Research Article

Sandwich Theorem of Cover Times

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Based on a connection between cover times of graphs and Talagrand's theory of majorizing measures, we establish sandwich theorems for cover times as well as blanket times.

1. Introduction

Let $G = (V, E)$ be a connected graph with n vertices and m edges. Consider a simple random walk on G : we start at a vertex $v \in V$; if at step t we are at vertex u , then we move from u to a neighbor at step $t + 1$ with probability d_u^{-1} , where d_u is the degree of u . Let E_v be the expectation operator governing the random walk started at vertex $v \in V$. The cover time of G is defined as

$$c(G) = \max_{v \in V} E_v \mu(G), \quad (1)$$

where $\mu(G)$ is the first time $t \geq 1$ that all vertices of G have been traversed [1].

Another relevant quantity is the strong δ -blanket time [2]. For $v \in V$, let $\pi_v = d_v/2m$ be the stationary measure of the random walk, and let $r_{v,t}$ be the number of visits to v up to time t . For $\delta \in (0, 1)$, the strong δ -blanket time is defined as

$$b(G, \delta) = \max_{v \in V} E_v \tau(\delta), \quad (2)$$

where $\tau(\delta)$ is the first time $t \geq 1$ such that

$$\frac{r_{u,t}/\pi_u}{r_{v,t}/\pi_v} \geq \delta, \quad (3)$$

holds for any two vertices u and v . Clearly, we have $b(G, \delta) \geq c(G)$ for any $\delta \in (0, 1)$. We refer the reader to [1, 3] for more background information on random walks.

Recently, Ding et al. [4] established an important connection between cover times (blanket times) and the γ_2 functional from Talagrand's theory of majorizing measures [5, 6];

see Theorem 1. We first review the γ_2 functional in brief. Consider a compact metric space (X, d) , and let $M_0 = 1$, $M_k = 2^{2^k}$ for $k \geq 1$. For a partition $\mathcal{P}$ of X and an element $x \in X$, denote by $\mathcal{P}(x)$ the unique set $S \in \mathcal{P}$ containing x . An admissible sequence $\{\mathcal{A}_k\}_{k \geq 0}$ of partitions of X is that $\mathcal{A}_{k+1}$ is a refinement of $\mathcal{A}_k$, and $|\mathcal{A}_k| \leq M_k$ for $k \geq 0$. The γ_2 functional is defined as

$$\gamma_2(X, d) = \inf_{\{\mathcal{A}_k\}} \sup_{x \in X} \sum_{k=0}^{\infty} 2^{k/2} \text{diam}(\mathcal{A}_k(x)), \quad (4)$$

where $\text{diam}(S)$ represents the diameter of S , that is, $\text{diam}(S) = \sup_{x, y \in S} d(x, y)$. Throughout the paper, we view graph G as a metric space with distance induced by the commute time $\kappa_G(u, v)$ between two vertices $u, v \in V$. Hence, (V, κ_G) is a compact metric space.

Theorem 1 (see [4]). *For any graph $G = (V, E)$ and $\delta \in (0, 1)$,*

$$c(G) \asymp (\gamma_2(V, \sqrt{\kappa_G}))^2 \asymp_\delta b(G, \delta), \quad (5)$$

where $A \asymp B$ means $C_1 B \leq A \leq C_2 B$ for some constants C_1, C_2 , and furthermore, $A \asymp_\delta B$ emphasizes that the constants may depend on δ . Here, $\sqrt{\kappa_G}(u, v) = \sqrt{\kappa_G(u, v)}$.

A comparison theorem for cover times is also presented.

Theorem 2 (see [4]). *Suppose that graphs G and G' are on the same vertex set V , and κ_G and $\kappa_{G'}$ are the distances induced by respective commute times. If there exists a number $L \geq 1$ such that $\kappa_G(u, v) \leq L \cdot \kappa_{G'}(u, v)$ for all $u, v \in V$, then*

$$c(G) \leq O(L) \cdot c(G'). \quad (6)$$

In this paper, we extend this nice comparison theorem and provide several applications.

2. Results

We have the following results.

Theorem 3. Suppose that three graphs G, G_1 , and G_2 are on the same vertex set V and that κ_G, κ_{G_1} , and κ_{G_2} are the distances induced by respective commute times. If there exist $\alpha \in (0, 1]$, $L_1 > 0$, and $L_2 > 0$ such that $L_1 \cdot \kappa_{G_1}(u, v)^\alpha \leq \kappa_G(u, v) \leq L_2 \cdot \kappa_{G_2}(u, v)$ for all $u, v \in V$, then

$$O(L_1) \cdot c(G_1)^\alpha \leq c(G) \leq O(L_2) \cdot c(G_2). \quad (7)$$

Proof. From the assumption, we have

$$\sqrt{L_1} \cdot \kappa_{G_1}(u, v)^{\alpha/2} \leq \sqrt{\kappa_G}(u, v) \leq \sqrt{L_2} \cdot \sqrt{\kappa_{G_2}}(u, v), \quad (8)$$

for all vertices u and v . Capitalizing on the C_r -inequality, which says that

$$\left(\sum_{i=1}^n x_i \right)^r \leq C_r \left(\sum_{i=1}^n x_i^r \right), \quad (9)$$

where $C_r = 1$ if $r \in (0, 1]$ and $C_r = n^{r-1}$ if $r \in (1, \infty)$, we obtain

$$\sqrt{L_1} \cdot \gamma_2(V, \sqrt{\kappa_{G_1}})^\alpha \leq \gamma_2(V, \sqrt{\kappa_G}) \leq \sqrt{L_2} \cdot \gamma_2(V, \sqrt{\kappa_{G_2}}) \quad (10)$$

by using definition (4).

It follows from Theorem 1 that

$$O(L_1) \cdot c(G_1)^\alpha \leq c(G) \leq O(L_2) \cdot c(G_2) \quad (11)$$

as desired. $\square$

Generally, the conditions imposed on commute times in Theorem 3 are thorny, if possible, to test, especially for complex and large-scale graphs. However, commute times for recursive graphs are likely to be estimated (see, e.g., [7]).

Based on this comparison characterization, we have the following bounds regarding the ratio of cover times.

Corollary 4. Maintaining the notations of Theorem 3, if $G_1 = G_2$, one has

$$\frac{O(L_1)}{(\max_{u,v \in V} H(u, v))^{1-\alpha} (1 + \ln n)^{1-\alpha}} \leq \frac{c(G)}{c(G_1)} \leq O(L_2), \quad (12)$$

where $H(u, v)$ is the expected hitting time from u to v in G_1 .

Proof. If $G_1 = G_2$, we obtain

$$O(L_1) \cdot c(G_1)^\alpha \leq c(G) \leq O(L_2) \cdot c(G_1). \quad (13)$$

An upper bound of cover time [8] yields

$$\begin{aligned} \frac{O(L_1)}{(\max_{u,v \in V} H(u, v))^{1-\alpha} (1 + \ln n)^{1-\alpha}} &\leq \frac{O(L_1)}{c(G_1)^{1-\alpha}} \\ &\leq \frac{c(G)}{c(G_1)} \leq O(L_2), \end{aligned} \quad (14)$$

where $n = |V|$ as mentioned before. $\square$

The following result is a “sandwich theorem” for monotonic graph sequences.

Corollary 5. Maintaining the notations of Theorem 3, if G_1 is a graph obtained by deleting an edge from G and G_2 is obtained by adding an edge to G , one has

$$O\left(\frac{m}{m+1}\right) \cdot c(G_2) \leq c(G) \leq O\left(\frac{m}{m-1}\right) \cdot c(G_1), \quad (15)$$

where $m = |E|$ is the number of edges in G .

Proof. Since G_1 and G_2 have $m - 1$ and $m + 1$ edges, respectively, we obtain by [3, Theorem 2.9] that

$$\frac{m}{m+1} \kappa_{G_2}(u, v) \leq \kappa_G(u, v) \leq \frac{m}{m-1} \kappa_{G_1}(u, v), \quad (16)$$

for any $u, v \in V$. Thus, the result follows directly from Theorem 3 by taking $\alpha = 1$. $\square$

We mention that it is recently shown in [9] that $c(G_2)/4 \leq c(G)$. We conclude the paper with a result on δ -strong blanket times analogous to our main theorem.

Corollary 6. Maintaining the notations of Theorem 3, for any $\delta \in (0, 1)$, one has

$$O_\delta(L_1) \cdot b(G_1, \delta)^\alpha \leq b(G, \delta) \leq O_\delta(L_2) \cdot b(G_2, \delta). \quad (17)$$

Proof. The same proof in Theorem 3 applies by using Theorem 1. $\square$

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