Research Article

Optimal Lehmer Mean Bounds for the Combinations of Identric and Logarithmic Means

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For any $\alpha \in (0, 1)$, we answer the questions: what are the greatest values p and λ and the least values q and μ , such that the inequalities $L_p(a, b) < I^\alpha(a, b)L^{1-\alpha}(a, b) < L_q(a, b)$ and $L_\lambda(a, b) < \alpha I(a, b) + (1 - \alpha)L(a, b) < L_\mu(a, b)$ hold for all $a, b > 0$ with $a \neq b$? Here, $I(a, b)$, $L(a, b)$, and $L_p(a, b)$ denote the identric, logarithmic, and p th Lehmer means of two positive numbers a and b , respectively.

1. Introduction

For $a, b > 0$ with $a \neq b$, the logarithmic mean $L(a, b)$ and identric mean $I(a, b)$ are defined by

$$L(a, b) = \frac{a - b}{\log a - \log b}, \quad (1)$$

$$I(a, b) = \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{1/(b-a)}, \quad (2)$$

respectively. In the recent past, both mean values have been the subject of intensive research. In particular, many remarkable inequalities for L and I can be found in the literature [1–19]. In [14, 17, 20], inequalities between L , I , and the classical arithmetic-geometric mean of Gauss are proved. The ratio of identric means leads to the weighted geometric mean

$$\frac{I(a^2, b^2)}{I(a, b)} = (a^a b^b)^{1/(a+b)}, \quad (3)$$

which has been investigated in [11, 13, 21]. It might be surprising that the logarithmic mean has applications in physics, economics, and even in meteorology [22–24]. In [22], the authors study a variant of Jensen's functional equation involving L , which appears in a heat conduction

problem. A representation of L as an infinite product and an iterative algorithm for computing the logarithmic mean as the common limit of two sequences of special geometric and arithmetic means are given in [5]. In [25, 26], it is shown that L can be expressed in terms of Gauss' hypergeometric function ${}_2F_1$. And in [26], the authors prove that the reciprocal of the logarithmic mean is strictly totally positive; that is, every $n \times n$ determinant with elements $1/L(a_i, b_j)$, where $0 < a_1 < a_2 < \dots < a_n$ and $0 < b_1 < b_2 < \dots < b_n$, is positive for all $n \geq 1$.

For $p \in \mathbb{R}$, the power mean $M_p(a, b)$ and Lehmer mean $L_p(a, b)$ of order p of two positive numbers a and b with $a \neq b$ are defined by

$$M_p(a, b) = \left(\frac{a^p + b^p}{2} \right)^{1/p}, \quad (4)$$

$$L_p(a, b) = \frac{a^{p+1} + b^{p+1}}{a^p + b^p}. \quad (5)$$

It is well known that $M_p(a, b)$ and $L_p(a, b)$ are strictly increasing with respect to $p \in \mathbb{R}$ for fixed $a, b > 0$ with $a \neq b$. The main properties for M_p and L_p are given in [27–32].

Let $A(a, b) = (a + b)/2$, $G(a, b) = \sqrt{ab}$, and $H(a, b) = 2ab/(a + b)$ be the arithmetic, geometric, and harmonic means

of two positive numbers a and b , respectively. Then it is well known that

$$\begin{aligned} \min \{a, b\} &< H(a, b) = L_{-1}(a, b) \\ &= M_{-1}(a, b) < G(a, b) = L_{-1/2}(a, b) \\ &= M_0(a, b) < L(a, b) < I(a, b) < A(a, b) \\ &= L_0(a, b) = M_1(a, b) < \max \{a, b\} \end{aligned} \quad (6)$$

for all $a, b > 0$ with $a \neq b$.

The following sharp bounds for L , I , $(LI)^{1/2}$, and $(L+I)/2$ in terms of power means are proved in [2–4, 6, 8, 9, 33]:

$$M_{2/3}(a, b) < I(a, b) < M_{\log 2}(a, b), \quad (7)$$

$$M_0(a, b) < L(a, b) < M_{1/3}(a, b),$$

$$M_0(a, b) < \sqrt{I(a, b)L(a, b)} < M_{1/2}(a, b), \quad (8)$$

$$\frac{1}{2}(I(a, b) + L(a, b)) < M_{1/2}(a, b)$$

for all $a, b > 0$ with $a \neq b$.

Alzer and Qiu [19] proved that

$$\begin{aligned} \alpha A(a, b) + (1 - \alpha)G(a, b) \\ < I(a, b) < \beta A(a, b) + (1 - \beta)G(a, b), \quad (9) \\ M_p(a, b) < \frac{1}{2}(I(a, b) + L(a, b)) \end{aligned}$$

for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 2/3$, $\beta \geq 2/e = 0.73575\dots$, and $p \leq \log 2/(1 + \log 2) = 0.40938\dots$

The following sharp upper and lower Lehmer mean bounds for L , I , $(LI)^{1/2}$, and $(L+I)/2$ are presented in [34]:

$$L_{-1/3}(a, b) < L(a, b) < L_0(a, b), \quad (10)$$

$$M_{-1/6}(a, b) < I(a, b) < L_0(a, b), \quad (11)$$

$$L_{-1/4}(a, b) < I^{1/2}(a, b)L^{1/2}(a, b) < L_0(a, b), \quad (12)$$

$$L_{-1/4}(a, b) < \frac{1}{2}(I(a, b) + L(a, b)) < L_0(a, b)$$

for all $a, b > 0$ with $a \neq b$.

The purpose of this paper is to present the best possible upper and lower Lehmer mean bounds of the product $I^\alpha(a, b)L^{1-\alpha}(a, b)$ and the sum $\alpha I(a, b) + (1 - \alpha)L(a, b)$ for any $\alpha \in (0, 1)$ and all $a, b > 0$ with $a \neq b$.

2. Lemmas

In order to establish our main result, we need several lemmas, which we present in this section.

Lemma 1. *If $p \geq -1/3$, then the following statements are true:*

- (1) $18p^2 + 21p + 8 > 0$;
- (2) $270p^3 + 567p^2 + 384p + 104 > 0$;

$$(3) 1134p^4 + 3429p^3 + 3879p^2 + 1950p + 448 > 0;$$

$$(4) 1458p^5 + 4941p^4 + 6903p^3 + 4809p^2 + 1714p + 320 > 0;$$

$$(5) 5346p^6 + 18225p^5 + 27297p^4 + 21996p^3 + 9672p^2 + 2264p + 320 > 0;$$

$$(6) 9p^3 + 33p^2 + 51p + 14 > 0.$$

Proof. (1) We clearly see that

$$\begin{aligned} 18p^2 + 21p + 8 &> 18p^2 + \left(-\frac{1}{3}\right) \times 21 + 8 \\ &= 18p^2 + 1 > 0. \end{aligned} \quad (13)$$

(2) Let

$$g_1(p) = 270p^3 + 567p^2 + 384p + 104. \quad (14)$$

Then simple computations lead to

$$\begin{aligned} g_1\left(-\frac{1}{3}\right) &= 29 > 0, \\ g_1'(p) &= 6(135p^2 + 189p + 64) \\ &> 6\left[135p^2 + 189 \times \left(-\frac{1}{3}\right) + 64\right] \\ &= 6(135p^2 + 1) > 0. \end{aligned} \quad (15)$$

Therefore, Lemma 1(2) follows from (14) and (15).

(3) Let

$$g_2(p) = 1134p^4 + 3429p^3 + 3879p^2 + 1950p + 448. \quad (16)$$

Then simple computations yield

$$\begin{aligned} g_2\left(-\frac{1}{3}\right) &= 116 > 0, \\ g_2'(p) &= 4536p^3 + 10287p^2 + 7758p + 1950, \\ g_2'\left(-\frac{1}{3}\right) &= 339 > 0, \\ g_2''(p) &= 13608p^2 + 20574p + 7758 \\ &> 13608p^2 + 20574 \times \left(-\frac{1}{3}\right) + 7758 \\ &= 13608p^2 + 900 > 0. \end{aligned} \quad (17)$$

Therefore, Lemma 1(3) follows from (16) and (17).

(4) Let

$$\begin{aligned} g_3(p) &= 1458p^5 + 4941p^4 + 6903p^3 \\ &\quad + 4809p^2 + 1714p + 320. \end{aligned} \quad (18)$$

Then simple computations lead to

$$\begin{aligned}
 g_3\left(-\frac{1}{3}\right) &= \frac{247}{3} > 0, \\
 g_3'(p) &= 7290p^4 + 19764p^3 + 20709p^2 \\
 &\quad + 9618p + 1714, \\
 g_3'\left(-\frac{1}{3}\right) &= 167 > 0, \\
 g_3''(p) &= 29160p^3 + 59292p^2 + 41418p + 9618, \quad (19) \\
 g_3''\left(-\frac{1}{3}\right) &= 1320 > 0, \\
 g_3'''(p) &= 87480p^2 + 118584p + 41418 \\
 &> 87480p^2 + 41418 + 118584 \times \left(-\frac{1}{3}\right) \\
 &= 87480p^2 + 1890 > 0.
 \end{aligned}$$

Therefore, Lemma 1(4) follows from (18) and (19).
(5) Let

$$\begin{aligned}
 g_4(p) &= 5346p^6 + 18225p^5 + 27297p^4 + 21996p^3 \\
 &\quad + 9672p^2 + 2264p + 320. \quad (20)
 \end{aligned}$$

Then simple computations yield

$$g_4\left(-\frac{1}{3}\right) = \frac{284}{3} > 0, \quad (21)$$

$$\begin{aligned}
 g_4'(p) &= 32076p^5 + 91125p^4 + 109188p^3 \\
 &\quad + 65988p^2 + 19344p + 2264, \quad (22)
 \end{aligned}$$

$$g_4'\left(-\frac{1}{3}\right) = 97 > 0, \quad (23)$$

$$\begin{aligned}
 g_4''(p) &= 160380p^4 + 364500p^3 \\
 &\quad + 327564p^2 + 131976p + 19344, \quad (24)
 \end{aligned}$$

$$g_4''\left(-\frac{1}{3}\right) = 228 > 0, \quad (25)$$

$$g_4'''(p) = 108(5940p^3 + 10125p^2 + 6066p + 1222), \quad (26)$$

$$g_4'''\left(-\frac{1}{3}\right) = 11340 > 0, \quad (27)$$

$$g_4^{(4)}(p) = 1944(990p^2 + 1125p + 337). \quad (28)$$

It follows from (28) and the discriminant $\Delta = (1125)^2 - 4 \times 990 \times 337 = -68895 < 0$ of the quadratic function $f(p) = 990p^2 + 1125p + 337$ that

$$g_4^{(4)}(p) > 0. \quad (29)$$

Therefore, Lemma 1(5) follows from (20)–(27) and (29).

(6) Let

$$g_5(p) = 9p^3 + 33p^2 + 51p + 14. \quad (30)$$

Then we have

$$\begin{aligned}
 g_5\left(-\frac{1}{3}\right) &= \frac{1}{3} > 0, \\
 g_5'(p) &= 27p^2 + 66p + 51 \quad (31) \\
 &> 27p^2 - 66 \times \left(\frac{1}{3}\right) + 51 = 27p^2 + 29 > 0.
 \end{aligned}$$

Therefore, Lemma 1(6) follows from (30) and (31). $\square$

Lemma 2. Suppose that $h(t) = (6p+1)t^{6p+6} - 2t^{6p+4} - 2(3p+2)t^{6p+3} + (3p+1)t^{3p+6} - 3pt^{3p+5} - 3(p+1)t^{3p+4} + 3(p+1)t^{3p+2} + 3pt^{3p+1} - (3p+1)t^{3p} + 2(3p+2)t^3 + 3t^2 - 6p - 1$. If $p \in (-1/3, -1/6)$, then $h(t) > 0$ for $t > 1$.

Proof. Let $h_1(t) = h'(t)/t$, $h_2(t) = t^{4-3p}h_1''(t)$, $h_3(t) = h_2'''(t)/t$, and $h_4(t) = t^{4-3p}h_3'''(t)/[9p(p+1)(3p+1)(3p+2)^2(6p+1)]$; then elaborated computations lead to

$$h(1) = 0, \quad (32)$$

$$\begin{aligned}
 h_1(t) &= 6(p+1)(6p+1)t^{6p+4} - 6(3p+2)t^{6p+2} \\
 &\quad - 6(2p+1)(3p+2)t^{6p+1} \\
 &\quad + 3(p+2)(3p+1)t^{3p+4} \quad (33) \\
 &\quad - 3p(3p+5)t^{3p+3} - 3(p+1)(3p+4)t^{3p+2} \\
 &\quad + 3(p+1)(3p+2)t^{3p} + 3p(3p+1)t^{3p-1} \\
 &\quad - 3p(3p+1)t^{3p-2} + 6(3p+2)t + 6,
 \end{aligned}$$

$$h_1(1) = 0, \quad (34)$$

$$\begin{aligned}
 h_1'(t) &= 12(p+1)(3p+2)(6p+1)t^{6p+3} \\
 &\quad - 12(3p+1)(3p+2)t^{6p+1} \\
 &\quad - 6(2p+1)(3p+2)(6p+1)t^{6p} \\
 &\quad + 3(p+2)(3p+1)(3p+4)t^{3p+3} \\
 &\quad - 9p(p+1)(3p+5)t^{3p+2} \quad (35) \\
 &\quad - 3(p+1)(3p+2)(3p+4)t^{3p+1} \\
 &\quad + 9p(p+1)(3p+2)t^{3p-1} \\
 &\quad + 3p(3p-1)(3p+1)t^{3p-2} \\
 &\quad - 3p(3p-2)(3p+1)t^{3p-3} \\
 &\quad + 6(3p+2),
 \end{aligned}$$

$$h_1'(1) = 0, \quad (36)$$

$$\begin{aligned} h_2(t) = & 36(p+1)(2p+1)(3p+2)(6p+1)t^{3p+6} \\ & - 12(3p+1)(3p+2)(6p+1)t^{3p+4} \\ & - 36p(2p+1)(3p+2)(6p+1)t^{3p+3} \\ & + 9(p+1)(p+2)(3p+1)(3p+4)t^6 \\ & - 9p(p+1)(3p+2)(3p+5)t^5 \\ & - 3(p+1)(3p+1)(3p+2)(3p+4)t^4 \\ & + 9p(p+1)(3p-1)(3p+2)t^2 \\ & + 3p(3p-2)(3p-1)(3p+1)t \\ & - 9p(p-1)(3p-2)(3p+1), \end{aligned} \quad (37)$$

$$h_2(1) = 12(3p+1)(18p^2+21p+8), \quad (38)$$

$$\begin{aligned} h_2'(t) = & 108(p+1)(p+2)(2p+1)(3p+2)(6p+1)t^{3p+5} \\ & - 12(3p+1)(3p+2)(3p+4)(6p+1)t^{3p+3} \\ & - 108p(p+1)(2p+1)(3p+2)(6p+1)t^{3p+2} \\ & + 54(p+1)(p+2)(3p+1)(3p+4)t^5 \\ & - 45p(p+1)(3p+2)(3p+5)t^4 \\ & - 12(p+1)(3p+1)(3p+2)(3p+4)t^3 \\ & + 18p(p+1)(3p-1)(3p+2)t \\ & + 3p(3p-1)(3p-2)(3p+1), \end{aligned} \quad (39)$$

$$h_2'(1) = 12(3p+1)(9p+7)(18p^2+21p+8), \quad (40)$$

$$\begin{aligned} h_2''(t) = & 108(p+1)(p+2)(2p+1) \\ & \times (3p+2)(3p+5)(6p+1)t^{3p+4} \\ & - 36(p+1)(3p+1)(3p+2) \\ & \times (3p+4)(6p+1)t^{3p+2} \\ & - 108p(p+1)(2p+1)(3p+2)^2(6p+1)t^{3p+1} \\ & + 270(p+1)(p+2)(3p+1)(3p+4)t^4 \\ & - 180p(p+1)(3p+2)(3p+5)t^3 \\ & - 36(p+1)(3p+1)(3p+2)(3p+4)t^2 \\ & + 18p(p+1)(3p-1)(3p+2), \end{aligned} \quad (41)$$

$$h_2''(1) = 36(3p+1)(p+1)(270p^3+567p^2+384p+104), \quad (42)$$

$$\begin{aligned} h_3(t) = & 108(p+1)(p+2)(2p+1)(3p+2) \\ & \times (3p+4)(3p+5)(6p+1)t^{3p+2} \\ & - 36(p+1)(3p+1)(3p+2)^2 \\ & \times (3p+4)(6p+1)t^{3p} \\ & - 108p(p+1)(2p+1)(3p+1)(3p+2)^2 \\ & \times (6p+1)t^{3p-1} + 1080(p+1)(p+2) \\ & \times (3p+1)(3p+4)t^2 \\ & - 540p(p+1)(3p+2)(3p+5)t \\ & - 72(p+1)(3p+1)(3p+2)(3p+4), \\ h_3(1) = & 36(3p+1)(p+1) \\ & \times (1134p^4+3429p^3+3879p^2+1950p+448), \end{aligned} \quad (43)$$

$$\begin{aligned} h_3'(t) = & 108(p+1)(p+2)(2p+1)(3p+2)^2 \\ & \times (3p+4)(3p+5)(6p+1)t^{3p+1} \\ & - 108p(p+1)(3p+1)(3p+2)^2 \\ & \times (3p+4)(6p+1)t^{3p-1} \\ & - 108p(p+1)(2p+1)(3p-1)(3p+1)(3p+2)^2 \\ & \times (6p+1)t^{3p-2} + 2160(p+1)(p+2) \\ & \times (3p+1)(3p+4)t \\ & - 540p(p+1)(3p+2)(3p+5), \end{aligned} \quad (45)$$

$$\begin{aligned} h_3'(1) = & 108(p+1)(3p+1) \\ & \times (1458p^5+4941p^4+6903p^3 \\ & + 4809p^2+1714p+320), \end{aligned} \quad (46)$$

$$\begin{aligned} h_3''(t) = & 108(p+1)(p+2)(2p+1)(3p+1) \\ & \times (3p+2)^2(3p+4)(3p+5)(6p+1)t^{3p} \\ & - 108p(p+1)(3p-1)(3p+1)(3p+2)^2(3p+4) \\ & \times (6p+1)t^{3p-2} - 108p(p+1)(2p+1)(3p-2) \\ & \times (3p-1)(3p+1)(3p+2)^2(6p+1)t^{3p-3} \\ & + 2160(p+1)(p+2)(3p+1)(3p+4), \end{aligned} \quad (47)$$

$$\begin{aligned} h_3''(1) = & 108(p+1)(3p+1) \\ & \times (5346p^6+18225p^5+27297p^4+21996p^3 \\ & + 9672p^2+2264p+320), \end{aligned} \quad (48)$$

$$\begin{aligned}
h_4(t) &= 36(p+2)(2p+1)(3p+4)(3p+5)t^3 \\
&\quad - 12(1-3p)(2-3p)(3p+4)t \\
&\quad + 36(1-p)(1-3p)(2-3p)(2p+1) \\
&> 36(p+2)(2p+1)(3p+4)(3p+5)t \\
&\quad - 12(1-3p)(2-3p)(3p+4)t \\
&\quad + 36(1-p)(1-3p)(2-3p)(2p+1) \\
&= 24(3p+4)(9p^3+33p^2+51p+14)t \\
&\quad + 36(1-p)(1-3p)(2-3p)(2p+1).
\end{aligned} \tag{49}$$

It follows from Lemma 1(6) and (49) that

$$h_4(t) > 0 \tag{50}$$

for $t > 1$.

From Lemma 1(1)–(5) and (38)–(48), we clearly see that

$$\begin{aligned}
h_2(1) &> 0, \\
h_2'(1) &> 0, \\
h_2''(1) &> 0, \\
h_3(1) &> 0, \\
h_3'(1) &> 0, \\
h_3''(1) &> 0.
\end{aligned} \tag{51}$$

Therefore, Lemma 2 follows from $9p(p+1)(3p+1)(3p+2)^2(6p+1) > 0$ and (32)–(36) together with (50) and (51). $\square$

Lemma 3. Inequality $[M_{2/3}(a, b)]^\alpha [L_{-1/3}(a, b)]^{1-\alpha} > L_{(\alpha-2)/6}(a, b)$ holds for any $\alpha \in (0, 1)$ and all $a, b > 0$ with $a \neq b$.

Proof. Without loss of generality, we assume that $a > b$. Let $t = (a/b)^{1/3} > 1$ and $p = (\alpha - 2)/6$; then $p \in (-1/3, -1/6)$, $\alpha = 6p + 2$, and from (1) and (4) we have

$$\begin{aligned}
&[M_{2/3}(a, b)]^\alpha [L_{-1/3}(a, b)]^{1-\alpha} - L_{(\alpha-2)/6}(a, b) \\
&= \left(b \left[(t^{3p} + 1)(t^2 + 1)^{3p+2}(t + 1)^{6p+1} \right. \right. \\
&\quad \left. \left. - 2^{9p+3} t^{6p+1} (t^{3p+3} + 1) \right] \right)^\alpha \\
&\quad \times \left(2^{9p+3} t^{6p+1} (t^{3p} + 1) \right)^{1-\alpha}.
\end{aligned} \tag{52}$$

Let

$$\begin{aligned}
f(t) &= \log(t^{3p} + 1) + (3p + 2) \log(t^2 + 1) \\
&\quad + (6p + 1) \log(t + 1) - (6p + 1) \log t \\
&\quad - \log(t^{3p+3} + 1) - 3(3p + 1) \log 2.
\end{aligned} \tag{53}$$

Then simple computations lead to

$$f(1) = 0, \tag{54}$$

$$f'(t) = \frac{h(t)}{t(t+1)(t^2+1)(t^{3p}+1)(t^{3p+3}+1)}, \tag{55}$$

where $h(t)$ is defined as in Lemma 2.

From (54) and (55) together with Lemma 2, we clearly see that

$$f(t) > 0 \tag{56}$$

for $t > 1$.

Therefore, Lemma 3 follows from (52) and (53) together with (56). $\square$

3. Main Results

Theorem 4. Inequality $L_{(\alpha-2)/6}(a, b) < I^\alpha(a, b)L^{1-\alpha}(a, b) < L_0(a, b)$ holds for any $\alpha \in (0, 1)$ and all $a, b > 0$ with $a \neq b$, and $L_{(\alpha-2)/6}(a, b)$ and $L_0(a, b)$ are the best possible lower and upper Lehmer mean bounds for the product $I^\alpha(a, b)L^{1-\alpha}(a, b)$.

Proof. Inequality $I^\alpha(a, b)L^{1-\alpha}(a, b) < L_0(a, b)$ follows directly from (10) and (11).

For the other inequality, we note that

$$\begin{aligned}
&I^\alpha(a, b)L^{1-\alpha}(a, b) - L_{(\alpha-2)/6}(a, b) \\
&= I^\alpha(a, b)L^{1-\alpha}(a, b) - [M_{2/3}(a, b)]^\alpha [L_{-1/3}(a, b)]^{1-\alpha} \\
&\quad + [M_{2/3}(a, b)]^\alpha [L_{-1/3}(a, b)]^{1-\alpha} - L_{(\alpha-2)/6}(a, b).
\end{aligned} \tag{57}$$

Therefore, $I^\alpha(a, b)L^{1-\alpha}(a, b) > L_{(\alpha-2)/6}(a, b)$ follows from (7) and (10) together with Lemma 3 and (57).

Next, we prove that $L_{(\alpha-2)/6}(a, b)$ and $L_0(a, b)$ are the best possible lower and upper Lehmer mean bounds for the product $I^\alpha(a, b)L^{1-\alpha}(a, b)$.

For any $0 < \varepsilon < 1$ and $x > 0$, from (1) and (2) we have

$$\begin{aligned}
&\lim_{x \rightarrow +\infty} \frac{I^\alpha(1, x)L^{1-\alpha}(1, x)}{L_{-\varepsilon}(1, x)} \\
&= \lim_{x \rightarrow +\infty} \left[\frac{x^{-\varepsilon} + 1}{x^{\varepsilon-1} + 1} e^{\alpha((\log x)/(x-1)-1)} \right. \\
&\quad \left. \times \left(1 - \frac{1}{x} \right)^{1-\alpha} \frac{x^\varepsilon}{(\log x)^{1-\alpha}} \right] = +\infty,
\end{aligned} \tag{58}$$

$$\begin{aligned}
&L_{(\alpha-2)/6+\varepsilon}(1, 1+x) - I^\alpha(1, 1+x)L^{1-\alpha}(1, 1+x) \\
&= \frac{(1+x)^{(\alpha+4)/6+\varepsilon} + 1}{(1+x)^{(\alpha-2)/6+\varepsilon} + 1} - e^{\alpha[(1+x)\log(1+x)/(x-1)]} \\
&\quad \times \frac{x^{1-\alpha}}{[\log(1+x)]^\alpha}.
\end{aligned}$$

Letting $x \rightarrow 0$ and making use of Taylor expansion, one has

$$\begin{aligned}
 & \frac{(1+x)^{(\alpha+4)/6+\varepsilon} + 1}{(1+x)^{(\alpha-2)/6+\varepsilon} + 1} - e^{\alpha[(1+x)\log(1+x)/x-1]} \\
 & \quad \times \frac{x^{1-\alpha}}{[\log(1+x)]^\alpha} \\
 & = 1 + \frac{1}{2}x + \left(\frac{\alpha}{24} - \frac{1}{12} + \frac{\varepsilon}{4}\right)x^2 + o(x^2) \\
 & \quad - \left[1 + \frac{\alpha}{2}x + \alpha\left(\frac{1}{8}\alpha - \frac{1}{6}\right)x^2 + o(x^2)\right] \\
 & \quad \times \left[1 + \left(\frac{1}{2} - \frac{\alpha}{2}\right)x + \left(\frac{1}{8}\alpha^2 - \frac{1}{24}\alpha - \frac{1}{12}\right)x^2 + o(x^2)\right] \\
 & = 1 + \frac{1}{2}x + \left(\frac{\alpha}{24} - \frac{1}{12} + \frac{\varepsilon}{4}\right)x^2 \\
 & \quad - \left[1 + \frac{1}{2}x + \left(\frac{\alpha}{24} - \frac{1}{12}\right)x^2\right] + o(x^2) \\
 & = \frac{\varepsilon}{4}x^2 + o(x^2). \tag{59}
 \end{aligned}$$

Equations (58) and (59) imply that for any $0 < \varepsilon < 1$, there exist $X_1 = X_1(\alpha, \varepsilon) > 1$ and $\delta_1 = \delta_1(\alpha, \varepsilon) > 0$, such that $I^\alpha(1, x)L^{1-\alpha}(1, x) > L_{-\varepsilon}(1, x)$ for $x \in (X_1, +\infty)$ and $L_{(\alpha-2)/6+\varepsilon}(1, 1+x) > I^\alpha(1, 1+x)L^{1-\alpha}(1, 1+x)$ for $x \in (0, \delta_1)$. $\square$

Theorem 5. Inequality $L_{(\alpha-2)/6}(a, b) < \alpha I(a, b) + (1 - \alpha)L(a, b) < L_0(a, b)$ holds for any $\alpha \in (0, 1)$ and all $a, b > 0$ with $a \neq b$, and $L_{(\alpha-2)/6}(a, b)$ and $L_0(a, b)$ are the best possible lower and upper Lehmer mean bounds for the sum $\alpha I(a, b) + (1 - \alpha)L(a, b)$.

Proof. Inequality $\alpha I(a, b) + (1 - \alpha)L(a, b) < L_0(a, b)$ follows directly from (10) and (11), and inequality $L_{(\alpha-2)/6}(a, b) < \alpha I(a, b) + (1 - \alpha)L(a, b)$ follows from Theorem 4.

Next, we prove that $L_{(\alpha-2)/6}(a, b)$ and $L_0(a, b)$ are the best possible lower and upper Lehmer mean bounds for the sum $\alpha I(a, b) + (1 - \alpha)L(a, b)$.

For any $0 < \varepsilon < 1$ and $x > 0$, from (1) and (2), we have

$$\begin{aligned}
 & \lim_{x \rightarrow +\infty} \frac{\alpha I(1, x) + (1 - \alpha)L(1, x)}{L_{-\varepsilon}(1, x)} \\
 & = \lim_{x \rightarrow +\infty} \left[\frac{x^{-\varepsilon} + 1}{x^{\varepsilon-1} + 1} \right. \\
 & \quad \times \left(\alpha e^{(\log x)/(x-1)} x^\varepsilon + \frac{(1 - \alpha)(1 - 1/x)}{\log x} x^\varepsilon \right) \Bigg] \\
 & = +\infty, \tag{60}
 \end{aligned}$$

$$\begin{aligned}
 & L_{(\alpha-2)/6+\varepsilon}(1, 1+x) - \alpha I(1, 1+x) - (1 - \alpha)L(1, 1+x) \\
 & = \frac{(1+x)^{(\alpha+4)/6+\varepsilon} + 1}{(1+x)^{(\alpha-2)/6+\varepsilon} + 1} - \alpha e^{(1+x)\log(1+x)/x-1} \\
 & \quad - \frac{(1 - \alpha)x}{\log(1+x)}. \tag{61}
 \end{aligned}$$

Letting $x \rightarrow 0$ and making use of Taylor expansion, one has

$$\begin{aligned}
 & \frac{(1+x)^{(\alpha+4)/6+\varepsilon} + 1}{(1+x)^{(\alpha-2)/6+\varepsilon} + 1} - \alpha e^{(1+x)\log(1+x)/x-1} - \frac{(1 - \alpha)x}{\log(1+x)} \\
 & = 1 + \frac{1}{2}x + \left(\frac{\alpha}{24} - \frac{1}{12} + \frac{\varepsilon}{4}\right)x^2 + o(x^2) \\
 & \quad - \alpha \left[1 + \frac{1}{2}x - \frac{1}{24}x^2 + o(x^2)\right] \\
 & \quad - (1 - \alpha) \left[1 + \frac{1}{2}x - \frac{1}{12}x^2 + o(x^2)\right] \\
 & = \frac{\varepsilon}{4}x^2 + o(x^2). \tag{62}
 \end{aligned}$$

Equations (60)-(62) imply that for any $0 < \varepsilon < 1$, there exist $X_2 = X_2(\alpha, \varepsilon) > 1$ and $\delta_2 = \delta_2(\alpha, \varepsilon) > 0$, such that $\alpha I(1, x) + (1 - \alpha)L(1, x) > L_{-\varepsilon}(1, x)$ for $x \in (X_2, +\infty)$ and $L_{(\alpha-2)/6+\varepsilon}(1, 1+x) > \alpha I(1, 1+x) + (1 - \alpha)L(1, 1+x)$ for $x \in (0, \delta_2)$. $\square$

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References

- [1] G. Allasia, C. Giordano, and J. Pečarić, "On the arithmetic and logarithmic means with applications to Stirling's formula," *Atti del Seminario Matematico e Fisico dell'Università di Modena*, vol. 47, no. 2, pp. 441-445, 1999.
- [2] H. Alzer, "Ungleichungen für $(e/a)^a(b/e)^b$," *Elemente der Mathematik*, vol. 40, pp. 120-123, 1985.
- [3] H. Alzer, "Ungleichungen für Mittelwerte," *Archiv der Mathematik*, vol. 47, no. 5, pp. 422-426, 1986.
- [4] F. Burk, "The geometric, logarithmic and arithmetic mean inequality," *The American Mathematical Monthly*, vol. 94, no. 6, pp. 527-528, 1987.
- [5] B. C. Carlson, "The logarithmic mean," *The American Mathematical Monthly*, vol. 79, no. 6, pp. 615-618, 1972.
- [6] T. P. Lin, "The power mean and the logarithmic mean," *The American Mathematical Monthly*, vol. 81, no. 8, pp. 879-883, 1974.
- [7] J. Maloney, J. Heidel, and J. Pečarić, "A reverse Hölder type inequality for the logarithmic mean and generalizations," *Journal of the Australian Mathematical Society Series B-Applied Mathematics*, vol. 41, no. 3, pp. 401-409, 2000.

- [8] A. O. Pittenger, "Inequalities between arithmetic and logarithmic means," *Univerzitet u Beogradu. Publikacije Elektrotehnikov Fakulteta. Serija Matematika*, vol. 678–715, pp. 15–18, 1980.
- [9] A. O. Pittenger, "The symmetric, logarithmic and power means," *Univerzitet u Beogradu. Publikacije Elektrotehnikov Fakulteta. Serija Matematika*, vol. 678–715, pp. 19–23, 1980.
- [10] J. Sandor, "Inequalities for means," in *Proceedings of the 3rd Symposium of Mathematics and its Applications (Timisoara, 1989)*, pp. 87–90, Romanian Academy, Timisoara, Romania, 1990.
- [11] J. Sándor, "On the identric and logarithmic means," *Aequationes Mathematicae*, vol. 40, no. 1, pp. 261–270, 1990.
- [12] J. Sándor, "A note on some inequalities for means," *Archiv der Mathematik*, vol. 56, no. 5, pp. 471–473, 1991.
- [13] J. Sandor, "On certain identities for means," *Studia Universitatis Babes-Bolyai Mathematica*, vol. 38, no. 4, pp. 7–14, 1993.
- [14] J. Sandor, "On certain inequalities for means," *Journal of Mathematical Analysis and Applications*, vol. 189, no. 2, pp. 602–606, 1995.
- [15] J. Sandor, "On refinements of certain inequalities for means," *Archiv der Mathematik*, vol. 31, no. 4, pp. 279–282, 1995.
- [16] J. Sandor, "Two inequalities for means," *International Journal of Mathematics and Mathematical Sciences*, vol. 18, no. 3, pp. 621–623, 1995.
- [17] J. Sandor, "On certain inequalities for means II," *Journal of Mathematical Analysis and Applications*, vol. 199, no. 2, pp. 629–635, 1996.
- [18] J. Sándor, "On certain inequalities for means, III," *Archiv der Mathematik*, vol. 76, no. 1, pp. 34–40, 2001.
- [19] H. Alzer and S.-L. Qiu, "Inequalities for means in two arivari-ables," *Archiv der Mathematik*, vol. 80, no. 2, pp. 201–215, 2003.
- [20] M. K. Vamanamurthy and M. Vuorinen, "Inequalities for means," *Journal of Mathematical Analysis and Applications*, vol. 183, no. 1, pp. 155–166, 1994.
- [21] J. Sandor and I. Rasa, "Inequalities for certain means in two arguments," *Nieuw Archief voor Wiskunde*, vol. 15, no. 41, pp. 51–55, 1997.
- [22] P. Kahlig and J. Matkowski, "Functional equations involving the logarithmic mean," *ZAMM Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 76, no. 7, pp. 385–390, 1996.
- [23] A. O. Pittenger, "The logarithmic mean in n variables," *American Mathematical Monthly*, vol. 92, no. 2, pp. 99–104, 1985.
- [24] G. Polya and G. Szego, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, Princeton, NJ, USA, 1951.
- [25] B. C. Carlson, "Algorithms involving arithmetic and geometric means," *American Mathematical Monthly*, vol. 78, pp. 496–505, 1971.
- [26] B. C. Carlson and J. L. Gustafson, "Total positivity of mean values and hypergeometric functions," *SIAM Journal on Mathematical Analysis*, vol. 14, no. 2, pp. 389–395, 1983.
- [27] P. S. Bullen, D. S. Mitrinovic, and P. M. Vasic, *Means and Their Inequalities*, D. Reidel, Dordrecht, The Netherlands, 1988.
- [28] K. B. Stolarsky, "Hölder means, Lehmer means, and $x^{-1} \log \cosh x$," *Journal of Mathematical Analysis and Applications*, vol. 202, no. 3, pp. 810–818, 1996.
- [29] Z. Liu, "Remark on inequalities between Hölder and Lehmer means," *Journal of Mathematical Analysis and Applications*, vol. 247, no. 1, pp. 309–313, 2000.
- [30] E. F. Beckenbach, "A class of mean value functions," *American Mathematical Monthly*, vol. 57, pp. 1–6, 1950.
- [31] E. F. Beckenbach and R. Bellman, *Inequalities*, Springer, New York, NY, USA, 1965.
- [32] Z. Pales, "Inequalities for sums of powers," *Journal of Mathematical Analysis and Applications*, vol. 131, no. 1, pp. 265–270, 1988.
- [33] K. B. Stolarsky, "The power and generalized logarithmic means," *American Mathematical Monthly*, vol. 87, no. 7, pp. 545–548, 1980.
- [34] H. Alzer, "Bestmögliche Abschätzungen für spezielle Mittelwerte," *Prirodoslovno-Matematički Fakultet Sveučilišta u Zagrebu*, vol. 23, no. 1, pp. 331–346, 1993.