta}{\alpha^2} \alpha' + \alpha\beta k_\gamma \right)^2 + 4(k_\gamma \cos \theta + \beta')^2 = 4. \quad (20)$$

Since $g(X, \nabla_T X) = 0$, from (15) and (19) we have

$$T = 2 \cos \theta X - \frac{2}{g(\nabla_T X, \nabla_T X)} \left(\frac{\cos \theta}{\alpha} \alpha' + \alpha^2 \beta k_\gamma \right) \nabla_T X. \quad (21)$$

It follows that

$$\begin{aligned}R(T, X, X, \nabla_T X) &= -\frac{2 \left((\cos \theta / \alpha) \alpha' + \alpha^2 \beta k_\gamma \right) R(\nabla_T X, X, X, \nabla_T X)}{g(\nabla_T X, \nabla_T X)} \\ &= -2 \left(\frac{\cos \theta}{\alpha} \alpha' + \alpha^2 \beta k_\gamma \right) K_M(s).\end{aligned}\quad (22)$$

It follows from (19) and (22) that (17) turns into

$$k_\Gamma \eta(N_\Gamma) + 2 \left(\frac{\cos \theta}{\alpha} \alpha' + \alpha^2 \beta k_\gamma \right) (K_M - 1) = 0. \quad (23)$$

For the Legendrian curves, we have the following theorem.

Theorem 1. Let $\Gamma(s) = (\gamma(s))$ and let $X(s)$ be a Legendrian geodesic parameterized by arc length in $(U(TM), G, \phi, \xi, \eta)$ with domain $s \in [a, b]$. Suppose that the set consisting of points $s \in [a, b]$ such that the sectional curvature of M at $\gamma(s)$ satisfying $K_M(s) = 1$ is discrete. Then $\gamma(s)$ is a geodesic of velocity 2 and $X(s)$ is the normal direction of $\gamma(s)$ in M .

Proof. Since $\Gamma(s)$ is a Legendrian geodesic, we have

$$\cos \theta = 0, \quad \beta = \pm \frac{1}{2}, \quad X = \pm N, \quad k_\Gamma = 0. \quad (24)$$

Substituting these into (20) and (23), we have

$$\begin{aligned}\alpha^2 (1 + k_\gamma^2) &= 4, \\ \alpha^2 k_\gamma (K_M - 1) &= 0.\end{aligned}\quad (25)$$

By the second equation of (25) and the assumption of $K(s)$ we have $k_\gamma = 0$ which means $\gamma(s)$ is a geodesic in M . Substituting $k_\gamma = 0$ into the first equation of (25) we have $\alpha = 2$. This completes the proof of Theorem 1. $\square$

For the non-Legendrian slant geodesics, we have the following theorem.

Theorem 2. Let $\Gamma(s) = (\gamma(s), X(s))$ be a slant geodesic in $U(TM)$ which is not Legendrian. Under the assumptions of K_M as in Theorem 1, we have the following.

- (1) If $X(s) \neq \pm E(s)$, then $\gamma(s)$ is a geodesic of velocity 2 and X is a parallel vector field along $\gamma(s)$.
- (2) If $X(s) = \pm E(s)$, then $\gamma(s)$ is a curve of velocity $2|\cos \theta|$ with constant curvature $k_\gamma = \pm \tan \theta$.

Proof. By the assumption we know that $\theta \neq \pi/2$, and from which and (18) we have $\beta \neq \pm 1/2$. Since $g(X, X) = 1$, from (18) we have

$$4\left(\frac{\cos \theta}{\alpha}\right)^2 + 4\beta^2 = 1, \quad (26)$$

from which we derive that

$$\frac{\cos \theta}{\alpha} = \begin{cases} \frac{\sqrt{1-4\beta^2}}{2}, & \theta \in \left[0, \frac{\pi}{2}\right), \\ -\frac{\sqrt{1-4\beta^2}}{2}, & \theta \in \left(\frac{\pi}{2}, \pi\right]. \end{cases} \quad (27)$$

Suppose that $\theta \in [0, \pi/2)$, then we have

$$\alpha = \frac{2 \cos \theta}{\sqrt{1-4\beta^2}}. \quad (28)$$

Taking derivative on both side of this equation with respect to s , we have

$$\frac{\cos \theta}{\alpha^2} \alpha' = \frac{2\beta\beta'}{\sqrt{1-4\beta^2}}. \quad (29)$$

It follows that

$$\frac{\cos \theta}{\alpha} \alpha' + \alpha^2 \beta k_\gamma = \frac{4\beta \cos \theta}{1-4\beta^2} (\beta' + k_\gamma \cos \theta). \quad (30)$$

Substituting (30) into (23), we have

$$k_\Gamma \eta(N_\Gamma) + \frac{8\beta \cos \theta}{1-4\beta^2} (\beta' + k_\gamma \cos \theta) (K_M - 1) = 0. \quad (31)$$

Remark 4. (31) also holds for $\theta \in (\pi/2, \pi]$.

For proving Theorem 2, we need the following lemma.

Lemma 5. *If there is some $s_0 \in [a, b]$ such that $\beta(s_0) = 0$, then β is vanishing everywhere.*

Proof. Substituting (30) into (20), we have

$$\frac{1}{1-4\beta^2} (\cos^2 \theta + (k_\gamma \cos \theta + \beta')^2) = 1, \quad (32)$$

and from which we derive that

$$k_\gamma \cos \theta + \beta' = \pm \sqrt{1-4\beta^2 - \cos^2 \theta}. \quad (33)$$

Since $k_\Gamma = 0$ and under the assumption of $K(s)$, from (31) and (33) we have

$$\beta(1-4\beta^2 - \cos^2 \theta) = 0. \quad (34)$$

Let I be a subset of $[a, b]$ such that $\beta|_I = 0$. Suppose that I is not empty. From (34) we have

$$\beta(s) = \frac{\pm \sqrt{1 - \cos^2 \theta}}{2}, \quad (35)$$

for any $s \in I - [a, b]$. It is a contradiction to the continuity of β unless $\cos \theta = \pm 1$; that is, $\theta = 0$ or π . Substituting $\theta = 0$ or π into (34) we have $\beta = 0$ which means $I = [a, b]$. This completes the proof of the Lemma.

Firstly, we consider $I = \emptyset$ which implies $\beta \neq 0$. It follows from which and (18) we have $X \neq \pm E$.

From (34) we have

$$\beta \equiv \frac{\pm \sin \theta}{2}. \quad (36)$$

It follows from which and (27) we have $\alpha = 2$. Substituting β into (33) we have $k_\gamma = 0$.

Substituting α, β , and k_γ into (19), we have $\nabla_T X = 0$. This proves Case (1) in Theorem 2.

Secondly, we consider $I = [a, b]$ which implies $\beta \equiv 0$. It follows from which and (18) we have $X = \pm E$ and $\alpha = 2|\cos \theta|$.

Substituting α and β into (33) we have

$$k_\gamma = \pm \sqrt{\frac{1}{\cos^2 \theta} - 1} = \pm \tan \theta, \quad (37)$$

which proves Case (2) in Theorem 2. This completes the proof of Theorem 2. $\square$

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