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Research Article

A Generalization of Prešić Type Mappings in 0-Complete Ordered Partial Metric Spaces

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The purpose of this paper is to prove some coincidence and common fixed point results for mappings satisfying Prešić type contraction condition in 0-complete ordered partial metric spaces. The results proved in this paper generalize and extend the results of Prešić and Matthews in 0-complete ordered partial metric spaces. Some examples are included which show that the generalization is proper.

1. Introduction

Banach contraction mapping principle is one of the most interesting and useful tools in applied mathematics. In recent years many generalizations of Banach contraction mapping principle have appeared. In 1965 Prešić [1, 2] extended Banach contraction mapping principle to mappings defined on product spaces and proved the following theorem.

Theorem 1. *Let (X, d) be a complete metric space, k a positive integer, and $T : X^k \to X$ a mapping satisfying the following contractive type condition:*

$$d\left(T\left(x_1, x_2, \ldots, x_k\right), T\left(x_2, x_3, \ldots, x_{k+1}\right)\right) \leq \sum_{i=1}^{k} q_i d\left(x_i, x_{i+1}\right), \tag{1}$$

for every $x_1, x_2, \ldots, x_{k+1} \in X$, where $q_1, q_2, \ldots, q_k$ are nonnegative constants such that $q_1 + q_2 + \cdots + q_k < 1$. Then there exists a unique point $x \in X$ such that $T(x, x, \ldots, x) = x$. Moreover if $x_1, x_2, \ldots, x_k$ are arbitrary points in X and for $n \in \mathbb{N}, x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$, then the sequence $\{x_n\}$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \ldots, \lim x_n)$.

Note that condition (1) in the case $k = 1$ reduces to the well-known Banach contraction mapping principle. So, Theorem 1 is a generalization of the Banach fixed point theorem. Some generalization of Theorem 1 can be seen in [1–11].

On the other hand in 1994 Matthews [12] introduced the notion of a partial metric space as a part of the study of denotational semantics of dataflow networks, showing that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. In partial metric space the usual metric was replaced by partial metric, with a property that the self-distance of any point may not be zero. Result of Matthews is generalized by several authors in different directions (see, e.g., [13–29]). Romaguera [30] introduced the notion of 0-Cauchy sequence and 0-complete partial metric spaces and proved some characterizations of partial metric spaces in terms of completeness and 0-completeness. Some results on 0-complete partial metric spaces can be seen in [30–32].

The existence of fixed point in partially ordered sets was investigated by Ran and Reurings [33] and then by Nieto and Rodríguez-López [34, 35]. Fixed point results in ordered partial metric spaces were obtained by several authors (see, e.g., [14, 16–19, 27, 29]). Very recently, in [7] (see also [36]) authors introduced the ordered Prešić type contraction and generalized the result of Prešić and proved some fixed point theorems for such mappings. In this paper, we generalize

and extend the result of Prešić [1, 2] in 0-complete ordered partial metric spaces. A generalization of result of Prešić in 0-complete partial metric spaces is also established. Some examples are included which show that the generalization is proper.

First we recall some definitions and properties of partial metric space [12, 30, 32, 37, 38].

Definition 2. A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ ($\mathbb{R}^+$ stands for nonnegative reals) such that for all $x, y, z \in X$:

(P1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(P2) $p(x,x) \leq p(x,y)$,

(P3) $p(x,y) = p(y,x)$,

(P4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

It is clear that if $p(x,y) = 0$, then from (P1) and (P2) $x = y$. But if $x = y$, $p(x,y)$ may not be 0. Also every metric space is a partial metric space, with zero self-distance.

Example 3. If $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $p(x,y) = \max\{x,y\}$, for all $x, y \in \mathbb{R}^+$, then $(\mathbb{R}^+, p)$ is a partial metric space.

Each partial metric on X generates a T_0 topology τ_p on X which has a base the family of open p-balls $\{B_p(x,\epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Theorem 4 (see [12]). *For each partial metric $p : X \times X \to \mathbb{R}^+$ the pair (X,d), where $d(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ for all $x, y \in X$, is a metric space.*

Here (X,d) is called induced metric space and d is induced metric. In further discussion until unless specified (X,d) will represent induced metric space.

Let (X,p) be a partial metric space.

(1) A sequence $\{x_n\}$ in (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n,x)$.

(2) A sequence $\{x_n\}$ in (X,p) is called Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n,x_m)$.

(3) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

(4) A sequence $\{x_n\}$ in (X,p) is called 0-Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m) = 0$. The space (X,p) is said to be 0-complete if every 0-Cauchy sequence in X converges with respect to τ_p to a point $x \in X$ such that $p(x,x) = 0$.

Lemma 5 (see [12, 30, 32, 37]). *Let (X,p) be a partial metric space and $\{x_n\}$ any sequence in X.*

(i) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in metric space (X,d).*

(ii) *(X,p) is complete if and only if the metric space (X,d) is complete. Furthermore, $\lim_{n\to\infty} d(x_n,x) = 0$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.*

(iii) *Every 0-Cauchy sequence in (X,p) is Cauchy in (X,d).*

(iv) *If (X,p) is complete, then it is 0-complete.*

The converse assertions of (iii) and (iv) do not hold. Indeed the partial metric space $(\mathbb{Q} \cap [0,\infty), p)$, where $\mathbb{Q}$ denotes the set of rational numbers and the partial metric p is given by $p(x,y) = \max\{x,y\}$, provides an easy example of a 0-complete partial metric space which is not complete. Also, it is easy to see that every closed subset of a 0-complete partial metric space is 0-complete.

Definition 6. Let X be a nonempty set, k a positive integer, and $f : X^k \to X$ a mapping. If $f(x,x,\ldots,x) = x$, then $x \in X$ is called a fixed point of f.

Definition 7. Let X be a nonempty set, k a positive integer, and $f : X^k \to X$ and $g : X \to X$ mappings.

(a) An element $x \in X$ is said to be a coincidence point of f and g if $gx = f(x,\ldots,x)$.

(b) If $w = gx = f(x,\ldots,x)$, then w is called a point of coincidence of f and g.

(c) If $x = gx = f(x,\ldots,x)$, then x is called a common fixed point of f and g.

(d) Mappings f and g are said to be commuting if $g(f(x,\ldots,x)) = f(gx,\ldots,gx)$ for all $x \in X$.

(e) Mappings f and g are said to be weakly compatible if they commute at their coincidence points.

Lemma 8 (see [8]). *Let X be a nonempty set, k a positive integer, and $f : X^k \to X$, $g : X \to X$ two weakly compatible mappings. If f and g have a unique point of coincidence $y = f(x,x,\ldots,x) = g(x)$, then y is the unique common fixed point of f and g.*

Remark that the previous definition in the case $k = 1$ reduces to the usual definitions of commuting and weakly compatible mappings in the sense of [39] (for details, see Introduction from [39]).

Definition 9. Let a nonempty set X be equipped with a partial order "$\sqsubseteq$" such that (X,p) is a partial metric space; then $(X,\sqsubseteq,p)$ is called an ordered partial metric space. A sequence $\{x_n\}$ in X is said to be nondecreasing with respect to "$\sqsubseteq$" if $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_n \sqsubseteq \cdots$. Let k be a positive integer and $f : X^k \to X$ a mapping; then f is said to be nondecreasing with respect to "$\sqsubseteq$" if for any finite nondecreasing sequence $\{x_n\}_{n=1}^{k+1}$ we have $f(x_1,x_2,\ldots,x_k) \sqsubseteq f(x_2,x_3,\ldots,x_{k+1})$. Let $g : X \to X$ be a mapping. f is said to be g-nondecreasing with respect to "$\sqsubseteq$" if for any finite nondecreasing sequence $\{gx_n\}_{n=1}^{k+1}$ we have $f(x_1,x_2,\ldots,x_k) \sqsubseteq f(x_2,x_3,\ldots,x_{k+1})$.

Remark 10. For $k = 1$ previous definitions reduce to usual definitions of fixed point and nondecreasing mapping in partial metric space.

Definition 11. Let X be a nonempty set equipped with partial order "$\sqsubseteq$", and let $g : X \to X$ be a mapping. A nonempty subset $\mathscr{A}$ of X is said to be well ordered if every two elements of $\mathscr{A}$ are comparable. Elements $a, b \in \mathscr{A}$ are called g-comparable if ga and gb are comparable. $\mathscr{A}$ is called g-well ordered if for all $a, b \in \mathscr{A}$, a and b are g-comparable; that is, ga and gb are comparable.

Example 12. Let $X = \{0, 1, 2, 3\}$, "$\sqsubseteq$" a partial order relation on X defined by $\sqsubseteq = \{(0,0), (1,1), (2,2), (3,3), (1,2), (2,3), (1,3)\}$. Let $\mathscr{A} = \{0, 1, 3\}$ and $g : X \to X$ be defined by $g0 = 1$, $g1 = 2$, $g2 = 3$, $g3 = 3$. Then it is clear that $\mathscr{A}$ is not well ordered but it is g-well ordered.

Let $(X, \sqsubseteq, p)$ be an ordered partial metric space, k a positive integer. Let $f : X^k \to X$ be a mapping; f is called ordered Prešić contraction if

$$p\left(f\left(x_1, x_2 \ldots, x_k\right), f\left(x_2, x_3, \ldots, x_{k+1}\right)\right) \leq \sum_{i=1}^{k} \alpha_i p\left(x_i, x_{i+1}\right), \tag{2}$$

for all $x_1, x_2, \ldots, x_{k+1} \in X$ with $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_{k+1}$, where α_i are nonnegative constant such that $\sum_{i=1}^{k} \alpha_i < 1$.

If (2) is satisfied for all $x_1, x_2, \ldots, x_{k+1} \in X$, then f is called Prešić contraction.

Note that in ordered partial metric spaces a Prešić contraction is necessarily an ordered Prešić contraction, but converse may not be true (see Example 21 of this paper).

Remark 13. Since every metric space is a partial metric space and there exist mappings which are Prešić contraction in partial metric spaces but not in metric spaces (see Example 20 of this paper), therefore the class of Prešić contraction in partial metric spaces is wider than that in metric spaces.

The following lemma will be useful in sequel.

Lemma 14. *Let $(X, \sqsubseteq, p)$ be any ordered partial metric space, k a positive integer. Let $f : X^k \to X$, $g : X \to X$ be two mappings satisfying*

$$\begin{aligned} &p\left(f\left(x_1, x_2 \ldots, x_k\right), f\left(x_2, x_3, \ldots, x_{k+1}\right)\right) \\ &\quad \leq \sum_{i=1}^{k} \alpha_i p\left(gx_i, gx_{i+1}\right), \end{aligned} \tag{3}$$

for all $x_1, x_2, \ldots, x_{k+1} \in X$ with $gx_1 \sqsubseteq gx_2 \sqsubseteq \cdots \sqsubseteq gx_{k+1}$ where α_i are nonnegative constant such that $\sum_{i=1}^{k} \alpha_i < 1$. If f and g have a point of coincidence v, then $p(v, v) = 0$.

Proof. Suppose that v is a point of coincidence of f and g with coincidence point $u \in X$, that is, $gu = f(u, \ldots, u) = v$. As $gu \sqsubseteq gu$, using (3) we obtain

$$\begin{aligned} p(v, v) &= p\left(f(u, \ldots, u), f(u, \ldots, u)\right) \\ &\leq \sum_{i=1}^{k} \alpha_i p(gu, gu) \\ &= \left[\sum_{i=1}^{k} \alpha_i\right] p(v, v); \end{aligned} \tag{4}$$

as $\sum_{i=1}^{k} \alpha_i < 1$, we obtain $p(v, v) = 0$. □

Now we can state our main results.

2. Main Results

Theorem 15. *Let $(X, \sqsubseteq, p)$ be any 0-complete ordered partial metric space and k a positive integer. Let $f : X^k \to X$, $g : X \to X$ be two mappings such that $f(X^k) \subset g(X)$ and $g(X)$ is a closed subspace of X. Suppose the following conditions hold:*

(I)

$$\begin{aligned} &p\left(f\left(x_1, x_2 \ldots, x_k\right), f\left(x_2, x_3, \ldots, x_{k+1}\right)\right) \\ &\quad \leq \sum_{i=1}^{k} \alpha_i p\left(gx_i, gx_{i+1}\right), \end{aligned} \tag{5}$$

for all $x_1, x_2, \ldots, x_{k+1} \in X$ with $gx_1 \sqsubseteq gx_2 \sqsubseteq \cdots \sqsubseteq gx_{k+1}$, where α_i are nonnegative constant such that $\sum_{i=1}^{k} \alpha_i < 1$;

(II) *there exist $x_1, x_2, \ldots, x_k \in X$ such that $gx_1 \sqsubseteq gx_2 \sqsubseteq \cdots \sqsubseteq gx_k \sqsubseteq f(x_1, x_2, \ldots, x_k)$;*

(III) *f is g-nondecreasing;*

(IV) *if a nondecreasing sequence $\{gx_n\}$ converges to $gu \in X$, then $gx_n \sqsubseteq gu$ for all $n \in \mathbb{N}$ and $gu \sqsubseteq ggu$.*

Then f and g have a point of coincidence. If in addition f and g are weakly compatible, then f and g have a common fixed point $v \in X$ with $p(v, v) = 0$. Moreover, the set of common fixed points of f and g is g-well ordered if and only if f and g have a unique common fixed point.

Proof. Starting with given $x_1, x_2, \ldots, x_k \in X$ we define a sequence $\{y_n\}$ as follows: let $y_n = gx_n$, $n = 1, 2, \ldots, k$. As $f(X^k) \subset g(X)$ and $gx_1 \sqsubseteq gx_2 \sqsubseteq \cdots \sqsubseteq gx_k \sqsubseteq f(x_1, x_2, \ldots, x_k)$, define $y_{n+k} = gx_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1})$, $n = 1, 2, \ldots$. Then $gx_1 \sqsubseteq gx_2 \sqsubseteq \cdots \sqsubseteq gx_k \sqsubseteq gx_{k+1}$ and f is g-nondecreasing, so $f(x_1, x_2, \ldots, x_k) \sqsubseteq f(x_2, x_3, \ldots, x_{k+1})$, that is, $gx_{k+1} \sqsubseteq gx_{k+2}$. Continuing this procedure, we obtain

$$gx_1 \sqsubseteq gx_2 \sqsubseteq \cdots \sqsubseteq gx_k \sqsubseteq gx_{k+1} \sqsubseteq \cdots, \tag{6}$$

that is,

$$y_1 \sqsubseteq y_2 \sqsubseteq \cdots \sqsubseteq y_k \sqsubseteq y_{k+1} \sqsubseteq \cdots. \tag{7}$$

Thus $\{y_n\} = \{gx_n\}$ is a nondecreasing sequence with respect to "$\sqsubseteq$."

For simplicity set $p_n = p(y_n, y_{n+1}) = p(gx_n, gx_{n+1})$, $\mu = \max\{p_1/\theta, p_2/\theta^2, \ldots, p_k/\theta^k\}$, where $\theta = [\sum_{i=1}^{k} \alpha_i]^{1/k}$.

By mathematical induction we will show that

$$p_n \le \mu\theta^n, \quad \forall n \in \mathbb{N}. \tag{8}$$

According to the definition of μ it is clear that (8) is true for $n = 1, 2, \ldots, k$. Let the following k inequalities

$$p_n \le \mu\theta^n, p_{n+1} \le \mu\theta^{n+1}, \ldots, p_{n+k-1} \le \mu\theta^{n+k-1} \tag{9}$$

be the induction hypothesis.

As $gx_n \sqsubseteq gx_{n+1} \sqsubseteq \cdots \sqsubseteq gx_{n+k}$, so using (5) we obtain

$$\begin{aligned}
p_{n+k} &= p(y_{n+k}, y_{n+k+1}) \\
&= p(f(x_n, \ldots, x_{n+k-1}), f(x_{n+1}, \ldots, x_{n+k})) \\
&\le \alpha_1 p(gx_n, gx_{n+1}) \\
&\quad + \alpha_2 p(gx_{n+1}, gx_{n+2}) + \cdots + \alpha_k p(gx_{n+k-1}, gx_{n+k}) \\
&\le \alpha_1 p_n + \alpha_2 p_{n+1} + \cdots + \alpha_k p_{n+k-1} \\
&\le \alpha_1 \mu\theta^n + \alpha_2 \mu\theta^{n+1} + \cdots + \alpha_k \mu\theta^{n+k-1} \\
&\le \mu\theta^n + \alpha_2 \mu\theta^n \\
&\quad + \cdots + \alpha_k \mu\theta^n \quad \left(\text{as } \theta = \left[\sum_{i=1}^{k} \alpha_i\right]^{1/k} < 1\right) \\
&= \left[\sum_{i=1}^{k} \alpha_i\right] \mu\theta^n \\
&= \mu\theta^{n+k},
\end{aligned} \tag{10}$$

and the inductive proof of (8) is complete.

Let $n, m \in \mathbb{N}$ with $m > n$; then

$$\begin{aligned}
&p(y_n, y_m) \\
&\le p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m) \\
&\quad - [p(y_{n+1}, y_{n+1}) + p(y_{n+2}, y_{n+2}) + \cdots + p(y_{m-1}, y_{m-1})] \\
&\le p_n + p_{n+1} + \cdots + p_{m-1} \\
&\le \mu\theta^n + \mu\theta^{n+1} + \cdots + \mu\theta^{m-1} \\
&\le [1 + \theta + \theta^2 + \cdots] \mu\theta^n \\
&= \frac{\mu\theta^n}{1-\theta}.
\end{aligned} \tag{11}$$

As $\theta < 1$, previous inequality implies that

$$\lim_{n,m\to\infty} p(y_n, y_m) = 0. \tag{12}$$

Thus $\{y_n\} = \{gx_n\}$ is a 0-Cauchy sequence in (X, p). Since $g(X)$ is 0-complete as it is closed subspace, it follows that $\{y_n\} = \{gx_n\}$ converges in $(g(X), p)$. Therefore

$$\lim_{n\to\infty} p(y_n, v) = \lim_{n\to\infty} p(y_n, gu) = p(v, v) = p(gu, gu) = 0, \tag{13}$$

where $v = gu \in g(X)$ for some $u \in X$.

We will show that u is coincidence point of and f and g.

Then for any integer n, we have

$$\begin{aligned}
&p(gu, f(u, \ldots, u)) \\
&\le p(gu, gx_{n+k}) + p(gx_{n+k}, f(u, \ldots, u)) \\
&\quad - p(gx_{n+k}, gx_{n+k}) \\
&\le p(gu, gx_{n+k}) + p(f(x_n, \ldots, x_{n+k-1}), f(u, \ldots, u)) \\
&\le p(gu, gx_{n+k}) \\
&\quad + p(f(x_n, \ldots, x_{n+k-1}), f(x_{n+1}, \ldots, x_{n+k-1}, u)) \\
&\quad + p(f(x_{n+1}, \ldots, x_{n+k-1}, u), f(x_{n+2}, \ldots, x_{n+k-1}, u, u)) \\
&\quad + \cdots + p(f(x_{n+k-1}, u, \ldots, u), f(u, \ldots, u)).
\end{aligned} \tag{14}$$

By assumption (IV) it follows that $gx_n \sqsubseteq gu$ for all $n \in \mathbb{N}$ and $gu \sqsubseteq ggu$, so using (5), previous inequality implies that

$$\begin{aligned}
&p(gu, f(u, \ldots, u)) \\
&\quad \le p(gu, y_{n+k}) + \alpha_1 p_n \\
&\qquad + \cdots + \alpha_{k-1} p_{n+k-2} + \alpha_k p(y_{n+k-1}, gu) \\
&\qquad + \alpha_1 p_{n+1} + \cdots + \alpha_{k-2} p_{n+k-2} \\
&\qquad + \alpha_{k-1} p(y_{n+k-1}, gu) + \alpha_k p(gu, gu) \\
&\qquad + \cdots + \alpha_1 p(y_{n+k-1}, gu) \\
&\qquad + [\alpha_2 + \cdots + \alpha_k] p(gu, gu).
\end{aligned} \tag{15}$$

Using (13) and the fact that $\lim_{n\to\infty} p_n = 0$, previous inequality implies that

$$p(gu, f(u, \ldots, u)) = 0 \quad \text{that is } gu = f(u, \ldots, u) = v. \tag{16}$$

Thus v is point of coincidence, and u is coincidence point of f and g.

Suppose f and g are weakly compatible, so

$$\begin{aligned}
f(v, \ldots, v) &= f(gu, \ldots, gu) \\
&= g(f(u, \ldots, u)) = gv = w \quad (\text{say});
\end{aligned} \tag{17}$$

so w is a point of coincidence of f and g, and by Lemma 14, we have $p(w,w) = p(gv,gv) = 0$. Again since $gu \sqsubseteq ggu = gv$ we obtain from (5) that

$$\begin{aligned}
&p(v, f(v,\dots,v)) \\
&= p(f(u,\dots,u), f(v,\dots,v)) \\
&\le p(f(u,\dots,u), f(u,\dots,u,v)) \\
&\quad + p(f(u,\dots,u,v), f(u,\dots,u,v,v)) + \cdots \\
&\quad + p(f(u,v,\dots,v), f(v,\dots,v)) \\
&\le [\alpha_1 + \cdots + \alpha_{k-1}]\, p(gu,gu) + \alpha_k p(gu,gv) \\
&\quad + [\alpha_1 + \cdots + \alpha_{k-2}]\, p(gu,gu) + \alpha_{k-1} p(gu,gv) \\
&\quad + \alpha_k p(gv,gv) + \cdots + \alpha_1 p(gu,gv) \\
&\quad + [\alpha_2 + \cdots + \alpha_k]\, p(gv,gv).
\end{aligned} \tag{18}$$

As $p(gu,gu) = p(gv,gv) = 0$, $gu = v$, and $gv = f(v,\dots,v)$ it follows from previous inequality that

$$p(v, f(v,\dots,v)) \le \sum_{i=1}^{k} \alpha_i p(v, f(v,\dots,v)); \tag{19}$$

as $\sum_{i=1}^{k} \alpha_i < 1$, we obtain

$$p(v, f(v,\dots,v)) = 0 \quad \text{that is } v = f(v,\dots,v) = gv. \tag{20}$$

Thus v is common fixed point of f and g.

Suppose that the set of common fixed points is g-well ordered. We will show that common fixed point is unique. Assume on the contrary that v' is another common fixed point of f and g, that is, $v' = f(v',\dots,v') = gv'$ and $v \ne v'$. As v and v' are g-comparable let, for example, $gv \sqsubseteq gv'$. From (5), it follows that

$$\begin{aligned}
&p(v,v') \\
&= p(f(v,\dots,v), f(v',\dots,v')) \\
&\le p(f(v,\dots,v), f(v,\dots,v,v')) \\
&\quad + p(f(v,\dots,v,v'), f(v,\dots,v',v')) \\
&\quad + \cdots + p(f(v,v',\dots,v'), f(v',\dots,v')) \\
&\le [\alpha_1 + \cdots + \alpha_{k-1}]\, p(gv,gv) + \alpha_k p(gv,gv') \\
&\quad + [\alpha_1 + \cdots + \alpha_{k-2}]\, p(gv,gv) + \alpha_{k-1} p(gv,gv') \\
&\quad + \alpha_k p(gv',gv') + \cdots + \alpha_1 p(gv,gv') \\
&\quad + [\alpha_2 + \cdots + \alpha_k]\, p(gv',gv').
\end{aligned} \tag{21}$$

As $p(gv,gv) = p(v,v) = 0$ and from Lemma 14, $p(v',v') = p(gv',gv') = 0$, it follows from previous inequality that

$$\begin{aligned}
p(v,v') &\le \sum_{i=1}^{k} \alpha_i p(gv,gv') \\
&= \sum_{i=1}^{k} \alpha_i p(v,v') < p(v,v') \quad \left(\text{as } \sum_{i=1}^{k} \alpha_i < 1\right),
\end{aligned} \tag{22}$$

a contradiction. Therefore common fixed point is unique. For converse, if common fixed point of f and g is unique, then the set of common fixed points of f and g is singleton therefore g-well ordered. □

Remark 16. For $k = 1$ previous theorem is a generalization and extension of result of Matthews [12] in ordered partial metric spaces.

Taking $g = I_X$, that is, identity mapping of X, we get the following fixed point result for ordered Prešić contraction in partial metric spaces.

Corollary 17. *Let $(X, \sqsubseteq, p)$ be any 0-complete ordered partial metric space and k a positive integer. Let $f : X^k \to X$ be a mapping and the following conditions hold:*

(I) *f is an ordered Prešić contraction;*

(II) *there exist $x_1, x_2, \dots, x_k \in X$ such that $x_1 \sqsubseteq x_2 \sqsubseteq \cdots \sqsubseteq x_k \sqsubseteq f(x_1, x_2, \dots, x_k)$;*

(III) *f is nondecreasing;*

(IV) *if a nondecreasing sequence $\{x_n\}$ converges to $u \in X$, then $x_n \sqsubseteq u$ for all $n \in \mathbb{N}$.*

Then f has a fixed point $v \in X$ with $p(v,v) = 0$. Moreover, the set of fixed points of f is well ordered if and only if f has a unique fixed point.

Following theorem generalizes and extends the result of Prešić in partial metric spaces.

Theorem 18. *Let (X, p) be any 0-complete partial metric space and k a positive integer. Let $f : X^k \to X$, $g : X \to X$ be two mappings such that $f(X^k) \subset g(X)$, $g(X)$ is a closed subspace of X, and*

$$\begin{aligned}
&p(f(x_1, x_2 \dots, x_k), f(x_2, x_3, \dots, x_{k+1})) \\
&\quad \le \sum_{i=1}^{k} \alpha_i p(gx_i, gx_{i+1}),
\end{aligned} \tag{23}$$

for all $x_1, x_2, \dots, x_{k+1} \in X$, where α_i are nonnegative constant such that $\sum_{i=1}^{k} \alpha_i < 1$. Then f and g have a point of coincidence. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point $v \in X$ with $p(v,v) = 0$.

Proof. We note that the inequality (23) is true for all $x_1, x_2, \dots, x_{k+1} \in X$; therefore the proof of theorem follows from similar process as used in the proof of Theorem 15. □

Taking $g = I_X$ that is, identity mapping in the previous theorem, we obtain the following corollary.

Corollary 19 (Prešić type). *Let (X, p) be any 0-complete partial metric space and k a positive integer. Let $f : X^k \to X$ be a mapping satisfying*

$$p\left(f\left(x_1, x_2 \ldots, x_k\right), f\left(x_2, x_3, \ldots, x_{k+1}\right)\right) \leq \sum_{i=1}^{k} \alpha_i p\left(x_i, x_{i+1}\right), \tag{24}$$

for all $x_1, x_2, \ldots, x_{k+1} \in X$, where α_i are nonnegative constant such that $\sum_{i=1}^{k} \alpha_i < 1$. Then f has a unique fixed point $v \in X$ with $p(v, v) = 0$.

Following is an example which illustrates that a Prešić type contraction in partial metric space need not to be a Prešić type contraction in metric space.

Example 20. Let $X = [0, 1]$ and defined $p : X \times X \to \mathbb{R}^+$ by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then (X, p) is a 0-complete partial metric space. For $k = 2$, suppose $f : X^2 \to X$ be defined by

$$f(x, y) = \begin{cases} 0 & \text{if } x = y = 1, \\ \dfrac{x+y}{5} & \text{otherwise.} \end{cases} \tag{25}$$

Then f is Prešić type contraction in partial metric space (X, p), that is, it satisfies (24) with $\alpha_1 = \alpha_2 = 1/5$, while it is not a Prešić type contraction in usual metric space (X, d) where $d(x, y) = |x - y|$ for all $x, y \in X$ (note that d is also the induced metric).

Proof. For $k = 2$ and $\alpha_1 = \alpha_2 = \alpha$ condition (24) becomes

$$p\left(f\left(x_1, x_2\right), f\left(x_2, x_3\right)\right) \leq \alpha\left[p\left(x_1, x_2\right) + p\left(x_2, x_3\right)\right], \tag{26}$$

for all $x_1, x_2, x_3 \in X$, where α is a nonnegative constant such that $\alpha < 1/2$. Note that if $x_1 = x_2 = x_3 = 1$, then (26) is satisfied trivially.

If $x_1, x_2, x_3 \in [0, 1)$ and $x_1 \leq x_2 \leq x_3$, then (26) is valid for $\alpha \in [1/5, 1/2)$. Indeed since

$$\begin{aligned} p\left(f\left(x_1, x_2\right), f\left(x_2, x_3\right)\right) &= p\left(\frac{x_1 + x_2}{5}, \frac{x_2 + x_3}{5}\right) \\ &= \max\left\{\frac{x_1 + x_2}{5}, \frac{x_2 + x_3}{5}\right\} \\ &= \frac{x_2 + x_3}{5}, \\ \alpha\left[p\left(x_1, x_2\right) + p\left(x_2, x_3\right)\right] &= \alpha\left[\max\left\{x_1, x_2\right\} + \max\left\{x_2, x_3\right\}\right] \\ &= \alpha\left[x_2 + x_3\right], \end{aligned} \tag{27}$$

so result follows.

If any two of x_1, x_2, x_3 are equal to 1, for example, if $x_1 = x_2 = 1$ and $x_3 \in [0, 1)$, then (26) is valid. In this case

$$\begin{aligned} p\left(f(1,1), f\left(1, x_3\right)\right) &= p\left(0, \frac{1 + x_3}{5}\right) \\ &= \max\left\{0, \frac{1 + x_3}{5}\right\} = \frac{1 + x_3}{5}, \\ \alpha\left[p(1,1) + p\left(1, x_3\right)\right] &= \alpha\left[\max\{1, 1\} + \max\left\{1, x_3\right\}\right] \\ &= 2\alpha. \end{aligned} \tag{28}$$

Hence, if $\alpha \in [1/5, 1/2)$, then $p(f(1,1), f(1, x_3)) \leq \alpha[p(1,1) + p(1, x_3)]$.

Similarly in all possible cases (26) is satisfied for $\alpha = 1/5$. Note that all other conditions of Corollary 19 are satisfied and f has a unique fixed point 0, that is, $f(0, 0) = 0$, with $p(0, 0) = 0$.

Again for $k = 2$, f will be Prešić contraction in (X, d) if

$$d\left(f\left(x_1, x_2\right), f\left(x_2, x_3\right)\right) \leq \alpha_1 d\left(x_1, x_2\right) + \alpha_2 d\left(x_2, x_3\right), \tag{29}$$

for all $x_0, x_1, x_2 \in X$, where α_1, α_2 are nonnegative constant such that $\alpha_1 + \alpha_2 < 1$. We will show that (29) is not satisfied for certain points in $[0, 1]$. Let $x_1 = 10/11$, $x_2 = x_3 = 1$; then

$$\begin{aligned} d\left(f\left(x_1, x_2\right), f\left(x_2, x_3\right)\right) &= d\left(f\left(\frac{10}{11}, 1\right), f(1,1)\right) \\ &= d\left(\frac{21}{55}, 0\right) = \frac{21}{55}, \\ \alpha_1 d\left(x_1, x_2\right) + \alpha_2 d\left(x_2, x_3\right) &= \alpha_1 d\left(\frac{10}{11}, 1\right) + \alpha_2 d(1,1) \\ &= \frac{1}{11}\alpha_1 \end{aligned} \tag{30}$$

but $\alpha_1 + \alpha_2 < 1$; therefore for these values of x_1, x_2, and x_3 (29) does not hold. Thus f is not a Prešić contraction in (X, d), and so we cannot apply the result of Prešić to conclude the existence of fixed point of f. □

Following example illustrates that an ordered Prešić contraction need not to be a Prešić contraction in ordered partial metric spaces, and the fixed point of ordered Prešić contraction may not be unique (when set of fixed points of f is not well ordered).

Example 21. Let $X = [0, 2]$ and let order relation "$\sqsubseteq$" be defined by

$$\begin{aligned} \sqsubseteq \; &= \{(x, y) : x, y \in [0, 1) \text{ with } y \leq x\} \\ &\cup \{(x, y) : x, y \in [1, 2) \text{ with } y \leq x\} \\ &\cup \{(2, 2)\}; \end{aligned} \tag{31}$$

suppose $p : X \times X \to \mathbb{R}^+$ be defined by

$$p(x, y) = \begin{cases} 0 & \text{if } x = y = 2, \\ \max\{x, y\} & \text{otherwise.} \end{cases} \tag{32}$$

Then $(X, \sqsubseteq, p)$ is an ordered complete partial metric space. For $k = 2$, suppose $f : X^2 \to X$ be a mapping defined by

$$f(x, y) = \begin{cases} x + y - 1 & \text{if } (x, y) \in [0, 1) \times [1, 2] \cup [1, 2] \times [0, 1), \\ 2 & \text{if } x = y = 2, \\ \dfrac{x + y}{4} & \text{otherwise.} \end{cases} \tag{33}$$

Then

(a) f is an ordered Prešić contraction that, satisfies (2) with $\alpha_1 = \alpha_2 = 1/4$,

(b) f is not a Prešić contraction; that is, f does not satisfy (2) for all $x_1, x_2 \in X$.

Proof. (a) For $k = 2$ and $\alpha_1 = \alpha_2 = \alpha$, (2) becomes

$$p(f(x_1, x_2), f(x_2, x_3)) \le \alpha [p(x_1, x_2) + p(x_2, x_3)], \tag{34}$$

for all $x_1, x_2, x_3 \in X$ with $x_1 \sqsubseteq x_2 \sqsubseteq x_3$, where α is nonnegative constant such that $\alpha < 1/2$. Note that if $x_1 = x_2 = x_3 = 2$, then (34) is satisfied trivially.

Now we have to check the validity of inequality (34) only for $x_1, x_2, x_3 \in [0, 1)$ and $x_1, x_2, x_3 \in [1, 2)$.

Let $x_1, x_2, x_3 \in [0, 1)$ with $x_1 \sqsubseteq x_2 \sqsubseteq x_3$, that is, $x_3 \le x_2 \le x_1$; then (34) is valid for $\alpha \in [1/4, 1/2)$. Indeed since

$$\begin{aligned} p(f(x_1, x_2), f(x_2, x_3)) &= p\left(\frac{x_1 + x_2}{4}, \frac{x_2 + x_3}{4}\right) \\ &= \max\left\{\frac{x_1 + x_2}{4}, \frac{x_2 + x_3}{4}\right\} \\ &= \frac{x_1 + x_2}{4}, \\ \alpha [p(x_1, x_2) + p(x_2, x_3)] &= \alpha [\max\{x_1, x_2\} + \max\{x_2, x_3\}] \\ &= \alpha [x_1 + x_2], \end{aligned} \tag{35}$$

so result follows.

Similarly in all possible cases (34) is satisfied for $\alpha = 1/4$.

Note that all other conditions of Corollary 17 are satisfied (except that the set of fixed points of f is well ordered) and f has two fixed points 0 and 2, that is, $f(0, 0) = 0$, $f(2, 2) = 2$, with $p(0, 0) = p(2, 2) = 0$. Note that set of fixed points of $f, \mathscr{A} = \{0, 2\}$ is not well ordered as $(0, 2), (2, 0) \notin \sqsubseteq$.

(b) For $k = 2$, (2) becomes

$$p(f(x_1, x_2), f(x_2, x_3)) \le \alpha_1 p(x_1, x_2) + \alpha_2 p(x_2, x_3). \tag{36}$$

We will show that (36) is not satisfied for all $x_1, x_2, x_3 \in X$, with $\alpha_1 + \alpha_2 < 1$, in particular, let $x_1 = 2$, $x_2 = 2$ and $x_3 = 0$; then

$$\begin{aligned} p(f(x_1, x_2), f(x_2, x_3)) &= p(f(2, 2), f(2, 0)) \\ &= p(2, 2 + 0 - 1) \\ &= \max\{2, 1\} = 2, \\ \alpha_1 p(x_1, x_2) + \alpha_2 p(x_2, x_3) &= \alpha_1 p(2, 2) + \alpha_2 p(2, 0) \\ &= \alpha_2 \max\{2, 0\} = 2\alpha_2, \end{aligned} \tag{37}$$

but $\alpha_1 + \alpha_2 < 1$; therefore for these values of x_1, x_2, and x_3 (36) does not hold. Thus f is not Prešić contraction in (X, p). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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