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Research Article

Analytical Homoclinic Solution of a Two-Dimensional Nonlinear System of Differential Equations

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Analytical solution of the homoclinic orbit of a two-dimensional system of differential equations that describes the Hamiltonian part of the slow flow of a three-degree-of-freedom dissipative system of linear coupled oscillators with an essentially nonlinear attachment is described.

1. Introduction

A homoclinic orbit is the trajectory of a flow of dynamical system that joins a saddle equilibrium point to itself; that is, the homoclinic trajectory $h(t)$ converges to the equilibrium point as $t \to \pm\infty$ [1].

The analytical solutions of homoclinic orbits are very important for many applications as in the use of the homoclinic Melnikov function, in order to prove the existence of transversal homoclinic orbits and chaotic behavior.

In what follows, we find the analytical solution of the homoclinic orbit of a one-degree-of-freedom system of differential equations that describes the Hamiltonian part of the slow flow of a three-degree-of-freedom dissipative system of linear coupled oscillators with an essentially nonlinear attachment [2].

The aim of the work in [2] was to study the asymptotic behavior of the system. Specifically, the initial dissipative system composed of two linear and one nonlinear oscillators was reduced to a nonautonomous damped strongly nonlinear second-order differential equation. With the use of the complexification-averaging technique (CX-A), we obtained the slow flow of the system, that is, a system of two, first-order, differential equations governed by slow time

$$a' = -\frac{\epsilon\lambda}{2}a + \frac{b}{2} - \frac{3C}{8}\left(a^2+b^2\right)b - \frac{J}{2} + \frac{B}{2}\sin\left(\frac{\epsilon\overline{B}}{\omega_{20}}t\right),$$
$$b' = -\frac{\epsilon\lambda}{2}b - \frac{a}{2} + \frac{3C}{8}\left(a^2+b^2\right)a - \frac{A}{2} - \frac{B}{2}\cos\left(\frac{\epsilon\overline{B}}{\omega_{20}}t\right), \quad (1)$$

where a, b are the variables and A, B, J, ϵ, $\overline{B}$, ω_{20} are parameters.

From the study of the dynamics of the slow flow [3], we concluded that the slow flow may do regular or chaotic oscillations. The computation of the analytical solution of the homoclinic orbit of the unperturbed problem is the first step in order to investigate the chaotic behavior, of the above system, with the use of the homoclinic Melnikov function.

2. Main Results

The unperturbed part of the above system is

$$a' = \frac{b}{2} - \frac{3C}{8}\left(a^2+b^2\right)b - \frac{J}{2} + \frac{B}{2}\sin\gamma,$$
$$b' = -\frac{a}{2} + \frac{3C}{8}\left(a^2+b^2\right)a - \frac{A}{2} - \frac{B}{2}\cos\gamma, \quad (2)$$

and γ is a parameter. The equilibrium points are found considering $a' = 0$, $b' = 0$. After some simple algebraic manipulations, we have

$$a = \frac{A + B\cos\gamma}{-J + B\sin\gamma}b, \quad (3)$$

and the third-order equation

$$b - \frac{3C}{4}\frac{(A + B\cos\gamma)^2 + (J - B\sin\gamma)^2}{(J - B\sin\gamma)^2}b^3 - J + B\sin\gamma = 0. \tag{4}$$

When (4) has three real roots, we have three equilibrium points. It is well known that, in order (4) to have three real roots, it must hold that the determinant $q^2 + 4p^3 < 0$, where $p = (3ac - b^2)/(9a^2)$, $q = (2b^3 - 9abc + 27a^2d)/(27a^3)$, $a = -(3C/4)(((A + B\cos\gamma)^2 + (J - B\sin\gamma)^2)/(J - B\sin\gamma)^2)$, $b = 0$, $c = 1$, and $d = -J + B\sin\gamma$. This holds for

$$D < \frac{16}{81C}, \tag{5}$$

where $D = (A + B\cos\gamma)^2 + (J - B\sin\gamma)^2$.

The equilibrium points are

$$\begin{aligned}
a_1 &= -\frac{4(A + B\cos\gamma)}{3\sqrt{CD}}\cos\left(\frac{\varpi}{3}\right),\\
b_1 &= \frac{4(J - B\sin\gamma)}{3\sqrt{CD}}\cos\left(\frac{\varpi}{3}\right),\\
a_2 &= -\frac{4(A + B\cos\gamma)}{3\sqrt{CD}}\cos\left(\frac{\varpi}{3} + \frac{2\pi}{3}\right),\\
b_2 &= \frac{4(J - B\sin\gamma)}{3\sqrt{CD}}\cos\left(\frac{\varpi}{3} + \frac{2\pi}{3}\right),\\
a_3 &= -\frac{4(A + B\cos\gamma)}{3\sqrt{CD}}\cos\left(\frac{\varpi}{3} + \frac{4\pi}{3}\right),\\
b_3 &= \frac{4(J - B\sin\gamma)}{3\sqrt{CD}}\cos\left(\frac{\varpi}{3} + \frac{4\pi}{3}\right),
\end{aligned} \tag{6}$$

where $\varpi = (1/3)\cos^{-1}((-9\sqrt{CD})/4)$.

The Hamiltonian of system (2) is given by

$$\begin{aligned}
h = {} & \frac{a^2 + b^2}{4} - \frac{3C}{32}\left(a^2 + b^2\right)^2 + \frac{b(B\sin\gamma - J)}{2}\\
& + \frac{a}{2}(A + B\cos\gamma).
\end{aligned} \tag{7}$$

We perform the canonical transformation $a = \sqrt{2\rho}\cos\vartheta$, $b = \sqrt{2\rho}\sin\vartheta$, and the hamiltonian becomes

$$\begin{aligned}
h = {} & \frac{\rho}{2} - \frac{3C}{8}\rho^2 + \frac{\sqrt{2\rho}}{2}(\sin\vartheta(B\sin\gamma - J)\\
& + \cos\vartheta(A + B\cos\gamma)),
\end{aligned} \tag{8}$$

$$\rho' = \frac{\sqrt{2\rho}}{2}((B\sin\gamma - J)\cos\vartheta - \sin\vartheta(A + B\cos\gamma)). \tag{9}$$

From the square of (9) by adding in both sides of the equation the quantity $((\sqrt{2\rho}/2)((B\sin\gamma - J)\sin\vartheta + \cos\vartheta(A + B\cos\gamma)))^2$, we have

$$\begin{aligned}
(\rho')^2 + {} & \left(\frac{\sqrt{2\rho}}{2}((B\sin\gamma - J)\sin\vartheta + \cos\vartheta(A + B\cos\gamma))\right)^2\\
& = \frac{\rho}{2}D.
\end{aligned} \tag{10}$$

From the hamiltonian (8), (10) becomes

$$\begin{aligned}
(\rho')^2 = {} & \left\{\frac{\sqrt{\rho D}}{\sqrt{2}} - \left(h - \frac{\rho}{2} + \frac{3C}{8}\rho^2\right)\right\}\\
& \times \left\{\frac{\sqrt{\rho D}}{\sqrt{2}} + \left(h - \frac{\rho}{2} + \frac{3C}{8}\rho^2\right)\right\}.
\end{aligned} \tag{11}$$

We denote by ρ^*, ϑ^* the unstable equilibrium point ($(\rho^*)' = 0$), and (10) becomes

$$((B\sin\gamma - J)\sin\vartheta^* + \cos\vartheta^*(A + B\cos\gamma)) = \pm\sqrt{D}. \tag{12}$$

By substituting (for the case $-\sqrt{D}$) in the hamiltonian, we have

$$h = \frac{\rho^*}{2} - \frac{3C}{8}(\rho^*)^2 - \sqrt{\frac{\rho^* D}{2}}, \tag{13}$$

and (11) becomes

$$\begin{aligned}
(\rho')^2 = {} & \left\{\frac{\sqrt{\rho D}}{\sqrt{2}} - \left(\frac{\rho^*}{2} + \frac{3C}{8}(\rho^*)^2\right.\right.\\
& \left.\left. - \sqrt{\frac{\rho^* D}{2}} - \frac{\rho}{2} + \frac{3C}{8}\rho^2\right)\right\}\\
& \times \left\{\frac{\sqrt{\rho D}}{\sqrt{2}} + \left(\frac{\rho^*}{2} + \frac{3C}{8}(\rho^*)^2\right.\right.\\
& \left.\left. - \sqrt{\frac{\rho^* D}{2}} - \frac{\rho}{2} + \frac{3C}{8}\rho^2\right)\right\}.
\end{aligned} \tag{14}$$

For the right hand side of the previous equation after some simple algebra manipulations, we have

$$\begin{aligned}
&\left\{\frac{\sqrt{\rho D}}{\sqrt{2}}-\left(\frac{\rho^*}{2}+\frac{3C}{8}(\rho^*)^2-\sqrt{\frac{\rho^* D}{2}}-\frac{\rho}{2}+\frac{3C}{8}\rho^2\right)\right\}\\
&\quad\times\left\{\frac{\sqrt{\rho D}}{\sqrt{2}}+\left(\frac{\rho^*}{2}+\frac{3C}{8}(\rho^*)^2\right.\right.\\
&\qquad\left.\left.-\sqrt{\frac{\rho^* D}{2}}-\frac{\rho}{2}+\frac{3C}{8}\rho^2\right)\right\}\\
&\quad=(\rho-\rho^*)\left\{\sqrt{\frac{D}{2}}\left(\sqrt{\frac{D}{2}}-\sqrt{\rho^*}+\frac{3C}{2}\left(\sqrt{\rho^*}\right)^3\right.\right.\\
&\qquad\left.+\frac{3C}{4}\sqrt{\rho^*}\rho-\frac{3C}{4}\sqrt{\rho^*}\rho^*\right)\\
&\qquad-\frac{\rho-\rho^*}{4}+\frac{3C}{8}(\rho-\rho^*)(\rho+\rho^*)\\
&\qquad\left.-\frac{9C^2}{64}(\rho-\rho^*)(\rho+\rho^*)^2\right\}.
\end{aligned}\tag{15}$$

We calculate $ba'-ab'$ and derive

$$\begin{aligned}
-2\vartheta'\rho&=\rho-\frac{3C}{2}\rho^2\\
&\quad+\sqrt{\frac{\rho}{2}}\{\sin\vartheta(B\sin\gamma-J)+\cos\vartheta(A+B\cos\gamma)\},
\end{aligned}\tag{16}$$

and for the equilibrium point $((\vartheta^*)'=0)$ we have

$$\begin{aligned}
&\rho^*-\frac{3C}{2}\rho^{*2}+\sqrt{\frac{\rho^*}{2}}\{\sin\vartheta^*(B\sin\gamma-J)\\
&\qquad+\cos\vartheta^*(A+B\cos\gamma)\}=0.
\end{aligned}\tag{17}$$

Using (12) and the above equality, (15) becomes

$$\begin{aligned}
&\left\{\frac{\sqrt{\rho D}}{\sqrt{2}}-\left(\frac{\rho^*}{2}+\frac{3C}{8}(\rho^*)^2-\sqrt{\frac{\rho^* D}{2}}-\frac{\rho}{2}+\frac{3C}{8}\rho^2\right)\right\}\\
&\times\left\{\frac{\sqrt{\rho D}}{\sqrt{2}}+\left(\frac{\rho^*}{2}+\frac{3C}{8}(\rho^*)^2-\sqrt{\frac{\rho^* D}{2}}-\frac{\rho}{2}+\frac{3C}{8}\rho^2\right)\right\}\\
&=(\rho-\rho^*)^2\left\{\sqrt{\frac{D}{2}}\frac{3C}{4}\sqrt{\rho^*}-\frac{1}{4}+\frac{3C}{8}(\rho+\rho^*)\right.\\
&\qquad\left.-\frac{9C^2}{64}(\rho+\rho^*)^2\right\}.
\end{aligned}\tag{18}$$

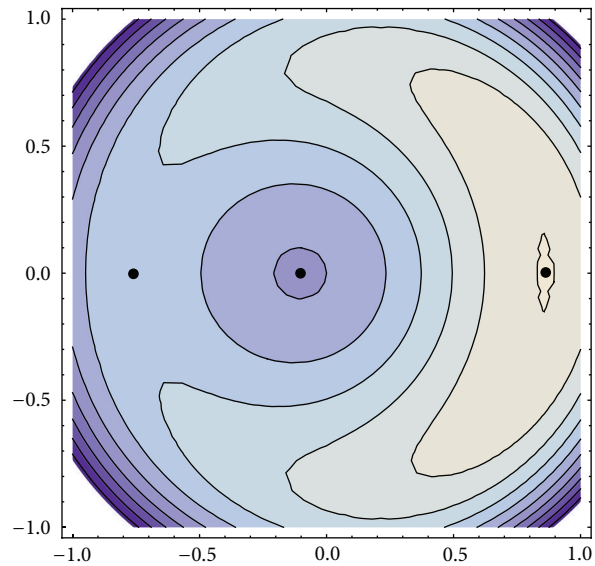

Figure 1: Phase space of the hamiltonian for $A=0.1$, $B=0.001$, $J=0.00001$, $\gamma=(\pi/9)$, and $C=2.0$.

Then from (14) and (18), we have

$$\begin{aligned}
&(d\rho)\left(|(\rho-\rho^*)|\left(\sqrt{(D/2)}(3C/4)\sqrt{\rho^*}\right.\right.\\
&\qquad-(1/4)+(3C/8)(\rho+\rho^*)\\
&\qquad\left.\left.-\left((9C^2)/64\right)(\rho+\rho^*)^2\right)^{1/2}\right)^{-1}=dt,
\end{aligned}\tag{19}$$

which is our main differential equation and is easily solved [4].

As it is seen in Figure 1 when our system has three equilibrium points, then depending on the parameters, we may have two homoclinic orbits. In our analysis, this result is given by the absolute value in (19).

For the case $\rho^*>\rho$, the homoclinic solution is

$$\begin{aligned}
\rho(t)&=\left(\left(128e^{Q(t)/2}+27C^3\sqrt{2D\rho^*}\right)\rho^*\right.\\
&\qquad-16e^{Q(t)/4}\left(3C\left(6\rho^*-3C\rho^{*2}+4\sqrt{2D\rho^*}\right)-8\right)\Big)\\
&\quad\times\left(128e^{Q(t)/2}+27C^3\sqrt{2D\rho^*}\right.\\
&\qquad\left.-48e^{Q(t)/4}(3C\rho^*-2)\right)^{-1},
\end{aligned}\tag{20}$$

where $Q(t)=t\sqrt{-4+3C(2\sqrt{2D\rho^*}+\rho^*(4-3C\rho^*))}$. After substituting the solution (20) in (16) and integrate we derive

$$\begin{aligned}
\vartheta&=W_1\tan^{-1}\left(g_1+g_2e^{Q(t)/4}\right)\\
&\quad+W_2\tan^{-1}\left(g_3+g_4e^{Q(t)/4}\right)+W_3t,
\end{aligned}\tag{21}$$

where

$$W_1 = \left(3C\left(8 - 6(1+3C)\rho^* + 9C(1+C)\rho^* - 12C\sqrt{2D\rho^*}\right)\right) \times \left(\sqrt{-9C^2\left(\rho^* - \frac{2}{3C}\right)^2 + 6C^3\sqrt{2D\rho^*}} \times \sqrt{-9C^2\left(\rho^2 - \frac{2}{3C}\right)^2 + 6C\sqrt{2D\rho^*}}\right)^{-1},$$

$$W_2 = \frac{h}{\rho^*\sqrt{-9C^2(\rho^* - (2/3C))^2 + 6C^3\sqrt{2D\rho^*}}} \times \left(6(3C-1)\rho^* - 9C(C-1)\rho^* + 12C\sqrt{2D\rho^*} - 8\right) \times \left(54C^3\rho^{*2}\sqrt{2D\rho^*} - \left(8 + 3C\left(3\rho^*(C\rho^* - 2) - 4\sqrt{2D\rho^*}\right)\right)^2\right)^{-1/2},$$

$$W_3 = \frac{9C\rho^{*2} - 16\rho^* - 8h}{64\rho^*},$$

$$g_1 = \frac{2 - 3C\rho^*}{\sqrt{-9C^2(\rho^* - (2/3C))^2 + 6C^3\sqrt{2D\rho^*}}},$$

$$g_2 = \frac{16}{3\sqrt{-9C^2(\rho^* - (2/3C))^2 + 6C^3\sqrt{2D\rho^*}}},$$

$$g_3 = \left(8 + 9C^2\rho^{*2} - 6C\left(3\rho^* + 2\sqrt{2D\rho^*}\right)\right) \times \left(54C^3\rho^{*2}\sqrt{2D\rho^*} - \left(8 + 3C\left(3\rho^*(C\rho^* - 2) - 4\sqrt{2D\rho^*}\right)\right)^2\right)^{-1/2},$$

$$g_4 = (16\rho^*) \times \left(54C^3\rho^{*2}\sqrt{2D\rho^*} - \left(8 + 3C\left(3\rho^*(C\rho^* - 2) - 4\sqrt{2D\rho^*}\right)\right)^2\right)^{-1/2}. \tag{22}$$

For the case $\rho > \rho^*$, the homoclinic solution is

$$\rho(t) = \left(\left(128e^{-Q(t)/2} + 27C^3\sqrt{2D\rho^*}\right)\rho^* - 16e^{-Q(t)/4}\left(3C\left(6\rho^* - 3C\rho^{*2} + 4\sqrt{2D\rho^*}\right) - 8\right)\right) \times \left(128e^{-Q(t)/2} + 27C^3\sqrt{2D\rho^*} - 48e^{-Q(t)/4}\left(3C\rho^* - 2\right)\right)^{-1},$$

$$\vartheta = W_3 t - W_1\tan^{-1}\left(g_1 + g_2 e^{-Q(t)/4}\right) - W_2\tan^{-1}\left(g_3 + g_4 e^{-Q(t)/4}\right). \tag{23}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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