Research Article

A Characterization of 4-Centralizer Groups

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A finite or infinite group is called an n -centralizer group if it has n numbers of distinct centralizers. In this paper, we prove that a finite or infinite group G is a 4-centralizer group if and only if $G/Z(G)$ is isomorphic to $C_2 \times C_2$. This extends a result of Belcastro and Sherman.

1. Introduction

Given a finite or infinite group G and $x \in G$, the set $C(x) = \{y \in G \mid xy = yx\}$ is called the *centralizer* of x in G . The set of all centralizers in G is denoted by $\text{Cent}(G)$. A group G is called an n -centralizer group if $|\text{Cent}(G)| = n$. It is easy to see that one-centralizer groups are precisely the abelian groups. Characterization of finite groups in terms of the number of distinct centralizers has been an interesting topic of research in recent years (see [1–8]). In [7], Belcastro and Sherman have proved the nonexistence of finite n -centralizer group for $n = 2, 3$. However, their proof also shows the nonexistence of infinite 2, 3-centralizer groups. Belcastro and Sherman [7] also characterize all finite 4-centralizer groups. More precisely, they proved that a finite group G is a 4-centralizer group if and only if $G/Z(G)$ is isomorphic to $C_2 \times C_2$, where $Z(G)$ is the center of G and C_2 is the cyclic group having two elements. In this paper, we extend the same characterization for infinite groups using elementary techniques of group theory.

Throughout this paper G will denote a finite or infinite group. Recall that for any group G , its center $Z(G)$ is the intersection of all centralizers in G . Also G is the union of all the centralizers of noncentral elements of G . It may be mentioned here that a finite or infinite group can not be written as union of two of its proper subgroups. These facts have important role in proving the main theorem of this paper.

2. Main Result

In this section, we prove the following main theorem of this paper.

Theorem 1. *A finite or infinite group G is a 4-centralizer group if and only if $G/Z(G) \cong C_2 \times C_2$.*

Proof. Let G be a 4-centralizer finite or infinite group and $\text{Cent}(G) = \{G, C(p), C(q), C(r)\}$, where p, q , and r are noncentral elements of G . Then $G = C(p) \cup C(q) \cup C(r)$, since G is the union of its proper centralizers.

Let us consider the centralizer $C(pq)$. Then $C(pq)$ will be one of $G, C(p), C(q)$, or $C(r)$.

If $C(pq) = G$, then $pq \in Z(G)$. This implies that $p = zq^{-1}$ for some $z \in Z(G)$. Therefore, we get $C(p) = C(q)$, a contradiction. If $C(pq) = C(p)$, then $g \in C(p) = C(pq)$ gives

$$gpq = pqg \implies pgq = pqg \implies g \in C(q). \quad (1)$$

Therefore, $C(p) \subseteq C(q)$. Hence, $G = C(q) \cup C(r)$, a contradiction, as a group can not be written as union of two of its proper subgroups. Similarly, it can be seen that $C(pq) \neq C(q)$.

Thus, $C(pq) = C(r)$ and so $G = C(p) \cup C(q) \cup C(pq)$. In a similar way it can be seen that $C(qp) = C(r)$ and so $C(pq) = C(qp)$.

We will now show that $C(p) \cap C(q) = Z(G)$. Clearly, $Z(G) \subseteq C(p) \cap C(q)$. Let $g \in C(p) \cap C(q)$. Then

$$\begin{aligned} gp = pg &\implies (gp)q = (pg)q \implies gpq \\ &= pqg \implies g \in C(pq). \end{aligned} \quad (2)$$

Therefore, $g \in C(p) \cap C(q) \cap C(pq) = Z(G)$ and so $C(p) \cap C(q) \subseteq Z(G)$. Thus,

$$C(p) \cap C(q) = Z(G). \quad (3)$$

In a similar way, it can be seen that $C(q) \cap C(pq) = Z(G)$, and $C(p) \cap C(pq) = Z(G)$.

Let us consider the right cosets Z, Zp, Zq , and Zpq , where $Z := Z(G)$. As p is a noncentral element of G it follows that $Z \neq Zp, Zq$, and Zpq .

If $Zp = Zq$, then we have $p = zq$ for some $z \in Z$. Therefore, we get $C(p) = C(q)$, a contradiction. Again, if $Zp = Zpq$, then $(pq)^{-1}p \in Z$ which gives $q \in Z$, a contradiction. Similarly, it can be seen that $Zq \neq Zpq$. Thus the cosets Z, Zp, Zq , and Zpq are mutually disjoint. Clearly, $Z \cup Zp \cup Zq \cup Zpq \subseteq G$.

Let $g \in Z$ then $g \in Z \cup Zp \cup Zq \cup Zpq$. Suppose that $g \in G - Z$ then without any loss of generality we may assume that $g \in C(p)$. Let us consider the centralizer $C(gq)$. Then $C(gq)$ is one of $G, C(p), C(q)$, or $C(pq)$.

Case 1. Let $C(gq) = G$. In this case, $gq \in Z(G)$ and so $gqq = gqq$ which gives $g \in C(q)$. Therefore, $g \in C(p) \cap C(q) = Z(G)$, a contradiction.

Case 2. Let $C(gq) = C(p)$. In this case, $gqp = pgq$ which gives $p \in C(q)$. Therefore, $p \in C(p) \cap C(q) = Z(G)$, a contradiction.

Case 3. Let $C(gq) = C(q)$. In this case, $gqq = gqq$ which gives $g \in C(q)$. Therefore, $g \in C(p) \cap C(q) = Z(G)$, a contradiction.

Hence, $C(gq) = C(pq)$, since G is a 4-centralizer group. Now, $C(gq) = C(pq)$ implies that $gqp = pqg$ which gives $gp^{-1}q = qgp^{-1}$ since $g \in C(p)$. Thus, $gp^{-1} \in C(q)$. Also, $g \in C(p)$ implies that $gp^{-1} \in C(p)$. Hence, $gp^{-1} \in C(p) \cap C(q) = Z(G)$ and so $g = zp$ for some $z \in Z(G)$. In other words, $g \in Zp$. Therefore, $g \in Z \cup Zp \cup Zq \cup Zpq$. Hence, $G = Z \cup Zp \cup Zq \cup Zpq$. This shows that $|G/Z(G)| = 4$. Finally, the fact that G is nonabelian gives $G/Z(G) \cong C_2 \times C_2$.

Conversely, let $G/Z(G) \cong C_2 \times C_2$. Therefore, there are four right cosets of $Z = Z(G)$ in G , namely, Z, Zp, Zq , and Zr , where p, q , and r are distinct noncentral elements of G . So, for any element $g \in G$, either $g \in Z$ or Zp or Zq or Zr . If $g \in Z$, then $C(g) = G$. If $g \in Zp$, then $g = zp$ for some $z \in Z$, therefore $C(g) = C(zp) = C(p)$. Similarly, $g \in Zq$ gives $C(g) = C(q)$ and $g \in Zr$ gives $C(g) = C(r)$. Hence, G has at most four centralizers, namely, $G, C(p), C(q)$, and $C(r)$. Since $|\text{Cent}(G)| \geq 4$ we have $|\text{Cent}(G)| = 4$. This completes the proof. $\square$

We conclude this paper by the following remark. In [7], Belcastro and Sherman proved Theorem 1 for finite groups only. Note that their proof can not be extended to

infinite groups as they have used the finiteness of the group extensively. Belcastro and Sherman [7] have also obtained the structure of 5-centralizer finite groups as follows.

A finite group G is a 5-centralizer group if and only if $G/Z(G) \cong C_3 \times C_3$ or $G/Z(G) \cong S_3$, where C_3 is the cyclic group having three elements and S_3 the symmetric group on three symbols.

Note that the *if part* of the above result also holds for infinite groups. The same proof of Belcastro and Sherman [7] holds for infinite groups. However, the proof of the *only if part* of the above result is not known for infinite groups. It may be interesting to study the infinite n -centralizer groups for $n > 4$.

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