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Research Article

On Quasi-h-Dense Submodules and h-Pure Envelopes of QTAG Modules

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A module M over an associative ring R with unity is a QTAG module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. There are many fascinating properties of QTAG modules of which h-pure submodules and high submodules are significant. A submodule N is quasi-h-dense in M if M/K is h-divisible, for every h-pure submodule K of M, containing N. Here we study these submodules and obtain some interesting results. Motivated by h-neat envelope, we also define h-pure envelope of a submodule N as the h-pure submodule $K \supseteq N$ if K has no direct summand containing N. We find that h-pure envelopes of N have isomorphic basic submodules, and if M is the direct sum of uniserial modules, then all h-pure envelopes of N are isomorphic.

1. Introduction

All the rings R considered here are associative with unity, and right modules M are unital QTAG modules. An element $x \in M$ is uniform, if xR is a nonzero uniform (hence uniserial) module and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M$, $e(x) = d(xR)$, and $H_M(x) = \sup\{d(yR/xR) \mid y \in M, x \in yR \text{ and } y \text{ uniform}\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k, and $H^k(M)$ is the submodule of M generated by the elements of exponent at most k. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$, and it is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height.

The modules $H_k(M)$, $k = 0, 1, \ldots, \infty$ form a neighbourhood system of zero giving rise to h-topology. The closure of a submodule $N \subset M$ is defined as $\overline{N} = \bigcap_{k=0}^{\infty}(N + H_k(M))$, and it is closed with respect to h-topology if $N = \overline{N}$.

A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. For a limit ordinal α, $H_\alpha(M) = \bigcap_{\rho<\alpha} H_\rho(M)$, for all ordinals $\rho < \alpha$, and it is α-pure in M if $H_\sigma(N) = H_\sigma(M) \cap N$ for all ordinals $\sigma < \alpha$.

A module M is summable if $\mathrm{Soc}(M) = \oplus_{\alpha<\tau} S_\alpha$, where S_α is the set of all elements of $H_\alpha(M)$ which are not in $H_{\alpha+1}(M)$, where τ is the length of M. A submodule $N \subset M$ is nice [1, Definition 2.3] in M, if $H_\sigma(M/N) = (H_\sigma(M) + N)/N$ for all ordinals σ; that is, every coset of M modulo N may be represented by an element of the same height.

The cardinality of the minimal generating set of M is denoted by $g(M)$. For all ordinals α, $f_M(\alpha)$ is the αth-*Ulm* invariant of M and it is equal to $g(\mathrm{Soc}(H_\alpha(M))/\mathrm{Soc}(H_{\alpha+1}(M)))$.

For a QTAG module M, there is a chain of submodules $M^0 \supset M^1 \supset M^2 \supset \cdots \supset M^\tau = 0$, for some ordinal τ. $M^{\sigma+1} = (M^\sigma)^1$, where M^σ is the σth-*Ulm* submodule of M. Singh [2] proved that the results which hold for TAG modules also hold good for QTAG modules.

2. Quasi-h-Dense Submodules

In [3], we studied semi-h-pure submodules which are not h-pure but contained in h-pure submodules. Now we investigate the submodules $N \subseteq M$ such that M/K is h-divisible for every h-pure submodule $K \subseteq M$, containing K. These modules are called quasi-h-dense submodules.

We start with the following.

Definition 1. A submodule N of M is quasi-h-dense in M, if for every h-pure submodule $K \subseteq M$, containing N, M/K is h-divisible.

Lemma 2. *If* $\mathrm{Soc}\,(H_{k-1}(M)) \nsubseteq N + H_k(M)$ *for some integer* k*, then there exists a proper submodule* K *of* M *containing* N *and a bounded submodule* Q *of* M *such that*

(i) $M = K \oplus Q$;

(ii) $K \supseteq N + H_k(M)$;

(iii) $N + H_k(M) \supseteq \mathrm{Soc}\,(H_{k-1}(K))$.

Proof. The socle of $H_{k-1}(M)$ can be expressed as the direct sum of S_1, S_2 where $S_1 = (N + H_k(M)) \cap \mathrm{Soc}(H_{k-1}(M))$ and $S_2 \cap H_k(M) = 0$. Now there exists a h-pure submodule Q of M such that $\mathrm{Soc}(Q) = S_2$. Also $(Q \oplus H_k(M))/H_k(M)$ is h-pure in $M/H_k(M)$ and $\mathrm{Soc}((Q \oplus H_k(M))/H_k(M)) = (S_2 \oplus H_k(M))/H_k(M)$. Now $\mathrm{Soc}(H_K(M/H_k(M))) = 0$ which is contained in $(S_2 \oplus H_k(M))/H_k(M) \subseteq \mathrm{Soc}(H_{K-1}(M/H_k(M)))$; therefore $(Q \oplus H_k(M))/H_k(M)$ is an absolute direct summand of $M/H_k(M)$ [4]. Thus there exists a submodule K of M such that

$$\frac{M}{H_k(M)} = \left(\frac{K}{H_k(M)}\right) \oplus \left(\frac{Q \oplus H_k(M)}{H_k(M)}\right) \tag{1}$$

and $K \supset N + H_k(M)$; therefore $M = K \oplus Q$ and $N + H_k(K) \supseteq \mathrm{Soc}(H_{k-1}(K))$. □

Proposition 3. *A submodule* N *of a QTAG module* M *is quasi-h-dense in* M *if and only if for all integers* $k \geq 0, N + H_{k+1}(M) \supseteq \mathrm{Soc}\,(H_k(M))$.

Proof. Let k be the least positive integer such that $N + H_k(M) \not\supseteq \mathrm{Soc}(H_{k-1}(M))$. By Lemma 2, there exists a proper submodule $K \subseteq M$ containing N and a bounded submodule Q of M such that $M/K \cong Q$. This contradiction proves that $N + H_{k+1}(M) \supseteq \mathrm{Soc}(H_k(M))$, for all integers $k \geq 0$.

For the converse, consider an h-pure submodule $K \supseteq N$. If M/K is not h-divisible, then there exist submodules P and Q of M such that $M/K = (P/K) \oplus (Q/K)$, where Q/K is bounded. If $H_k(Q/K) = 0$ and $H_{k-1}(Q/K) \neq 0$, then $P \supseteq N + H_k(M) \supseteq \mathrm{Soc}(H_{k-1}(M))$. Since P is h-pure in M, we have $H_{k-1}(M) \subseteq P$. On repeating the process, after a finite number of steps, we get $P = M$ and $Q/K = 0$, which is a contradiction. Hence, M/K is h-divisible and N is quasi-h-dense in M. □

Remark 4. For a submodule N of a QTAG module M, the following are equivalent:

(i) N is quasi-h-dense in M;

(ii) $N + H_{k+1}(M) \supseteq \mathrm{Soc}\,(H_k(M)), \forall k > \omega$;

(iii) for every h-pure submodule K containing N, M/K is h-divisible.

Lemma 5. *If* K *is an h-divisible submodule of a module* M *and* N *is a* K*-high submodule of* M*, then* N *is h-pure in* M.

Proof. For an h-divisible submodule $K \subseteq M$, K-high submodules are h-neat [5]. Therefore, $N \cap H_1(M) = H_1(N)$. Assume that $N \cap H_i(M) = H_i(N)$ for all $i \leq k$. Let $x \in N \cap H_{k+1}(M)$. Now there exists $y \in M$ such that $d(yR/xR) = k+1$. If $y' \in M$ such that $d(yR/y'R) = k$, then $y' + z \in \mathrm{Soc}(M)$ for some $z \in N$ and $y' + z \in \mathrm{Soc}(M) = \mathrm{Soc}(K) \oplus \mathrm{Soc}(N) = \mathrm{Soc}(H_k(K)) \oplus \mathrm{Soc}(N)$. Now there exists $u \in \mathrm{Soc}(H_k(K))$ such that $d(xR/uR) = k$ and $y' + u \in N$. By induction $y \in H_{k+1}(N)$ and N is h-pure in M. □

Theorem 6. *Let* N *be a proper submodule of* M*. Then* N *is not semi-h-pure if and only if the following hold:*

(i) N *is quasi-h-dense in* M;

(ii) *there exists* $m \in \mathbb{Z}^+$ *such that* $\mathrm{Soc}\,(H_m(M)) \subseteq N$.

Proof. By Lemma 2 and Proposition 3, (i) is satisfied. Suppose $(\mathrm{Soc}(H_k(M)) + N)/N \neq 0$, for all $k \geq 0$. If there exists a nonnegative integer m such that

$$\frac{\mathrm{Soc}\,(H_m(M)) + N}{N} = \frac{\mathrm{Soc}\,(H_{m+t}(M)) + N}{N}, \quad \text{for every } t \geq 0, \tag{2}$$

then we have

$$\mathrm{Soc}\,(H_m(M)) = \mathrm{Soc}\,(N \cap H_m(M)) + \mathrm{Soc}\,(H_{m+t}(M)), \quad \forall t \geq 0. \tag{3}$$

Let K be the h-neat envelope [6] of $N \cap H_m(M)$ in $H_m(M)$. Then K is h-pure in $H_m(M)$. Since $N \cap H_m(M)$ is quasi-h-dense in $H_m(M)$, $H_m(M)/N$ is h-divisible. Thus, there exists a submodule Q of M such that $N \subseteq Q$ and

$$\frac{M}{K} = \frac{Q}{K} \oplus \frac{H_m(M)}{K}. \tag{4}$$

Now Q is h-pure in M, hence $H_m(M) = K$ and $\mathrm{Soc}(H_m(M)) = \mathrm{Soc}(K) = N \cap \mathrm{Soc}(H_m(M))$. Therefore, we may assume that there exists a chain of integers $0 \leq k_0 < k_1 < k_2 \cdots$ such that

$$\frac{\mathrm{Soc}\left(H_{k_i}(M)\right) + N}{N} \neq \frac{\mathrm{Soc}\left(H_{k_{i+1}}(M)\right) + N}{N}, \quad \text{for every } i \geq 0. \tag{5}$$

Then we may choose $x_i \in \mathrm{Soc}(H_{k_i}(M))$ such that $x_i \notin \mathrm{Soc}(H_{k_i+1}(M))$ for every i and we have $\{x_i\} \cap N = 0$. Let P be an N-high submodule of M such that $\Sigma x_i R \subseteq P$. There exists a proper unbounded basic submodule B of P and a submodule $T \subseteq M$, containing N such that $M/B = (T/B) \oplus (P/B)$. Now T is h-pure in M which is a contradiction proving (ii).

Conversely, suppose K is an h-pure submodule of M containing N. By (i), M/K is h-divisible. Since $\mathrm{Soc}\,(H_m(M)) \subseteq K$ and K is h-pure in M, we have $K = M$. □

Now we should mention the following notations used by Khan and Zubair [7]:

$$N^k(M) = (N + H_{k+1}(M)) \cap \operatorname{Soc}(H_k(M)),$$
$$N_k(M) = (N \cap \operatorname{Soc}(H_k(M))) + \operatorname{Soc}(H_{k+1}(M)), \tag{6}$$
$$Q_k(M,N) = \frac{N^k(M)}{N_k(M)}.$$

We prove the following.

Proposition 7. *Let $K \supseteq N$ be an h-pure submodule of a QTAG module M. Then $Q_k(M,N) \cong Q_k(K,N)$, for all $k \geq 0$.*

Proof. Since $N^k(M) \cap K = N^k(K)$ and $N_k(M) \cap K = N_k(K)$, there exists a homomorphism $f : Q_k(K,N) \to Q_k(M,N)$ such that $f(x + N_k(K)) = x + N_k(M)$. If $z \in N^k(M)$ and $z + N_k(M) \in Q_k(M,N)$, then there exist $u \in N \cap H_k(M)$ and $v \in H_{k+1}(M)$ such that $z = u + v$ and $e(z) = 1$. Moreover if $u' \in uR$, $v' \in vR$, $d(uR/u'R) = 1 = d(vR/v'R)$, then $u'R = v'R$ is contained in $N \cap H_{k+2}(M) \subseteq K \cap H_{k+2}(M) = H_{k+2}(K)$ and there exists $y \in K$ such that $u'R = v'R = y''R$, where $d(yR/y''R) = k+2$ and $d(y'R/y''R) = k + 1 = d(yR/u'R) = d(yR/v'R)$ for some $y', y'' \in H_{k+1}(K)$ and $H_{k+2}(K)$, respectively. Since $(u + y'')R \subseteq N^k(K)$ and $(v + y'')R \subseteq \operatorname{Soc}(H_{k+1}(M))$, we have $f(u + y' + N_k(K)) = z + N_k(M)$. Also f is one to one by definition; hence f is an isomorphism. □

Theorem 8. *Let N be a semi-h-pure submodule in M having an h-pure hull. Then there exists an integer $l \in \mathbb{Z}^+$ such that $Q_k(M,N) = 0$, for $k \geq l$.*

Proof. Let K be an h-pure hull of N in M. By Theorem 6, there exists a nonnegative integer l such that $\operatorname{Soc}(H_l(K)) \subseteq N$ and for all $k \geq l$,

$$\begin{aligned} N^k(K) &= (N + H_{k+1}(K)) \cap \operatorname{Soc}(H_k(K)) \\ &= \operatorname{Soc}(H_k(K)) = N_k(K). \end{aligned} \tag{7}$$

Therefore $Q_k(K,N) = 0$, for all $k \geq l$. Hence by Proposition 7, $Q_k(M,N) = 0$ for all $k \geq l$. □

Proposition 9. *Let K be an h-pure hull of a semi-h-pure submodule N of M and l a nonnegative integer. Then $\operatorname{Soc}(H_l(K)) \subseteq N$ and $\operatorname{Soc}(H_{l-1}(K)) \nsubseteq N$ if and only if $Q_k(M,N) = 0$ for all $k \geq l$ and $Q_{k-1}(M,N) \neq 0$.*

Proof. Since $N^k(K) = \operatorname{Soc}(H_k(K)) = N_k(K)$ for all $k \geq l$, by Proposition 7, $Q_k(M,N) = Q_k(K,N) = 0$ for all $k \geq l$. If $Q_{k-1}(M,N) = 0$, then $Q_{k-1}(K,N) = 0$ and $\operatorname{Soc}(H_{l-1}(K)) = N^{l-1}(K) = N_{l-1}(K) \subseteq N$. This contradiction implies that $Q_{l-1}(M,N) \neq 0$.

For the converse, consider the integer q such that $\operatorname{Soc}(H_q(K)) \subseteq N$ and $\operatorname{Soc}(H_{q-1}(K)) \nsubseteq N$. If $q > l$, then

$$\begin{aligned} N^l(K) &= \operatorname{Soc}(H_l(K)) = N_l(K) \\ &= \operatorname{Soc}(N \cap H_l(K)) + \operatorname{Soc}(H_q(K)) \subseteq N, \end{aligned} \tag{8}$$

which is a contradiction. If $q < l$, then

$$N^{l-1}(K) = \operatorname{Soc}(H_{l-1}(K)) = N_{l-1}(K). \tag{9}$$

Again it is a contradiction, hence $l = q$. □

kV modules are defined in [3], and we can observe the following.

Remark 10. If N is h-pure in M, then by Proposition 7, $Q_k(M,N) = Q_k(N,N) = 0$ and $N \subseteq \operatorname{Soc}(M)$ implies that $N^k(M) = N_k(M)$ for all $k \geq 0$. In other words, N is a V module in M.

Proposition 11. *Let N be an h-neat submodule of M. Then N is h-pure in M if and only if N is a V module in M.*

Proof. Suppose $N \cap H_k(M) = H_k(N)$. Consider $x \in N$ such that there exists $y \in M$ and $d(yR/xR) = k+1$. This implies the existence of y' such that $d(y'R/xR) = k$ and $d(yR/y'R) = 1$. Now there exists $z, z' \in N$ such that $y' - z' \in N^{k-1}(M)$, where $d(zR/z'R) = k-1$. As $N^{k-1}(M) = N_{k-1}(M)$, we have $y' - z' = u + v$ for some $u \in N \cap \operatorname{Soc}(H_{k-1}(N))$ and $v \in \operatorname{Soc}(H_k(N))$. Since $y' - v \in N \cap H_k(M) = H_k(N)$, there exists $\omega \in N$ such that $d(\omega R/xR) = k+1$ and N is h-pure in M. The converse is trivial. □

Proposition 12. *A submodule $N \subseteq M$ is a V module if and only if $\operatorname{Soc}(N + H_k(M)) = \operatorname{Soc}(N) + \operatorname{Soc}(H_k(M))$ for all $k \geq 1$.*

Proof. Suppose N is a V module in M. Therefore $\operatorname{Soc}(N + H_1(M)) = \operatorname{Soc}(N) + \operatorname{Soc}(H_1(M))$. We will prove the result by induction. Assume that $\operatorname{Soc}(N + H_k(M)) = \operatorname{Soc}(N) + \operatorname{Soc}(H_k(M))$ for all $k \leq l$. Since $\operatorname{Soc}(N + H_{l+1}(M)) \subseteq \operatorname{Soc}(N + H_l(M)) \subseteq \operatorname{Soc}(N) + \operatorname{Soc}(H_l(M))$, we have

$$\begin{aligned} \operatorname{Soc}(N + H_{l+1}(M)) &= (N + H_{l+1}(M)) \\ &\quad \cap (\operatorname{Soc}(N) + \operatorname{Soc}(H_l(M))) \\ &= \operatorname{Soc}(N) + N^l(M) \\ &= \operatorname{Soc}(N) + N_l(M) \\ &= \operatorname{Soc}(N) + \operatorname{Soc}(H_{l+1}(M)). \end{aligned} \tag{10}$$

The converse is trivial. □

Following results are the immediate consequences of the previous discussion.

Remark 13. (i) N is a V module in M if and only if $H_1(N \cap H_k(M)) = H_1(N) \cap H_{k+1}(M)$ for all $k \geq 1$.

(ii) If $H^n(N + H_k(M)) = H^n(H_k(M))$ for all $n, k \geq 0$, then N is a V module in M.

3. h-Pure Envelopes

For a proper submodule $N \subseteq M$, it is not always possible for N to have an h-pure hull in M. We study the situation

when there is a proper h-pure submodule $K \supseteq N$ of M, but no proper direct summand of K contains N. This motivates us to define h-pure envelopes like h-neat envelope defined earlier [6]. We find that the *Ulm*-Kaplansky invariants are same for all h-pure envelopes.

Definition 14. Let N be a submodule of a QTAG module M. An h-pure submodule $P \supseteq N$ is an h-pure envelope of N if P has no proper direct summand containing N.

Proposition 15. *In a module M, an h-pure submodule $P \supseteq N$ is an h-pure envelope of N if and only if P contains no h-divisible summand disjoint from N and for any k, no uniserial summand of decomposition length k disjoint from $N + H_k(P)$.*

Proof. If P is not an h-pure envelope of N, then $P = K \oplus L$ with $N \subseteq K, L \neq 0$. If L is not h-reduced, then it contains an h-divisible summand disjoint from N, and if L is h-reduced, then it contains a uniserial summand L' such that $d(L') = k$ for some $k > 0$. Without loss of generality, we may assume that $L = L'$. Now

$$H_k(P) = (K \cap H_k(P)) \oplus (L \cap H_k(P)) = K \cap H_k(P), \quad (11)$$

implying that $H_k(P) \subseteq K$. Therefore, $L \cap (N + H_k(P)) = 0$ and $N + H_k(P) \subseteq K$.

Conversely, if L is h-divisible submodule of P with $L \cap N = 0$, then L is a summand of P; therefore P cannot be an h-pure envelope of N. If $L \neq 0$, it is a uniserial summand such that $d(L) = k$ and $L \cap (N + H_k(P)) = 0$, then $P = K \oplus L$ such that $N + H_k(P) \subseteq K$. This K can be chosen to be any submodule of M which is maximal with respect to the properties of containing $N + H_k(P)$ and disjoint from L. This P can not be the h-pure envelope of N. □

Let xR be a uniserial summand of N such that $d(xR) = k + 1$. If $y \in xR$ such that $d(xR/yR) = k$, then $xR \cap (N + H_{k+1}(P)) = 0$ if and only if $yR \nsubseteq N + H_{k+1}(P)$. Since $y \in \mathrm{Soc}(H_k(P))$, this implies that $\mathrm{Soc}(H_k(P)) \nsubseteq N + H_{k+1}(N)$. This enables us to prove the following.

Proposition 16. *An h-pure submodule $P \subseteq M$ containing the submodule N of a separable module M is an h-pure envelope if and only if $\mathrm{Soc}(H_k(P)) \subseteq N + H_{k+1}(P)$, for all $k \in N$.*

Proof. Let N be a semi-h-pure submodule of a separable module M and P the h-pure hull of N. Now we have $\mathrm{Soc}(N \cap H_k(P)) + H_{k+1}(P) = \mathrm{Soc}(H_k(P))$. This implies that $f_k(P, N) = 0$ for every $k \geq 0$. This is true for h-pure envelopes, and if M is separable, this is sufficient also. □

Remark 17. Since

$$\frac{\mathrm{Soc}(H_k(P))}{\mathrm{Soc}(H_{k+1}(P) + N) \cap \mathrm{Soc}(H_k(P))} \cong \frac{\mathrm{Soc}(H_k(P)) + \mathrm{Soc}(H_{k+1}(P) + N)}{\mathrm{Soc}(H_{k+1}(P) + N)} = 0, \quad (12)$$

for all $k < \omega$, if and only if $\mathrm{Soc}(H_k(P)) \subseteq N + H_{k+1}(P)$ for every $k < \omega$, an h-pure submodule $P \supseteq N$ is an h-pure envelope of N if and only if $f_k(N, P) = 0$, for every $k < \omega$.

Remark 18. Since the union of a chain of h-pure envelopes of N may not contain any h-divisible direct summand disjoint from N or a uniserial summand of decomposable length k disjoint from $N + H_k(P)$, every h-pure envelope of $N \subseteq M$ is contained in a maximal h-pure envelope of N.

Now we investigate h-pure envelopes of N containing other h-pure envelopes of N.

Theorem 19. *Let N be a submodule of a separable module M and $P \subseteq Q$, h-pure submodules of M containing N. Then*

(i) *for every $k < \omega$, the natural embedding $P \to Q$ induces a monomorphism*

$$f_k : \frac{\mathrm{Soc}(H_k(P)) + \mathrm{Soc}(N + H_{k+1}(P))}{\mathrm{Soc}(N + H_{k+1}(P))} \longrightarrow \frac{\mathrm{Soc}(H_k(Q)) + \mathrm{Soc}(N + H_{k+1}(Q))}{\mathrm{Soc}(N + H_{k+1}(Q))}, \quad (13)$$

(ii) *the map f_k is onto for every $k > \omega$ if and only if P is an h-dense submodule of Q.*

Proof. The maps f_k send the coset $x + \mathrm{Soc}(N + H_{k+1}(P))$ upon the coset $x + \mathrm{Soc}(N + H_{k+1}(Q))$. Since $x \in H_k(P)$, there exists $y \in P$ such that $d(yR/xR) = k$ and $e(x) = 1$; therefore

$$f_k\left(\frac{\mathrm{Soc}(H_k(P)) + \mathrm{Soc}(N + H_{k+1}(P))}{\mathrm{Soc}(N + H_{k+1}(P))}\right) = \frac{\mathrm{Soc}(H_k(P)) + \mathrm{Soc}(N + H_{k+1}(Q))}{\mathrm{Soc}(N + H_{k+1}(Q))}. \quad (14)$$

If $x + \mathrm{Soc}(N + H_{k+1}(P)) \in \ker f_k$, then $f_k(x + \mathrm{Soc}(N + H_{k+1}(P))) = \mathrm{Soc}(N + H_{k+1}(Q))$ or $x = u + z, u \in N, z \in H_{k+1}(Q)$; that is, there exists $v \in Q$ such that $d(vR/zR) = k+1$. Now $x - u = z \in N \cap H_{k+1}(Q) = H_{k+1}(P)$. Therefore $x \in N + H_{k+1}(P)$ or $x \in \mathrm{Soc}(N + H_{k+1}(P))$ and f_k is a monomorphism, which completes part (i).

(ii) Suppose P is h-dense in Q, therefore $Q = P + H_k(Q)$ for $k < \omega$. For $x \in \mathrm{Soc}(H_k(Q))$ such that $x \notin \mathrm{Soc}(H_{k+1}(Q))$, there exists $y \in Q$ such that $d(yR/xR) = k$. Since P is h-dense in Q, $x = u + z$, $u \in P, z \in H_1(Q), x \in \mathrm{Soc}(H_k(P)) + \mathrm{Soc}(N + H_{k+1}(Q))$, and f_k is surjective.

Conversely, suppose each f_k is surjective, therefore $\mathrm{Soc}(H_k(Q)) \subseteq \mathrm{Soc}(H_k(P)) + \mathrm{Soc}(N + H_{k+1}(Q))$, for every $k < \omega$. Let $x \in \mathrm{Soc}(H_k(Q))$. Now there exists $y \in Q$ such that $d(yR/xR) = k$. Now $x = z + (u + v)$ with $z \in \mathrm{Soc}(H_k(P)), u \in N$, and $v \in Q$ such that there exists $v' \in Q$ with $d(v'R/vR) = k + 1$. Also $u \in H_k(N)$ and $z + u \in H_k(P)$. We will prove that $H^k(Q) \subseteq P + H_1(Q)$ implying that $Q = P + H_1(Q)$. For $k = 1$, if $y \in \mathrm{Soc}(Q)$, then $y \in P + H_1(Q)$, and if $y \in H^{k+1}(Q), x \in \mathrm{Soc}(Q)$ and $x = z + u + v$ implies that $y - z - u - v \in P + H_1(Q)$. Therefore $y \in P + H_1(Q)$, and we are done. □

Now we investigate the relation between *Ulm*-Kaplansky invariants of the submodules N and their h-pure envelopes.

Theorem 20. *Let P be an h-pure envelope of the submodule N of a separable module M. Then*

$$\begin{aligned}\frac{\operatorname{Soc}\left(P\cap H_k(M)\right)}{\operatorname{Soc}\left(P\cap H_{k+1}(M)\right)} &\cong \frac{\operatorname{Soc}\left(N\cap H_k(P)\right)}{\operatorname{Soc}\left(N\cap H_{k+1}(P)\right)} \\ &\oplus \frac{\operatorname{Soc}\left(H_{k+1}(P)+\left(N\cap H_k(P)\right)\right)}{\operatorname{Soc}\left(H_{k+1}(P)\right)+\operatorname{Soc}\left(N\cap H_k(P)\right)}.\end{aligned} \tag{15}$$

Proof. We have

$$\begin{aligned}\frac{\operatorname{Soc}\left(N\cap H_k(P)\right)}{\operatorname{Soc}\left(N\cap H_{k+1}(P)\right)} &= \frac{H_k(P)\cap\operatorname{Soc}(N)}{H_{k+1}(P)\cap\operatorname{Soc}(N)} \\ &\cong \frac{\left(\operatorname{Soc}\left(H_k(P)\right)\cap N\right)+\operatorname{Soc}\left(H_{k+1}(P)\right)}{\operatorname{Soc}\left(H_{k+1}(P)\right)} \\ &= \frac{\operatorname{Soc}\left(H_k(P)\right)\cap\left(\operatorname{Soc}(N)+\operatorname{Soc}\left(H_{k+1}(P)\right)\right)}{\operatorname{Soc}\left(H_{k+1}(P)\right)}.\end{aligned} \tag{16}$$

Since M is separable, by Remark 17, we have

$$\frac{\operatorname{Soc}\left(H_k(P)\right)}{\operatorname{Soc}\left(\left(N\cap\operatorname{Soc}\left(H_k(P)\right)\right)+H_{k+1}(P)\right)}=0, \tag{17}$$

therefore,

$$\operatorname{Soc}\left(H_k(P)\right)=\operatorname{Soc}\left(\left(N\cap\operatorname{Soc}\left(H_k(P)\right)\right)+H_{k+1}(P)\right), \tag{18}$$

and we are able to write

$$\begin{aligned}\frac{\operatorname{Soc}\left(\left(N\cap H_k(P)\right)+H_{k+1}(P)\right)}{\operatorname{Soc}\left(N\cap H_k(P)\right)+\operatorname{Soc}\left(H_{k+1}(P)\right)} &= \frac{\operatorname{Soc}\left(H_k(P)\right)}{\operatorname{Soc}\left(H_k(P)\right)\cap\left(\operatorname{Soc}(N)+\operatorname{Soc}\left(H_{k+1}(P)\right)\right)}.\end{aligned} \tag{19}$$

This implies that

$$\frac{\operatorname{Soc}\left(P\cap H_k(M)\right)}{\operatorname{Soc}\left(P\cap H_{k+1}(M)\right)}=\frac{\operatorname{Soc}\left(H_k(P)\right)}{\operatorname{Soc}\left(H_{k+1}(P)\right)} \tag{20}$$

is an extension of $\operatorname{Soc}(N\cap H_k(P))/\operatorname{Soc}(N\cap H_{k+1}(P))$ by $\operatorname{Soc}(H_{k+1}(P)+(N\cap H_k(P)))/(\operatorname{Soc}(H_{k+1}(P))+\operatorname{Soc}(N\cap H_k(P)))$, and the result follows. □

Following are the immediate consequences of the previous discussion.

Corollary 21. *Every h-pure envelope of the submodule N in a separable module M has same Ulm-Kaplansky invariants. Therefore h-pure envelopes of N have isomorphic basic submodules.*

Corollary 22. *Let M be the direct sum of uniserial modules. Then all h-pure envelopes of a submodule $N\subseteq M$ are isomorphic, if they exist.*

Proposition 23. *If N is h-pure submodule of a separable module M such that its closure $\overline{N}$ is not h-pure in M, then $\overline{N}$ has no h-pure envelope in M.*

Proof. An h-pure envelope Q of $\overline{N}$ must have a larger basic submodule than N, and any uniserial summand in this larger basic submodule not in $\overline{N}$ contradicts the Proposition 15, and the result follows. □

Theorem 24. *Every submodule of a separable module M admits h-pure envelopes if and only if M is quasicomplete; that is, the closure of every h-pure submodule of M is h-pure in M.*

Proof. Suppose on the contrary that M is not quasicomplete. Thus it contains an h-pure submodule N such that $\overline{N}$ is not h-pure in M. By Proposition 23, the submodule $\overline{N}\subseteq M$ does not have an h-pure envelope. This contradiction proves that M is quasicomplete.

For the converse, consider a submodule N of a quasicomplete module M. We construct a countable sequence of subsocles of N, $0\subseteq N_0\subseteq N_1\subseteq\cdots\subseteq N_k\subseteq\cdots$ with another sequence, $0\subseteq K_0\subseteq K_1\subseteq\cdots\subseteq K_k\subseteq\cdots$ of h-pure submodules of M such that $N_i\subseteq K_i$ and $H_i(K_i)=0$, for every i. Now we may say that N_{k+1} is the maximal submodule such that $N_k\subseteq N_{k+1}\subseteq\operatorname{Soc}(N)$ and $N_{k+1}\cap H_{k+1}(M)=0$. Put $T_k=N_{k+1}\cap H_k(M)$ such that the nonzero elements of T_k are of height k and $N_{k+1}=N_k\oplus T_k$. Now $T_k\cap N_k=0$, and $T_k\oplus K_k$ is bounded by k; therefore there exists a minimal h-pure submodule containing $T_k\oplus K_k$, which is K_{k+1} here. Now K_{k+1} is also bounded, hence a summand of M and by Proposition 16, $H_k(K_{k+1})=T_k$. By the same argument, we get $K'=\bigcup_{k\in\omega}K_k$, an h-pure submodule of M. Now

$$N'=\bigcup_{k\in\omega}N_k=\bigoplus_{k\in\omega}T_k \tag{21}$$

is an h-dense subsocle of N, and we have to show that K' does not have a uniserial summand of decomposition length k, which is disjoint from $N+H_k(K')$. Suppose xR is a uniserial summand of K' such that $d(xR)=k$ and $xR\cap(N+H_k(K'))=0$. Let m be the least positive integer such that $xR\cap(N+H_m(K'))=0$, $x\in K_{m+1}$. Since $xR\cap(T_m+K_m+H_k(K_{m+1}))\neq 0$ and there are elements $u\in T_m$, $y\in K_m$, $x'\in\operatorname{Soc}(xR)$, $v,v'\in K_{m+1}$ such that $d(v'R/vR)=k$, $x'=u+y+v$. Here $H(u)=m$, $y\neq 0$ because $xR\cap(N+H_k(K_{m+1}))=0$. Therefore $y\in K_m$ implies that $H(y)\leq m-1$. As $H(x')=k-1$, we have $k-1=m-1$; therefore there exist $z,z'\in K_{m+1}$ such that $d(zR/z'R)=k$, $uR=z'R$. Also there are elements $w,w'\in K_{m+1}$ such that $d(wR/w'R)=k-1$ and $w'R=yR$. Since K_m is h-pure, there is a uniserial summand $tR\subseteq K_m$ and $d(tR/yR)=k-1$ and $tR\cap(N+H_k(K'))\neq 0$, which is a contradiction implying that $xR\cap(N+H_k(K'))\neq 0$.

Suppose that K is the closure of K' with respect to h-topology. Then, K is h-pure in M and $K\supset\operatorname{Soc}(N)$.

Case i. When M is closed [8] with respect to h-topology [4], by the structure of M, $K = \overline{K}$ is a summand of M. Consider the decomposition $M = K \oplus L$. If Q is a submodule of M such that $\mathrm{Soc}(Q) = \mathrm{Soc}(K)$, then

$$\mathrm{Soc}\left(\frac{N+L}{N}\right) = \frac{(N + \mathrm{Soc}(L))}{N}. \tag{22}$$

If $x+N \in (N+\mathrm{Soc}(L))/N$ has height k in M/N, then $H_L(x) = k$ and $(N \oplus L)/N$ is h-pure in M/N. Thus,

$$\frac{M}{N} = \frac{Q}{N} \oplus \left(\frac{L \oplus N}{N}\right) \tag{23}$$

and $M = Q \oplus L$, and Q is an h-pure envelope of N in the closed module M.

Case ii. When M is an arbitrary QTAG module, again consider the decomposition $M = K \oplus L$. If K is bounded, then we are done; otherwise M/K is a closed module. By Case i, $(N + K)/K \subset M/K$ has an h-pure envelope L/K. Assume on the contrary that xR is a summand of L such that $d(xR) = k$, $xR \cap (N + H_k(L)) = 0$. If $\mathrm{Soc}(xR) \cap K = 0$, then $(xR + K)/K$ is a uniserial summand of L/K and

$$\left(\frac{xR+K}{K}\right) \bigcap \left(\frac{N+K}{K} + H_k\left(\frac{L}{K}\right)\right) \neq 0, \tag{24}$$

implying that $xR \cap (N + H_k(L)) \nsubseteq N$ which is a contradiction. Therefore $\mathrm{Soc}(xR) \subseteq N$, but then N contains a summand yR such that $\mathrm{Soc}(yR) = \mathrm{Soc}(xR)$ and $xR \cap (N + H_k(L)) = 0$ implies that $yR \cap (N + H_k(K)) = 0$, again a contradiction. Thus, no summand xR exists, and L is an h-pure envelope of N. □

Now we investigate the conditions under which every submodule of a QTAG module M has an h-pure envelope.

Proposition 25. *M^1 has an h-pure envelope in M if and only if it is h-divisible.*

Proof. If M^1 is not h-divisible, then the basic submodule of any h-pure submodule N of M containing M^1 is nontrivial and it has a nonzero uniserial summand disjoint from M^1. By Proposition 15, N can not be an h-pure envelope of M^1. The converse is trivial. □

The following theorem characterizes the module M whose every submodule has an h-pure envelope.

Theorem 26. *In a module M, every submodule has an h-pure envelope if and only if M is the direct sum of an h-divisible and a quasicomplete module.*

Proof. Suppose that every submodule N of M has an h-pure envelope. By Proposition 25, M^1 is h-divisible. Now $M = D \oplus K$, where D is the maximal h-divisible submodule of M and K is separable. Let $N \subseteq K$ and P, the h-pure envelope of N in M. Now if we project P into K, then the projection of P is an h-pure envelope of N in K. Therefore, all the submodules of K have h-pure envelopes and Theorem 24 implies that K is a quasicomplete module.

Conversely, suppose $M = D \oplus K$, where D is h-divisible and K a quasicomplete module. For $N \subseteq M$, $D+N = D \oplus N_0$, where $N_0 \subseteq K$ may be chosen. By Theorem 24, N_0 has an h-pure envelope P_0 in K. Now there exists an h-divisible submodule $D_0 \subseteq D$ that contains $N \cap D$ as an essential submodule. If we put $P = P_0 \oplus N_0$, then $N \subseteq P$ which is h-pure in M. P cannot have a uniserial or h-divisible summand disjoint from N, because such a summand would be disjoint from N_0 and $N \cap D$. Therefore, P is an h-pure envelope of N. □

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