Research Article

On Quasi-\(h\)-Dense Submodules and \(h\)-Pure Envelopes of QTAG Modules

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A module \(M\) over an associative ring \(R\) with unity is a QTAG module if every finitely generated submodule of any homomorphic image of \(M\) is a direct sum of uniserial modules. There are many fascinating properties of QTAG modules of which \(h\)-pure submodules and high submodules are significant. A submodule \(N\) is quasi-\(h\)-dense in \(M\) if \(M/K\) is \(h\)-divisible, for every \(h\)-pure submodule \(K\) of \(M\), containing \(N\). Here we study these submodules and obtain some interesting results. Motivated by \(h\)-neat envelope, we also define \(h\)-pure envelope of a submodule \(N\) as the \(h\)-pure submodule \(K \supseteq N\) if \(K\) has no direct summand containing \(N\). We find that \(h\)-pure envelopes of \(N\) have isomorphic basic submodules, and if \(M\) is the direct sum of uniserial modules, then all \(h\)-pure envelopes of \(N\) are isomorphic.

1. Introduction

All the rings \(R\) considered here are associative with unity, and right modules \(M\) are unital QTAG modules. An element \(x \in M\) is uniform, if \(xR\) is a nonzero uniform (hence uniserial) module and for any \(R\)-module \(M\) with a unique decomposition series, \(d(M)\) denotes its decomposition length. For a uniform element \(x \in M\), \(e(x) = d(xR)\), and \(H_M(x) = \{d(yR/xR) \mid y \in M, x \in yR \text{ and } y \text{ uniform}\}\) are the exponent and height of \(x\) in \(M\), respectively. \(H_k(M)\) denotes the submodule of \(M\) generated by the elements of height at least \(k\), and \(H^k(M)\) is the submodule of \(M\) generated by the elements of exponent at most \(k\). \(M\) is \(h\)-divisible if \(M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)\), and it is \(h\)-reduced if it does not contain any \(h\)-divisible submodule. In other words, it is free from the elements of infinite height.

The modules \(H_k(M), k = 0, 1, \ldots, \infty\) form a neighbour-
hood system of zero giving rise to \(h\)-topology. The closure of a submodule \(N \subseteq M\) is defined as \(\overline{N} = \bigcap_{k=0}^{\infty} (N + H_k(M))\), and it is closed with respect to \(h\)-topology if \(N = \overline{N}\).

A submodule \(N \subseteq M\) is \(h\)-pure in \(M\) if \(N \cap H_k(M) = H_k(N)\), for every integer \(k \geq 0\). For a limit ordinal \(\alpha, H_\alpha(M) = \bigcap_{\rho<\alpha} H_\rho(M)\), for all ordinals \(\rho < \alpha\), and if \(H_\alpha(N) = H_\alpha(M) \cap N\) for all ordinals \(\sigma < \alpha\).

A module \(M\) is summable if \(\text{Soc}(M) = \oplus_{\alpha<\tau} S_\alpha\), where \(S_\alpha\) is the set of all elements of \(H_\alpha(M)\) which are not in \(H_{\alpha+1}(M)\), where \(\tau\) is the length of \(M\). A submodule \(N \subseteq M\) is nice [1, Definition 2.3] in \(M\), if \(H_\gamma(M/N) = (H_\gamma(M) + N)/N\) for all ordinals \(\gamma\); that is, every coset of \(M \text{ modulo } N\) may be represented by an element of the same height.

The cardinality of the minimal generating set of \(M\) is denoted by \(g(M)\). For all ordinals \(\alpha, f_M(\alpha)\) is the \(\alpha\)-Ulm invariant of \(M\) and it is equal to \(g(\text{Soc}(H_\alpha(M))/\text{Soc}(H_{\alpha+1}(M)))\).

For a QTAG module \(M\), there is a chain of submodules \(M^0 \supset M^1 \supset M^2 \supset \cdots \supset M^\tau = 0\), for some ordinal \(\tau, M^{\tau+1} = (M^\tau)^1\), where \(M^\tau\) is the \(\alpha\)-Ulm submodule of \(M\). Singh [2] proved that the results which hold for TAG modules also hold good for QTAG modules.

2. Quasi-\(h\)-Dense Submodules

In [3], we studied semi-\(h\)-pure submodules which are not \(h\)-pure but contained in \(h\)-pure submodules. Now we investigate the submodules \(N \subseteq M\) such that \(M/K\) is \(h\)-divisible for every \(h\)-pure submodule \(K \subseteq M\), containing \(K\). These modules are called quasi-\(h\)-dense submodules.
We start with the following.

**Definition 1.** A submodule $N$ of $M$ is quasi-$h$-dense in $M$, if for every $h$-pure submodule $K \subseteq M$, containing $N$, $M/K$ is $h$-divisible.

**Lemma 2.** If $\text{Soc}(H_{k-1}(M)) \nsubseteq N + H_k(M)$ for some integer $k$, then there exists a proper submodule $K$ of $M$ containing $N$ and a bounded submodule $Q$ of $M$ such that

(i) $M = K \oplus Q$;
(ii) $K \supseteq N + H_k(M)$;
(iii) $N + H_k(K) \supseteq \text{Soc}(H_{k-1}(K))$.

**Proof.** The socle of $H_{k-1}(M)$ can be expressed as the direct sum of $S_1$, $S_2$ (where $S_1 = (N + H_k(M)) \cap \text{Soc}(H_{k-1}(M))$ and $S_2 \cap H_k(M) = 0$). Now there exists a $h$-pure submodule $Q$ of $M$ such that $\text{Soc}(Q) = S_2$. Also $(Q \oplus H_k(M))/H_k(M)$ is $h$-pure in $M/H_k(M)$ and $\text{Soc}(Q \oplus H_k(M))/H_k(M) = (S_2 \oplus H_k(M))/H_k(M)$. Now $\text{Soc}(H_k(M)/H_k(M)) = 0$ which is contained in $(S_2 \oplus H_k(M))/H_k(M) \subseteq \text{Soc}(H_{k-1}(M)/H_k(M))$; therefore $(Q \oplus H_k(M))/H_k(M)$ is an absolute direct summand of $M/H_k(M)$ [4]. Thus there exists a submodule $K$ of $M$ such that

$$\frac{M}{H_k(M)} = \left(\frac{K}{H_k(M)} \oplus \frac{Q}{H_k(M)}\right)$$

and $K \supseteq N + H_k(M)$; therefore $M = K \oplus Q$ and $N + H_k(K) \supseteq \text{Soc}(H_{k-1}(K))$.

**Proposition 3.** A submodule $N$ of a QTAG module $M$ is quasi-$h$-dense in $M$ if and only if for all integers $k \geq 0$, $N + H_{k+1}(M) \supseteq \text{Soc}(H_k(M))$.

**Proof.** Let $k$ be the least positive integer such that $N + H_k(M) \nsubseteq \text{Soc}(H_k(M))$. By Lemma 2, there exists a proper submodule $K \subseteq M$ containing $N$ and a bounded submodule $Q$ of $M$ such that $M/K \equiv Q$. This contradiction proves that $N + H_{k+1}(M) \supseteq \text{Soc}(H_k(M))$, for all integers $k \geq 0$.

For the converse, consider an $h$-pure submodule $K \subseteq N$. If $M/K$ is not $h$-divisible, then there exist submodules $P$ and $Q$ of $M$ such that $M/K = (P/K) \oplus (Q/K)$, where $Q/K$ is bounded. If $H_k(Q/K) = 0$ and $H_k(P/K) \neq 0$, then $P \supseteq N + H_k(M) \supseteq \text{Soc}(H_{k-1}(M))$. Since $P$ is $h$-pure in $M$, we have $H_k(M) \subseteq P$. On repeating the process, after a finite number of steps, we get $P = M$ and $Q/K = 0$, which is a contradiction. Hence $M/K$ is $h$-divisible and $N$ is quasi-$h$-dense in $M$.

**Remark 4.** For a submodule $N$ of a QTAG module $M$, the following are equivalent:

(i) $N$ is quasi-$h$-dense in $M$;
(ii) $N + H_{k+1}(M) \supseteq \text{Soc}(H_k(M))$, $\forall k > \omega$;
(iii) for every $h$-pure submodule $K$ containing $N$, $M/K$ is $h$-divisible.

**Lemma 5.** If $K$ is an $h$-divisible submodule of a module $M$ and $N$ is a $K$-high submodule of $M$, then $N$ is $h$-pure in $M$.

**Proof.** For an $h$-divisible submodule $K \subseteq M$, $K$-high submodules are $h$-neat [5]. Therefore, $N \cap H_t(M) = H_t(N)$. Assume that $N \cap H_t(M) = H_t(N)$ for all $i \leq k$. Let $x \in N \cap H_{k+1}(M)$. Now there exists $y \in M$ such that $d(yR/xR) = k+1$. If $y' \in M$ such that $d(y'R/y'R) = k$, then $y' + z \in \text{Soc}(M)$ for some $z \in N$ and $y' + z \in \text{Soc}(M) = \text{Soc}(K) \oplus \text{Soc}(N) = \text{Soc}(H_k(K)) \oplus \text{Soc}(N)$. Now there exists $u \in \text{Soc}(H_k(K))$ such that $d(xR/ur) = k$ and $y' + u \in N$. By induction $y \in H_{k+1}(N)$ and $N$ is $h$-pure in $M$.

**Theorem 6.** Let $N$ be a proper submodule of $M$. Then $N$ is not semi-$h$-pure if and only if the following hold:

(i) $N$ is quasi-$h$-dense in $M$;
(ii) there exists $m \in \mathbb{Z}^+$ such that $\text{Soc}(H_m(M)) \nsubseteq N$.

**Proof.** By Lemma 2 and Proposition 3, (i) is satisfied. Suppose $\text{Soc}(H_m(M)) + N/N \neq 0$, for all $k \geq 0$. If there exists a nonnegative integer $m$ such that

$$\text{Soc}(H_m(M)) + N = \text{Soc}(H_{m+t}(M)) + N,$$

for every $t \geq 0$,

then we have

$$\text{Soc}(H_m(M)) = \text{Soc}(N \cap H_m(M)) + \text{Soc}(H_{m+t}(M)),$$

for every $t \geq 0$.

Let $K$ be the $h$-neat envelope [6] of $N \cap H_m(M)$ in $H_m(M)$. Then $K$ is $h$-pure in $H_m(M)$. Since $N \cap H_m(M)$ is quasi-$h$-dense in $H_m(M)$, $H_m(M)/N$ is $h$-divisible. Thus, there exists a submodule $Q$ of $M$ such that $N \subseteq Q$ and

$$\frac{M}{K} = \frac{Q}{K} \oplus \frac{H_k(M)}{K}.$$
Now we should mention the following notations used by Khan and Zubair [7]:

\[ N^k(M) = (N + H_{k+1}(M)) \cap \text{Soc}(H_k(M)), \]

\[ N_k(M) = (N \cap \text{Soc}(H_k(M))) + \text{Soc}(H_{k+1}(M)), \]

\[ Q_k(M, N) = \frac{N^k(M)}{N_k(M)}. \]

We prove the following.

**Proposition 7.** Let \( K \supseteq N \) be an \( h \)-pure submodule of a QTAG module \( M \). Then \( Q_k(M, N) \equiv Q_k(K, N) \), for all \( k \geq 0 \).

**Proof.** Since \( N^k(M) \cap K = N^k(K) \) and \( N_k(M) \cap K = N_k(K) \), there exists a homomorphism \( f : Q_k(K, N) \to Q_k(M, N) \) such that \( f(x + N_k(K)) = x + N_k(M) \). If \( z \in N^k(M) \) and \( z + N_k(M) \in Q_k(M, N) \), then there exist \( u \in N \cap H_2(M) \) and \( v \in H_{k+1}(M) \) such that \( z = u + v \) and \( e(z) = 1 \). Moreover, if \( u' \in uR, v' \in vR, d(uR/u'R) = 1 \), then \( u'R = v'R \) is contained in \( N \cap H_{k+2}(M) \subseteq K \cap H_{k+2}(M) = H_{k+1}(K) \) and there exists \( y \in K \) such that \( u'R = v'R = y' + R \) and \( (y' + R)R = k + 2 \). Therefore Soc \( H_k(M) = \text{Soc}(H_k(K)) \), and \( H_{k+1}(K) \). Since \( (u + v')R \subseteq N^k(M) \) and \( (v' + y')R \subseteq \text{Soc}(H_{k+1}(M)) \), we have \( f(u + v' + N_k(K)) = z + N_k(M) \). Also \( f \) is one to one by definition; hence \( f \) is an isomorphism.

**Theorem 8.** Let \( N \) be a semi-\( h \)-pure submodule in \( M \) having an \( h \)-pure hull. Then there exists an integer \( l \in \mathbb{Z}^+ \) such that \( Q_k(M, N) = 0 \), for \( k \geq l \).

**Proof.** Let \( K \) be an \( h \)-pure hull of \( N \) in \( M \). By Theorem 6, there exists a nonnegative integer \( l \) such that \( \text{Soc}(H_l(K)) \subseteq \) \( N \) and for all \( k \geq l \),

\[ N^k(K) = (N + H_{k+1}(K)) \cap \text{Soc}(H_k(K)) \]

\[ = \text{Soc}(H_k(K)) = N_k(K). \]

Therefore \( Q_k(K, N) = 0 \), for all \( k \geq l \). Hence by Proposition 7, \( Q_k(M, N) = 0 \) for all \( k \geq l \).

**Proposition 9.** Let \( K \) be an \( h \)-pure hull of a semi-\( h \)-pure submodule \( N \) of \( M \) and \( l \) a nonnegative integer. Then \( \text{Soc}(H_{l+1}(K)) \subseteq N \) and \( \text{Soc}(H_{l+1}(K)) \subseteq N \) if and only if \( Q_k(M, N) = 0 \) for all \( k \geq l \) and \( Q_{l-1}(M, N) \neq 0 \).

**Proof.** Since \( N^k(K) = \text{Soc}(H_k(K)) = N_k(K) \) for all \( k \geq 0 \), by Proposition 7, \( Q_k(M, N) = Q_k(K, N) = 0 \) for all \( k \geq 1 \). If \( Q_{l-1}(M, N) = 0 \), then \( Q_{l-1}(K, N) = 0 \) and \( \text{Soc}(H_{l+1}(K)) = N_{l-1}(K) \subseteq N \). This contradiction implies that \( Q_{l-1}(M, N) \neq 0 \).

For the converse, consider the integer \( q \) such that \( \text{Soc}(H_q(K)) \subseteq N \) and \( \text{Soc}(H_{q-1}(K)) \subseteq N \). If \( q > l \), then

\[ N^l(K) = \text{Soc}(H_l(K)) = N_l(K) \]

\[ = \text{Soc}(N \cap H_l(K)) + \text{Soc}(H_{l+1}(K)) \subseteq N, \]

which is a contradiction. If \( q < l \), then

\[ N^{l-1}(K) = \text{Soc}(H_{l+1}(K)) = N_{l-1}(K). \]

Again it is a contradiction, hence \( l = q \).

**kV** modules are defined in [3], and we can observe the following.

**Remark 10.** If \( N \) is \( h \)-pure in \( M \), then by Proposition 7, \( Q_k(M, N) = Q_k(K, N) = 0 \) and \( N \subseteq \text{Soc}(M) \) implies that \( N^k(M) = N_k(M) \) for all \( k \geq 0 \). In other words, \( N \) is a \( V \) module in \( M \).

**Proposition 11.** Let \( N \) be an \( h \)-neat submodule of \( M \). Then \( N \) is \( h \)-pure in \( M \) if and only if \( N \) is a \( V \) module in \( M \).

**Proof.** Suppose \( N \cap H_k(M) = H_k(N) \). Consider \( x \in N \) such that there exists \( y \in M \) and \( d(yR/xR) = k+1 \). This implies the existence of \( y' \) such that \( d(y'R/x'R) = k \) and \( d(y'R/y'R) = 1 \). Now there exists \( z, z' \in N \) such that \( y' - z' \in N^{k+1}(M) \), where \( d(zR/z'R) = k-1 \). As \( N^{k+1}(M) = N_{k+1}(M) \), we have \( y' - z' = u + v \) for some \( u \in N \cap \text{Soc}(H_{k+1}(N)) \) and \( v \in \text{Soc}(H_k(N)) \).

Since \( y' - v \in N \cap H_k(M) = H_k(N) \), there exists \( \omega \in N \) such that \( d(\omega R/xR) = k+1 \) and \( N \) is \( h \)-pure in \( M \). The converse is trivial.

**Proposition 12.** A submodule \( N \subseteq M \) is a \( V \) module if and only if \( \text{Soc}(N + H_k(M)) = \text{Soc}(N) + \text{Soc}(H_k(M)) \) for all \( k \geq 1 \).

**Proof.** Suppose \( N \) is a \( V \) module in \( M \). Therefore \( \text{Soc}(N + H_k(M)) = \text{Soc}(N) + \text{Soc}(H_k(M)) \). We will prove the result by induction. Assume that \( \text{Soc}(N + H_k(M)) = \text{Soc}(N) + \text{Soc}(H_k(M)) \) for all \( k \leq l \). Since \( \text{Soc}(N + H_{l+1}(M)) \subseteq \text{Soc}(N + H_{l+1}(M)) \subseteq \text{Soc}(N) + \text{Soc}(H_{l+1}(M)) \), we have

\[ \text{Soc}(N + H_{l+1}(M)) = (N + H_{l+1}(M)) \cap (\text{Soc}(N) + \text{Soc}(H_{l+1}(M))) \]

\[ = \text{Soc}(N) + N^l(M) \]

\[ = \text{Soc}(N) + \text{Soc}(H_{l+1}(M)). \]

The converse is trivial.

Following results are the immediate consequences of the previous discussion.

**Remark 13.** (i) \( N \) is a \( V \) module in \( M \) if and only if \( H_l(N \cap H_k(M)) = H_l(N) \cap H_{k+1}(M) \) for all \( k \geq 1 \).

(ii) If \( H^k(N + H_k(M)) = H^k(H_k(M)) \) for all \( n, k \geq 0 \), then \( N \) is a \( V \) module in \( M \).

### 3. \( h \)-Pure Envelopes

For a proper submodule \( N \subseteq M \), it is not always possible for \( N \) to have an \( h \)-pure hull in \( M \). We study the situation
when there is a proper $h$-pure submodule $K \supseteq N$ of $M$, but no proper direct summand of $K$ contains $N$. This motivates us to define $h$-pure envelopes like $h$-neat envelope defined earlier [6]. We find that the Ulm-Kaplansky invariants are same for all $h$-pure envelopes.

**Definition 14.** Let $N$ be a submodule of a QTAG module $M$. An $h$-pure submodule $P \supseteq N$ of $M$ is an $h$-pure envelope of $N$ if $P$ has no proper direct summand containing $N$.

**Proposition 15.** In a module $M$, an $h$-pure submodule $P \supseteq N$ is an $h$-pure envelope of $N$ if and only if $P$ contains no $h$-divisible summand disjoint from $N$ and for any $k$, no uniserial summand of decomposition length $k$ disjoint from $N+H_k(P)$.

**Proof.** If $P$ is not an $h$-pure envelope of $N$, then $P = K \oplus L$ with $N \subseteq K, L \neq 0$. If $L$ is not $h$-reduced, then it contains an $h$-divisible summand disjoint from $N$, and if $L$ is $h$-reduced, then it contains a uniserial summand $L'$ such that $d(L') = k$ for some $k > 0$. Without loss of generality, we may assume that $L = L'$. Now

$$H_k(P) = (K \cap H_k(P)) \oplus (L \cap H_k(P)) = K \cap H_k(P),$$

(11)

implying that $H_k(P) \subseteq K$. Therefore, $L \cap (N+H_k(P)) = 0$ and $N+H_k(P) \subseteq K$.

Conversely, if $L$ is $h$-divisible submodule of $P$ with $L \cap N = 0$, then $L$ is a summand of $P$; therefore $P$ cannot be an $h$-pure envelope of $N$. If $L \neq 0$, it is a uniserial summand such that $d(L) = k$ and $L \cap (N+H_k(P)) = 0$, then $P = K \oplus L$ such that $N+H_k(P) \subseteq K$. This $K$ can be chosen to be any submodule of $M$ which is maximal with respect to the properties of containing $N+H_k(P)$ and disjoint from $L$. This $P$ can not be the $h$-pure envelope of $N$.

Let $xR$ be a uniserial summand of $N$ such that $d(xR) = k+1$. If $y \in xR$ such that $d(yR/xR) = k$, then $xR \cap (N+H_{k+1}(P)) = 0$ if and only if $yR \not\subseteq N+H_{k+1}(P)$. Since $y \in Soc(H_k(P))$, this implies that $Soc(H_k(P)) \not\subseteq N+H_{k+1}(N)$. This enables us to prove the following.

**Proposition 16.** An $h$-pure submodule $P \subseteq M$ containing the submodule $N$ of a separable module $M$ is an $h$-pure envelope if and only if $Soc(H_k(P)) \subseteq N+H_{k+1}(P)$, for all $k \in \mathbb{N}$.

**Proof.** Let $N$ be a semi-$h$-pure submodule of a separable module $M$ and $P$ the $h$-pure hull of $N$. Now we have $Soc(N \cap H_k(P)) + H_{k+1}(P) = Soc(H_k(P))$. This implies that $f_k(P,N) = 0$ for every $k \geq 0$. This is true for $h$-pure envelopes, and if $M$ is separable, this is sufficient also.

**Remark 17.** Since

$$\frac{Soc(H_k(P))}{Soc(H_{k+1}(P)+N) \cap Soc(H_k(P))} \equiv \frac{Soc(H_k(P))+Soc(H_{k+1}(P)+N)}{Soc(H_{k+1}(P)+N)} = 0,$$

for all $k < \omega$, if and only if $Soc(H_k(P)) \subseteq N+H_{k+1}(P)$ for every $k < \omega$, an $h$-pure submodule $P \supseteq N$ is an $h$-pure envelope of $N$ if and only if $f_k(P,N) = 0$, for every $k < \omega$.

**Remark 18.** Since the union of a chain of $h$-pure envelopes of $N$ may not contain any $h$-divisible direct summand disjoint from $N$ or a uniserial summand of decomposable length $k$ disjoint from $N+H_k(P)$, every $h$-pure envelope of $N \subseteq M$ is contained in a maximal $h$-pure envelope of $N$.

Now we investigate $h$-pure envelopes of $N$ containing other $h$-pure envelopes of $N$.

**Theorem 19.** Let $N$ be a submodule of a separable module $M$ and $P \subseteq Q$, $h$-pure submodules of $M$ containing $N$. Then

(i) for every $k < \omega$, the natural embedding $P \rightarrow Q$ induces a monomorphism

$$f_k : \frac{Soc(H_k(P)) + Soc(N+H_{k+1}(P))}{Soc(N+H_{k+1}(P))} \rightarrow \frac{Soc(H_k(Q)) + Soc(N+H_{k+1}(Q))}{Soc(N+H_{k+1}(Q))},$$

(13)

(ii) the map $f_k$ is onto for every $k > \omega$ if and only if $P$ is an $h$-dense submodule of $Q$.

**Proof.** The maps $f_k$ send the coset $x \leftrightarrow Soc(N+H_{k+1}(P))$ upon the coset $x \leftrightarrow Soc(N+H_{k+1}(Q))$. Since $x \in H_k(P)$, there exists $y \in P$ such that $d(yR/xR) = k$ and $e(x) = 1$; therefore

$$f_k \left( \frac{Soc(H_k(P)) + Soc(N+H_{k+1}(P))}{Soc(N+H_{k+1}(P))} \right) = \frac{Soc(H_k(P)) + Soc(N+H_{k+1}(Q))}{Soc(N+H_{k+1}(Q))}.$$  

(14)

If $x + Soc(N+H_{k+1}(P)) \in ker f_k$, then $f_k(x + Soc(N+H_{k+1}(P))) = Soc(N+H_{k+1}(Q))$ or $x = u+z$, $u \in N$, $z \in H_k(Q)$; that is, there exists $v \in Q$ such that $d(vR/zR) = k+1$. Now $x-u = z \in N \cap H_{k+1}(Q) = H_{k+1}(P)$. Therefore $x \in N+H_{k+1}(P)$ or $x \in Soc(N+H_{k+1}(P))$ and $f_k$ is a monomorphism, which completes part (i).

(iii) Suppose $P$ is $h$-dense in $Q$, therefore $Q = P + H_k(Q)$ for $k < \omega$. For $x \in Soc(H_k(Q))$ such that $x \notin Soc(H_{k+1}(Q))$, there exists $y \in Q$ such that $d(yR/xR) = k$. Since $P$ is $h$-dense in $Q$, $x = u+z$, $u \in P, z \in H_k(Q), x \in Soc(H_k(P)) + Soc(N+H_{k+1}(Q))$, and $f_k$ is surjective.

Conversely, suppose each $f_k$ is surjective, therefore $Soc(H_k(Q)) \subseteq Soc(H_k(P)) + Soc(N+H_{k+1}(Q))$, for every $k < \omega$. Let $x \in Soc(H_k(Q))$. Now there exists $y \in Q$ such that $d(yR/xR) = k$. Now $x = z + (u+v)$ with $z \in Soc(H_k(P)), u \in N$, and $v \in Q$ such that $v \notin Q$ with $d(vR/zR) = k+1$. Also $u \in H_k(N)$ and $z + u \in H_k(P)$. We will prove that $H_k(Q) \subseteq P + H_k(Q)$ implying that $Q = P + H_k(Q)$. For $k = 1$, if $y \in Soc(Q)$, then $y \in P + H_1(Q)$, and if $y \in H_{k+1}(Q), x \in Soc(Q)$ and $x = z + u + v$ implies that $x-z = u-v \in P + H_1(Q)$. Therefore $y \in P + H_1(Q)$, and we are done.

Now we investigate the relation between Ulm-Kaplansky invariants of the submodules $N$ and their $h$-pure envelopes.
**Theorem 20.** Let \( P \) be an \( h \)-pure envelope of the submodule \( N \) of a separable module \( M \). Then

\[
\frac{\text{Soc} \left( P \cap H_k(M) \right)}{\text{Soc} \left( P \cap H_{k+1}(M) \right)} = \frac{\text{Soc} \left( N \cap H_k(P) \right)}{\text{Soc} \left( N \cap H_{k+1}(P) \right)} \oplus \frac{\text{Soc} \left( H_{k+1}(P) + (N \cap H_k(P)) \right)}{\text{Soc} \left( H_{k+1}(P) + (N \cap H_k(P)) \right)}.
\]

\[\text{(15)}\]

**Proof.** We have

\[
\frac{\text{Soc} \left( N \cap H_k(P) \right)}{\text{Soc} \left( N \cap H_{k+1}(P) \right)} = \frac{H_k(P) \cap \text{Soc}(N)}{H_{k+1}(P) \cap \text{Soc}(N)} = \frac{(\text{Soc}(H_k(P)) \cap N) + \text{Soc}(H_{k+1}(P))}{\text{Soc}(H_{k+1}(P))}.
\]

Using the properties of \( h \)-pure envelopes, we can simplify this to

\[
\frac{\text{Soc}(H_k(P)) \cap (N \cap \text{Soc}(H_k(P))) + H_{k+1}(P)}{\text{Soc}(H_{k+1}(P))} = 0,
\]

\[\text{(16)}\]

Therefore,

\[
\text{Soc}(H_k(P)) = \text{Soc}(N \cap \text{Soc}(H_k(P))) + H_{k+1}(P),
\]

\[\text{(17)}\]

and we are able to write

\[
\frac{\text{Soc} \left( (N \cap H_k(P)) + H_{k+1}(P) \right)}{\text{Soc}(N \cap H_k(P)) + \text{Soc}(H_{k+1}(P))} = \frac{\text{Soc}(H_k(P))}{\text{Soc}(H_k(P)) \cap (N \cap \text{Soc}(H_k(P)))}.
\]

\[\text{(18)}\]

This implies that

\[
\frac{\text{Soc} \left( P \cap H_k(M) \right)}{\text{Soc} \left( P \cap H_{k+1}(M) \right)} = \frac{\text{Soc}(H_k(P))}{\text{Soc}(H_{k+1}(P))}.
\]

\[\text{(19)}\]

**Corollary 21.** Every \( h \)-pure envelope of the submodule \( N \) in a separable module \( M \) has same Ulm-Kaplansky invariants. Therefore \( h \)-pure envelopes of \( N \) have isomorphic basic submodules.

**Corollary 22.** Let \( M \) be the direct sum of uniserial modules. Then all \( h \)-pure envelopes of a submodule \( N \subseteq M \) are isomorphic, if they exist.

**Proposition 23.** If \( N \) is an \( h \)-pure submodule of a separable module \( M \) such that its closure \( \overline{N} \) is not \( h \)-pure in \( M \), then \( \overline{N} \) has no \( h \)-pure envelope in \( M \).

**Proof.** An \( h \)-pure envelope \( Q \) of \( \overline{N} \) must have a larger basic submodule than \( N \), and any uniserial summand in this larger basic submodule not in \( \overline{N} \) contradicts the Proposition 15, and the result follows.

**Theorem 24.** Every submodule of a separable module \( M \) admits \( h \)-pure envelopes if and only if \( M \) is quasicomplete; that is, the closure of every \( h \)-pure submodule of \( M \) is \( h \)-pure in \( M \).

**Proof.** Suppose on the contrary that \( M \) is not quasicomplete. Thus it contains an \( h \)-pure submodule \( N \) such that \( \overline{N} \) is not \( h \)-pure in \( M \). By Proposition 23, the submodule \( \overline{N} \subseteq M \) does not have an \( h \)-pure envelope. This contradiction proves that \( M \) is quasicomplete.

For the converse, consider a submodule \( N \) of a quasicomplete module \( M \). We construct a countable sequence of subsocles of \( N, 0 \subseteq N_0 \subseteq N_1 \subseteq \cdots \subseteq N_k \subseteq \cdots \) with another sequence, \( 0 \subseteq K_0 \subseteq K_1 \subseteq \cdots \subseteq K_k \subseteq \cdots \) of \( h \)-pure submodules of \( M \) such that \( N_0 \subseteq K_1 \) and \( H_i(K_i) = 0 \), for every \( i \). Now we may say that \( K_{k+1} \) is the maximal submodule such that \( N_k \subseteq K_{k+1} \subseteq \text{Soc}(N) \) and \( K_k \cap H_{k+1}(M) = 0 \). Put \( T_k = N_{k+1} \cap H_k(M) \) such that the nonzero elements of \( T_k \) are of height \( k \) and \( N_{k+1} = N_k + T_k \). Now \( T_k \cap N_k = 0 \), and \( T_k \cap K_k \) is bounded by \( k \); therefore there exists a minimal \( h \)-pure submodule containing \( T_k \cap K_k \), which is \( K_{k+1} \) here. Now \( K_{k+1} \) is also bounded, hence a uniserial in \( M \) and by Proposition 16, \( H_k(K_{k+1}) = T_k \). By the same argument, we get \( \mathcal{K'} = \bigcup_{k \in \omega} K_k \), an \( h \)-pure submodule of \( M \).

\[
\mathcal{N}' = \bigcup_{K \in \mathcal{N}} T_k = \bigoplus_{K \in \mathcal{N}} T_k
\]

is an \( h \)-dense subsocle of \( N \), and we have to show that \( K' \) does not have a uniserial summand of decomposition length \( k \), which is disjoint from \( (N + H_k(K')) \). Suppose \( xR \) is a uniserial summand of \( K' \) such that \( d(xR) = k \) and \( xR \cap (N + H_k(K')) = 0 \). Let \( m \) be the least positive integer such that \( xR \cap (N + H_m(K')) \) is \( 0 \), \( x \in K_{m+1} \). Since \( xR \cap (T_m + K_m + H_k(K_{m+1})) = 0 \) and there are elements \( u \in T_m, y \in K_m \cap xR \subseteq \text{Soc}(xR), v, v' \in K_{m+1} \) such that \( d(v' R/vR) = k, x' = u + y + v \). Here \( H(u) = m \), \( y \neq 0 \) because \( xR \cap (N + H_k(K_{m+1})) = 0 \). Therefore \( y \in K_m \) implies that \( H(y) \leq m - 1 \). As \( H(x') = k - 1 \), we have \( k - 1 = m - 1 \); therefore there exist \( z, z' \in K_{m+1} \) such that \( d(zR/z'R) = k, uR = z'R \). Also there are elements \( u, w' \in K_{m+1} \) such that \( d(w'R/w'R) = k - 1 \) and \( w'R = yR \). Since \( K_{m+1} \) is \( h \)-pure, there is a uniserial summand \( R \subseteq K_n \) and \( d(tR/yR) = k - 1 \) and \( tR \cap (N + H_k(K')) = 0 \), which is a contradiction implying that \( xR \cap (N + H_k(K')) \) is not \( h \)-pure in \( M \).
Case i. When $M$ is closed [8] with respect to $h$-topology [4], by the structure of $M$, $K = \overline{K}$ is a summand of $M$. Consider the decomposition $M = K \oplus L$. If $Q$ is a submodule of $M$ such that $\text{Soc}(Q) = \text{Soc}(K)$, then

$$\text{Soc}\left(\frac{N + L}{N}\right) = \frac{(N + \text{Soc}(L))}{N}. \quad (22)$$

If $x + N \in (N + \text{Soc}(L))/N$ has height $k$ in $M/N$, then $H_k(x) = k$ and $(N \oplus L)/N$ is $h$-pure in $M/N$. Thus,

$$M = \frac{Q \oplus (L \oplus N)}{N}. \quad (23)$$

and $M = Q \oplus L$, and $Q$ is an $h$-pure envelope of $N$ in the closed module $M$.

Case ii. When $M$ is an arbitrary QTAG module, again consider the decomposition $M = K \oplus L$. If $K$ is bounded, then we are done; otherwise $M/K$ is a closed module. By Case i, $(N + K)/K \subset M/K$ has an $h$-pure envelope $L/K$. Assume on the contrary that $xR$ is a summand of $L$ such that $d'(xR) = k$, $xR \cap (N + H_k(L)) = 0$. If $\text{Soc}(xR) \cap K = 0$, then $(xR + K)/K$ is a uniserial summand of $L/K$ and

$$\left(\frac{xR + K}{K}\right) \cap \left(\frac{N + K}{K} + H_k\left(\frac{L}{K}\right)\right) \neq 0, \quad (24)$$

implying that $xR \cap (N + H_k(L)) \not\subset N$ which is a contradiction. Therefore $\text{Soc}(xR) \subseteq N$, but then $N$ contains a summand $yR$ such that $\text{Soc}(yR) = \text{Soc}(xR)$ and $xR \cap (N + H_k(L)) = 0$ implies that $yR \cap (N + H_k(K)) = 0$, again a contradiction. Thus, no summand $xR$ exists, and $L$ is an $h$-pure envelope of $N$.

Now we investigate the conditions under which every submodule of a QTAG module $M$ has an $h$-pure envelope.

**Proposition 25.** $M^1$ has an $h$-pure envelope in $M$ if and only if it is $h$-divisible.

**Proof.** If $M^1$ is not $h$-divisible, then the basic submodule of any $h$-pure submodule $N$ of $M$ containing $M^1$ is nontrivial and it has a nonzero uniserial summand disjoint from $M^1$. By Proposition 15, $N$ cannot be an $h$-pure envelope of $M^1$. The converse is trivial.

The following theorem characterizes the module $M$ whose every submodule has an $h$-pure envelope.

**Theorem 26.** In a module $M$, every submodule has an $h$-pure envelope if and only if $M$ is the direct sum of an $h$-divisible and a quasicomplete module.

**Proof.** Suppose that every submodule $N$ of $M$ has an $h$-pure envelope. By Proposition 25, $M^1$ is $h$-divisible. Now $M = D \oplus K$, where $D$ is the maximal $h$-divisible submodule of $M$ and $K$ is separable. Let $N \subseteq K$ and $P$, the $h$-pure envelope of $N$ in $M$. Now if we project $P$ into $K$, then the projection of $P$ is an $h$-pure envelope of $N$ in $K$. Therefore, all the submodules of $K$ have $h$-pure envelopes and Theorem 24 implies that $K$ is a quasicomplete module.

Conversely, suppose $M = D \oplus K$, where $D$ is $h$-divisible and $K$ a quasicomplete module. For $N \subseteq M, D + N = D \oplus N_0$, where $N_0 \subseteq K$ may be chosen. By Theorem 24, $N_0$ has an $h$-pure envelope $P_0$ in $K$. Now there exists an $h$-divisible submodule $D_0 \subseteq D$ that contains $N \cap D$ as an essential submodule. If we put $P = P_0 \oplus N_0$, then $N \subseteq P$ which is $h$-pure in $M$. $P$ cannot have a uniserial or $h$-divisible summand disjoint from $N_0$, because such a summand would be disjoint from $N_0$ and $N \cap D$. Therefore, $P$ is an $h$-pure envelope of $N$. \qed

**References**


