Research Article

Hardy-Type Inequalities on Time Scale via Convexity in Several Variables

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We extend some Hardy-type inequalities with general kernels to arbitrary time scales using multivariable convex functions. Some classical and new inequalities are deduced seeking applications.

1. Introduction

The significant Hardy inequality is published in [1] (1952) (both in the continuous and discrete settings). More general Hardy integral inequalities have been studied in continuous cases. We notice only [2–6] and the references therein.

Recently, this inequality is studied in discrete case, and some variants of it are proved in case of time scales [7–9].

In [10], the authors study Hardy-type inequalities using convex functions of one variable with general kernels to arbitrary time scales.

The aim of this paper is to provide Hardy-type inequalities using multivariable convex function with general kernels to arbitrary time scales. Notice that time scales include continuous and discrete time cases under unified approach.

Firstly, we recall necessary preliminary facts needed afterward. The main results are given in Section 3. Section 4 is devoted to some inequalities with certain kernels. In the last section, we discuss some particular cases of Hardy-type inequalities.

2. Preliminaries

First, we recall the basic concepts used in the paper and refer the interested reader to [11] for the theory of time scales.

A time scale is any nonempty closed subset of the real line $\mathbb{T} \subset \mathbb{R}$. On nondensity points we define forward, respectively, backward jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ as

$$\begin{aligned}\sigma(t) &= \inf \{s \in \mathbb{T} : s > t\}, \\ \rho(t) &= \sup \{s \in \mathbb{T} : s < t\}.\end{aligned}\quad (1)$$

The point t is said to be right-scattered if $\sigma(t) > t$ and left-scattered if $\rho(t) < t$, respectively. Clearly, t is right-dense if $\sigma(t) = t$ and left dense if $\rho(t) = t$, respectively.

Let $n \in \mathbb{N}$; we define n -dimensional time scale by the Cartesian product of given time scales $\mathbb{T}_i$, $i \in \{1, \dots, n\}$, as

$$\Omega^n = \{\vec{a} = (a_1, a_2, \dots, a_n) : a_i \in \mathbb{T}_i, i \in \{1, \dots, n\}\}. \quad (2)$$

Evidently, Ω^n is a complete metric space with distance as follows:

$$d(\vec{a}, \vec{b}) = \left(\sum_{i=1}^n |b_i - a_i|^2 \right)^{1/2}, \quad a, b \in \Omega^n. \quad (3)$$

Now we are going to describe the construction of Lebesgue measure in Ω^n . We refer to [12–14] for the theory of measure spaces and measurable functions on time scales.

Let $\mathcal{F}$ be the family of all n -dimensional time scale intervals in Ω^n ; that is,

$$V = [\vec{a}, \vec{b}] = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n], \quad (4)$$

with $\vec{a} = (a_1, a_2, \dots, a_n), \vec{b} = (b_1, b_2, \dots, b_n) \in \Omega^n$, and $a_i \leq b_i$ for all $i \in \{1, \dots, n\}$. Let $m : \mathcal{F} \rightarrow [0, \infty)$ be the set function that assigns to each n -dimensional time scale interval $V = [a, b)$ its volume as follows:

$$m(V) = \prod_{i=1}^n (b_i - a_i). \quad (5)$$

Let $\emptyset \neq E \subset \Omega^n$. If there exists finite or countable system of pairwise disjoint n -dimensional time scale intervals $V_k = [\vec{a}_k, \vec{b}_k)$ with $E \subset \bigcup_{k \geq 1} V_k$, then the outer measure m^* of E is defined by

$$m^*(E) = \inf \left\{ \sum_{k \geq 1} m(V_k), E \subset \bigcup_{k \geq 1} V_k \right\}. \quad (6)$$

If there is no such covering of E , then $m^*(E) = \infty$.

A subset A of Ω^n is said to be measurable (or Δ -measurable) if

$$m^*(E) = m^*(E \cap A) + m^*(E \cap A^c) \quad (7)$$

holds for all $E \subset \Omega^n$, where $A^c = \Omega^n \setminus A$. The family $\mathcal{M}$ of all m^* -measurable subsets of Ω^n is a σ -algebra generated by $\mathcal{F}$. The restriction of m^* to $\mathcal{M}$, which we denote by μ_Δ , is a σ -additive measure on $\mathcal{M}$. Clearly, $\mathcal{F} \subset \mathcal{M}$ and $\mu_\Delta(V) = m(V)$ for each $V \in \mathcal{F}$. The measure μ_Δ (called the Lebesgue Δ -measure on Ω^n) is Carathéodory extension of the original measure m defined on $\mathcal{F}$. We call $(\Omega^n, \mathcal{M}, \mu_\Delta)$ an n -dimensional time scale measure space.

Denote $\tilde{a}_i = \inf_{t \in \mathbb{T}_i} t$. Similarly, $\tilde{b}_i = \sup_{t \in \mathbb{T}_i} t$. Let Ω_∞^n be the set of all points $\vec{b} = (b_1, b_2, \dots, b_n) \in \Omega^n$ for which there exists at least one b_i such that $b_i = \tilde{b}_i$. From [11, Theorem 3.1], we know that if $\vec{t} = (t_1, t_2, \dots, t_n) \in \Omega^n \setminus \Omega_\infty^n$, then the single-point set $\{t\}$ is Δ -measurable and

$$\mu_\Delta(\{t\}) = \prod_{i=1}^n (\sigma_i(t_i) - t_i). \quad (8)$$

Obviously, for all $i \in \{1, \dots, n\}$, the set $\Omega_0^n = \Omega^n \setminus \Omega_\infty^n$ can be represented as a finite or countable union of intervals of the family $\mathcal{F}$; hence it is Δ -measurable. Furthermore, the set $\Omega_\infty^n = \Omega^n \setminus \Omega_0^n$ is Δ -measurable being the difference of two Δ -measurable sets Ω^n and Ω_0^n , but Ω_∞^n does not have a finite or countable covering intervals of $\mathcal{F}$, that is, for any Δ -measurable subset A of Ω^n such that $A \cap \Omega_\infty^n \neq \emptyset$ has Δ -measure ∞ . In particular, if $a \in \mathbb{T}$, where $\mathbb{T}$ is an arbitrary time scale, then the set $[a, \infty) = \{t \in \mathbb{T} : a \leq t\}$ is Δ -measurable.

The function $f : \Omega^n \rightarrow \overline{\mathbb{R}} := [-\infty, \infty]$ is said to be Δ -measurable if for every $\alpha \in \mathbb{R}$, the set $f^{-1}([-\infty, \alpha)) = \{t = (t_1, \dots, t_n) \in \Omega^n : f(t) < \alpha\}$ is Δ -measurable. It is easy to see that f is Δ -measurable if and only if for each open set $G \subset \mathbb{R}$, the set $f^{-1}(G) = \{t \in \Omega^n : f(t) \in G\}$ is Δ -measurable. Moreover, if $f : \Omega^n \rightarrow \mathbb{R}$ is Δ -measurable and $\Phi : I \subset \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function, then $\Phi \circ f : \Omega^n \rightarrow \mathbb{R}$ is Δ -measurable.

Having the σ -additive measure μ_Δ on Ω^n , we possess the corresponding integration theory for functions $f : E \subset \Omega^n \rightarrow \mathbb{R}$, according to the general Lebesgue integration

theory (see, e.g., [15]). The Lebesgue integral associated with the measure μ_Δ on Ω^n is called the Lebesgue Δ -integral. For a Δ -measurable set $E \subset \Omega^n$ and a Δ -measurable function $f : E \rightarrow \mathbb{R}$, the corresponding Δ -integral of f over E will be denoted by

$$\int_E f(t_1, \dots, t_n) \Delta t_1 \cdots \Delta t_n \quad \text{or} \quad \int_E f(t) \Delta t. \quad (9)$$

Notice that all theorems of the general Lebesgue integration theory, including the Lebesgue dominated convergence theorem, hold also for Lebesgue Δ -integrals on Ω^n .

If $\mathbb{T}$ is a time scale and the interval $[a, b) \subset \mathbb{T}$ contains only isolated points, then

$$\int_a^b f(t) \Delta t = \sum_{t \in [a, b)} (\sigma(t) - t) f(t). \quad (10)$$

Let $(\Omega, \mathcal{M}, \mu_\Delta)$ and $(\Lambda, \mathcal{L}, \lambda_\Delta)$ be two finite dimensional time scale measure spaces. We define the product measure space $(\Omega \times \Lambda, \mathcal{M} \times \mathcal{L}, \mu_\Delta \times \lambda_\Delta)$, where $\mathcal{M} \times \mathcal{L}$ is the product σ -algebra generated by $\{E \times F : E \in \mathcal{M}, F \in \mathcal{L}\}$ and

$$(\mu_\Delta \times \lambda_\Delta)(E \times F) = \mu_\Delta(E) \lambda_\Delta(F). \quad (11)$$

Fubini's Theorem in Time Scales see ([16, Theorem 1.1]). If $f : \Omega \times \Lambda \rightarrow \mathbb{R}$ is a $\mu_\Delta \times \lambda_\Delta$ -integrable function and if we define the function $\varphi(y) = \int_\Omega f(x, y) \Delta x$ for a.e. $y \in \Lambda$ and $\psi(x) = \int_\Lambda f(x, y) \Delta y$ for a.e. $x \in \Omega$, then φ is λ_Δ -integrable on Λ , ψ is μ_Δ -integrable on Ω , and

$$\int_\Omega \Delta x \int_\Lambda f(x, y) \Delta y = \int_\Lambda \Delta y \int_\Omega f(x, y) \Delta x. \quad (12)$$

3. Inequalities with General Kernels

Let $\mathbf{f}(y) = (f_1(y), \dots, f_m(y))$ be m -tuple of functions such that $f_j(y)$ are λ_Δ -integrable for all $j \in \{1, \dots, m\}$. Then $\int_\Lambda \mathbf{f}(y) \Delta(y)$ denotes the m -tuple $(\int_\Lambda f_1(y) \Delta(y), \dots, \int_\Lambda f_m(y) \Delta(y))$; that is, Δ -integral acts on each component of $\mathbf{f}(y)$.

The following Jensen's inequality on time scales is given in [17, Theorem 2.1].

Theorem 1. *Let $(\Omega, \mathcal{M}, \mu_\Delta)$ and $(\Lambda, \mathcal{L}, \lambda_\Delta)$ be two time scale measure spaces. Suppose $U \subset \mathbb{R}^m$ is a closed convex set, $\Phi \in C(U, \mathbb{R})$ is convex, and $\mathbf{f}(\Lambda) \subset U$. Moreover, let $k : \Omega \times \Lambda \rightarrow \mathbb{R}$ be nonnegative such that $k(x, \cdot)$ is λ_Δ -integrable. Then one has*

$$\Phi \left(\frac{\int_\Lambda k(x, y) \mathbf{f}(y) \Delta y}{\int_\Lambda k(x, y) \Delta y} \right) \leq \frac{\int_\Lambda k(x, y) \Phi(\mathbf{f}(y)) \Delta y}{\int_\Lambda k(x, y) \Delta y}. \quad (13)$$

Throughout this section, we assume that the following hypotheses hold.

H1 $(\Omega, \mathcal{M}, \mu_\Delta)$ and $(\Lambda, \mathcal{L}, \lambda_\Delta)$ are two time scale measure spaces.

H2 $k : \Omega \times \Lambda \rightarrow \mathbb{R}_+$ is such that $K(x) := \int_\Lambda k(x, y) \Delta y < \infty, x \in \Omega$.

H3 $\xi : \Omega \rightarrow \mathbb{R}_+$ is such that $w(y) := \int_\Omega (k(x, y) \xi(x) / K(x)) \Delta x, y \in \Lambda$.

Theorem 2. If $U \subset \mathbb{R}^m$ is a closed convex set such that $\Phi : U \rightarrow \mathbb{R}$ is convex and continuous, then

$$\begin{aligned} & \int_{\Omega} \xi(x) \Phi \left(\frac{1}{K(x)} \int_{\Lambda} k(x, y) \mathbf{f}(y) \Delta y \right) \Delta x \\ & \leq \int_{\Lambda} w(y) \Phi(\mathbf{f}(y)) \Delta y \end{aligned} \quad (14)$$

holds for all λ_{Δ} -integrable functions $\mathbf{f} : \Lambda \rightarrow \mathbb{R}^m$ such that $\mathbf{f}(\Lambda) \subset U$.

Proof. Using Jensen's inequality (13) for several variables and the Fubini theorem on time scales, we find that

$$\begin{aligned} & \int_{\Omega} \xi(x) \Phi \left(\frac{1}{K(x)} \int_{\Lambda} k(x, y) \mathbf{f}(y) \Delta y \right) \Delta x \\ & = \int_{\Omega} \xi(x) \Phi \left(\frac{1}{K(x)} \int_{\Lambda} k(x, y) f_1(y) \Delta y, \dots, \frac{1}{K(x)} \right. \\ & \quad \left. \times \int_{\Lambda} k(x, y) f_m(y) \Delta y \right) \Delta x \\ & \leq \int_{\Omega} \frac{\xi(x)}{K(x)} \left(\int_{\Lambda} k(x, y) \Phi(\mathbf{f}(y)) \Delta y \right) \Delta x \\ & = \int_{\Lambda} \Phi(\mathbf{f}(y)) \left(\int_{\Omega} \frac{k(x, y) \xi(x)}{K(x)} \Delta x \right) \Delta y \\ & = \int_{\Lambda} w(y) \Phi(\mathbf{f}(y)) \Delta y. \end{aligned} \quad (15)$$

The proof is therefore complete. $\square$

Remark 3. If Φ is concave, then (14) holds in reverse direction.

Corollary 4. Let $\Psi : [l_1, \hat{l}_1) \times \dots \times [l_m, \hat{l}_m) \rightarrow \mathbb{R}_+$ be continuous function and define

$$\tilde{L}_j(f_j, \Lambda) = L_j^{-1} \left(\frac{\int_{\Lambda} k(x, y) L_j(f_j(y)) \Delta y}{K(x)} \right) \quad (16)$$

for all $j \in \{1, \dots, m\}$. If $\Phi(s_1, \dots, s_m) = \Psi(L_1^{-1}(s_1), \dots, L_m^{-1}(s_m))$ is convex, then

$$\begin{aligned} & \int_{\Omega} \xi(x) \Psi(\tilde{L}_1(f_1, \Lambda), \dots, \tilde{L}_m(f_m, \Lambda)) \Delta x \\ & \leq \int_{\Lambda} w(y) \Psi(f_1(y), \dots, f_m(y)) \Delta y \end{aligned} \quad (17)$$

holds for all $f_j(\Lambda) \subset [l_j, \hat{l}_j)$ and continuous monotone functions $L_j : [l_j, \hat{l}_j) \rightarrow \mathbb{R}$ such that $L_j \circ f_j$ are λ_{Δ} -integrable for all $j \in \{1, \dots, m\}$.

Proof. Replace in Theorem 2 $f_j(y)$ with $L_j(f_j(y))$ for all $j \in \{1, \dots, m\}$ and $\Phi(s_1, \dots, s_m)$ with $\Psi(L_1^{-1}(s_1), \dots, L_m^{-1}(s_m))$. $\square$

Remark 5. In case that $\mathbb{T} = \mathbb{N}$ and $m = 2$, Corollary 4 is as [18, Corollary 1.2].

Remark 6. In case that $m = 2$, we can use the results of Beck [19] (see also [20], page 194) as applications of Corollary 4, which corresponds to the generalizations of Hölder's and Minkoski's inequalities. In classical case, many authors have studied these types of generalizations; see, for example, [21–23].

Furthermore, in the paper, we use $p > 1$ with $(1/p) + (1/\hat{p}) = 1$.

Corollary 7. If $\mathbf{f}(x) = (f_1(x), f_2(x))$, then

$$\begin{aligned} & \int_{\Omega} \xi(x) \left(\frac{1}{K(x)} \int_{\Lambda} k(x, y) (f_1(y))^p \Delta y \right)^{1/p} \\ & \quad \times \left(\frac{1}{K(x)} \int_{\Lambda} k(x, y) (f_2(y))^{\hat{p}} \Delta y \right)^{1/\hat{p}} \Delta x \\ & \geq \int_{\Lambda} w(y) f_1(y) f_2(y) \Delta y \end{aligned} \quad (18)$$

holds for all λ_{Δ} -integrable $f_j : \Lambda \rightarrow \mathbb{R}_+$, where $j \in \{1, 2\}$.

Proof. Use $m = 2$, $\Psi(s_1, s_2) = s_1 s_2$, $L_1(t_1) = (t_1)^p$, $L_2(t_2) = (t_2)^{\hat{p}}$ in Corollary 4; then $\Phi(s_1, s_2) = s_1^{1/p} s_2^{1/\hat{p}}$ is concave in Theorem 2. $\square$

Corollary 8. If $\mathbf{f}(x) = (f_1(x), f_2(x))$, then

$$\begin{aligned} & \int_{\Omega} \xi(x) \left(\left(\frac{1}{K(x)} \int_{\Lambda} k(x, y) (f_1(y))^p \Delta y \right)^{1/p} \right. \\ & \quad \left. + \left(\frac{1}{K(x)} \int_{\Lambda} k(x, y) (f_2(y))^{\hat{p}} \Delta y \right)^{1/\hat{p}} \right)^p \Delta x \\ & \geq \int_{\Lambda} w(y) (f_1(y) + f_2(y))^p \Delta y \end{aligned} \quad (19)$$

holds for all λ_{Δ} -integrable $f_j : \Lambda \rightarrow \mathbb{R}_+$, where $j \in \{1, 2\}$.

Proof. Use $m = 2$, $\Psi(s_1, s_2) = (s_1 + s_2)^p$, $L_1(t_1) = (t_1)^p$, $L_2(t_2) = (t_2)^{\hat{p}}$ in Corollary 4; then $\Phi(s_1, s_2) = (s_1^{1/p} + s_2^{1/\hat{p}})^p$ is concave in Theorem 2. $\square$

Remark 9. If $p < 1$, then reverse inequalities hold in Corollaries 4, 7, and 8.

4. Inequalities with Special Kernels

Throughout this section and in the next section, we assume that the following hypothesis holds.

H4 $\Omega = \Lambda = [a_1, b_1]_{\mathbb{T}} \times [a_2, b_2]_{\mathbb{T}} \times \dots \times [a_n, b_n]_{\mathbb{T}}$, $0 \leq a_i < b_i \leq \infty$ for all $i \in \{1, \dots, n\}$, where $\mathbb{T}$ is an arbitrary time scale.

Corollary 10. Assume that

$\xi : \Omega \rightarrow \mathbb{R}_+$ is such that

$$w(y) = \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \frac{k(x_1, \dots, x_n, y) \xi(x_1, \dots, x_n)}{K(x_1, \dots, x_n)} \Delta x_1 \cdots \Delta x_n, \quad y \in \Lambda. \quad (20)$$

If $U \subset \mathbb{R}^m$ is a closed convex set such that $\Phi : U \rightarrow \mathbb{R}$ is convex and continuous; then

$$\begin{aligned} & \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \xi(x_1, \dots, x_n) \Phi((A_k \mathbf{f})(x_1, \dots, x_n)) \Delta x_1 \cdots \Delta x_n \\ & \leq \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} w(y_1, \dots, y_n) \Phi(\mathbf{f}(y_1, \dots, y_n)) \\ & \quad \times \Delta y_1 \cdots \Delta y_n \end{aligned} \quad (21)$$

holds for all λ_Δ -integrable $\mathbf{f} : \Lambda \rightarrow \mathbb{R}^m$ such that $\mathbf{f}(\Lambda) \subset U$, where

$$(A_k \mathbf{f})(x) = \frac{1}{K(x)} \int_{a_1}^{\sigma(x_1)} \cdots \int_{a_n}^{\sigma(x_n)} k(x, y_1, \dots, y_n) \times \mathbf{f}(y_1, \dots, y_n) \Delta y_1 \cdots \Delta y_n. \quad (22)$$

Proof. Statement follows from Theorem 2 using

$$k(x_1, \dots, x_n, y_1, \dots, y_n) = 0, \quad \text{if } a_i \leq y_i \leq \sigma(x_i) \quad (23)$$

for all $i \in \{1, \dots, n\}$,

since in this case

$$K(x) = \int_{a_1}^{\sigma(x_1)} \cdots \int_{a_n}^{\sigma(x_n)} k(x, y_1, \dots, y_n) \Delta y_1 \cdots \Delta y_n. \quad (24)$$

□

Corollary 11. Assume that

$\xi : \Omega \rightarrow \mathbb{R}_+$ is such that

$$w(y) = \int_{a_1}^{y_1} \cdots \int_{a_n}^{y_n} \frac{k(x_1, \dots, x_n, y) \xi(x_1, \dots, x_n)}{K(x_1, \dots, x_n)} \Delta x_1 \cdots \Delta x_n, \quad y \in \Lambda. \quad (25)$$

If $U \subset \mathbb{R}^m$ is a closed convex set such that $\Phi : U \rightarrow \mathbb{R}$ is convex and continuous, then (21) holds for all λ_Δ -integrable $\mathbf{f} : \Lambda \rightarrow \mathbb{R}^m$ such that $\mathbf{f}(\Lambda) \subset U$, and

$$(A_k \mathbf{f})(x) = \frac{1}{K(x)} \int_{\sigma(x_1)}^{b_1} \cdots \int_{\sigma(x_n)}^{b_n} k(x, y_1, \dots, y_n) \times \mathbf{f}(y_1, \dots, y_n) \Delta y_1 \cdots \Delta y_n. \quad (26)$$

Proof. Statement follows from Theorem 2 using

$$k(x_1, \dots, x_n, y_1, \dots, y_n) = 0, \quad \text{if } a_i \leq \sigma(x_i) \leq y_i \quad (27)$$

for all $i \in \{1, \dots, n\}$,

since in this case

$$K(x) = \int_{\sigma(x_1)}^{b_1} \cdots \int_{\sigma(x_n)}^{b_n} k(x, y_1, \dots, y_n) \Delta y_1 \cdots \Delta y_n. \quad (28)$$

□

Theorem 12. Assume that

$\xi : \Omega \rightarrow \mathbb{R}_+$ is such that

$$\bar{w}(y) = \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \frac{\xi(x_1, \dots, x_n)}{\prod_{i=1}^n (\sigma(x_i) - a_i)} \Delta x_1 \cdots \Delta x_n, \quad y \in \Lambda. \quad (29)$$

If $U \subset \mathbb{R}^m$ is a closed convex set such that $\Phi : U \rightarrow \mathbb{R}$ is convex and continuous, then

$$\begin{aligned} & \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \xi(x_1, \dots, x_n) \Phi((\bar{A}_k \mathbf{f})(x_1, \dots, x_n)) \Delta x_1 \cdots \Delta x_n \\ & \leq \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \bar{w}(y_1, \dots, y_n) \Phi(\mathbf{f}(y_1, \dots, y_n)) \\ & \quad \times \Delta y_1 \cdots \Delta y_n \end{aligned} \quad (30)$$

holds for all λ_Δ -integrable $\mathbf{f} : \Lambda \rightarrow \mathbb{R}^m$ such that $\mathbf{f}(\Lambda) \subset U$, where

$$\begin{aligned} (\bar{A}_k \mathbf{f})(x_1, \dots, x_n) &= \frac{1}{\prod_{i=1}^n (\sigma(x_i) - a_i)} \\ & \times \int_{a_1}^{\sigma(x_1)} \cdots \int_{a_n}^{\sigma(x_n)} \bar{\mathbf{f}}(y_1, \dots, y_n) \\ & \quad \times \Delta y_1 \cdots \Delta y_n. \end{aligned} \quad (31)$$

Proof. Statement follows from Theorem 2 using

$$k(x_1, \dots, x_n, y_1, \dots, y_n) = \begin{cases} 1 & \text{if } a_i \leq y_i < \sigma(x_i) \leq b_i, \quad i \in \{1, \dots, n\}, \\ 0 & \text{otherwise,} \end{cases} \quad (32)$$

since in this case

$$\begin{aligned} K(x_1, \dots, x_n) &= \int_{a_1}^{\sigma(x_1)} \cdots \int_{a_n}^{\sigma(x_n)} \Delta y_1 \cdots \Delta y_n \\ &= \prod_{i=1}^n (\sigma(x_i) - a_i). \end{aligned} \quad (33)$$

Thus $A_k = \bar{A}_k$, $w = \bar{w}$. □

Corollary 13. If $a_i = 0$ for all $i \in \{1, \dots, n\}$ in **H4** and $U \subset \mathbb{R}^m$ is a closed convex set such that $\Phi : U \rightarrow \mathbb{R}$ is convex and continuous, then

$$\begin{aligned} & \int_0^{b_1} \cdots \int_0^{b_n} \Phi((A_k \mathbf{f})(x_1, \dots, x_n)) \frac{\Delta x_1 \cdots \Delta x_n}{x_1 \cdots x_n} \\ & \leq \int_0^{b_1} \cdots \int_0^{b_n} \prod_{i=1}^n \left(\frac{1}{y_i} - \frac{1}{b_i} \right) \Phi(\mathbf{f}(y_1, \dots, y_n)) \Delta y_1 \cdots \Delta y_n \end{aligned} \quad (34)$$

holds for all λ_Δ -integrable $\mathbf{f} : \Lambda \rightarrow \mathbb{R}^m$ such that $\mathbf{f}(\Lambda) \subset U$, where

$$\begin{aligned} (A_k \mathbf{f})(x_1, \dots, x_n) &:= \frac{1}{\prod_{i=1}^n \sigma(x_i)} \\ &\times \int_0^{\sigma(x_1)} \cdots \int_0^{\sigma(x_n)} \mathbf{f}(y_1, \dots, y_n) \\ &\times \Delta y_1 \cdots \Delta y_n. \end{aligned} \quad (35)$$

Proof. The statement follows from Theorem 12 using $\xi(x_1, \dots, x_n) = 1/x_1 \cdots x_n$, since in this case

$$\begin{aligned} w(y_1, \dots, y_n) &= \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \frac{1}{\prod_{i=1}^n x_i \sigma(x_i)} \Delta x_1 \cdots \Delta x_n \\ &= \prod_{i=1}^n \left(\frac{1}{y_i} - \frac{1}{b_i} \right). \end{aligned} \quad (36)$$

□

Example 14. If $b_i = \infty$ for all $i \in \{1, \dots, n\}$ in addition to the assumptions of Corollary 13, then (34) takes the form

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty \Phi((A_k \mathbf{f})(x_1, \dots, x_n)) \frac{\Delta x_1 \cdots \Delta x_n}{x_1 \cdots x_n} \\ & \leq \int_0^\infty \cdots \int_0^\infty \Phi(\mathbf{f}(y_1, \dots, y_n)) \frac{\Delta y_1 \cdots \Delta y_n}{y_1 \cdots y_n}. \end{aligned} \quad (37)$$

Remark 15. Clearly, if the left-hand side is ∞ in (34), then right-hand side is also ∞ .

Remark 16. For $\mathbb{T} = \mathbb{R}$ and $m = 1$, inequality (37) is proved in [4, 24].

5. Some Particular Cases

In this section, firstly, we give Hilbert-type inequality on time scales.

Theorem 17. If $n = 1$, $a_1 = 0$, and $b_1 = \infty$ in **H4** and for $q > 1$ one defines

$$\begin{aligned} K_1(x) &= \int_0^\infty \frac{(y/x)^{-1/q}}{x+y} \Delta y, \\ K_2(y) &= \int_0^\infty \frac{(y/x)^{1-1/q}}{x+y} \Delta x, \end{aligned} \quad (38)$$

then

$$\begin{aligned} & \int_0^\infty \left(\int_0^\infty \frac{(y/x)^{1-1/q} (g_1(y))^p}{x+y} \Delta y \right)^{1/p} \\ & \times \left(\int_0^\infty \frac{(y/x)^{1-1/q} (g_2(y))^{p'}}{x+y} \Delta y \right)^{1/p'} \Delta x \\ & \geq \int_0^\infty K_2(y) g_1(y) g_2(y) \Delta y \end{aligned} \quad (39)$$

holds for all λ_Δ -integrable $g_j : \Lambda \rightarrow \mathbb{R}_+$, where $j \in \{1, 2\}$.

Proof. Use $\xi(x) = K_1(x)/x$ and $k(x, y) = \begin{cases} (y/x)^{-1/q}/(x+y), & x \neq 0, y \neq 0, x+y \neq 0, \\ 0, & \text{otherwise} \end{cases}$ in Corollary 7 to obtain

$$\begin{aligned} & \int_0^\infty \frac{K_1(x)}{x} \left(\frac{1}{K_1(x)} \int_0^\infty \frac{(y/x)^{-1/q} (f_1(y))^p}{x+y} \Delta y \right)^{1/p} \\ & \times \left(\frac{1}{K_1(x)} \int_0^\infty \frac{(y/x)^{-1/q} (f_2(y))^{p'}}{x+y} \Delta y \right)^{1/p'} \Delta x \\ & \geq \int_0^\infty w(y) f_1(y) f_2(y) \Delta y, \end{aligned} \quad (40)$$

where

$$\begin{aligned} w(y) &= \int_0^\infty \frac{k(x, y) \xi(x)}{K_1(x)} \Delta x = \int_0^\infty \frac{k(x, y)}{x} \Delta x \\ &= \frac{1}{y} \int_0^\infty \frac{(y/x)^{1-1/q}}{x+y} \Delta x = \frac{K_2(y)}{y}. \end{aligned} \quad (41)$$

Using this value in (40), we obtain

$$\begin{aligned} & \int_0^\infty \left(\int_0^\infty \frac{(y/x)^{-1/q} (f_1(y))^p}{x+y} \Delta y \right)^{1/p} \\ & \times \left(\int_0^\infty \frac{(y/x)^{-1/q} (f_2(y))^{p'}}{x+y} \Delta y \right)^{1/p'} \frac{\Delta x}{x} \\ & \geq \int_0^\infty K_2(y) f_1(y) f_2(y) \frac{\Delta y}{y}. \end{aligned} \quad (42)$$

Now, if we replace $f_1(y)$ with $g_1(y)y^{1/p}$ and $f_2(y)$ with $g_2(y)y^{1/p'}$, we obtain (39). □

Another Proof of (42). Consider L.H.S of (40) and applying Hölder inequality on time scale [25, Theorem 6.2] and Fubini theorem on time scale [16, Theorem 1.1], we have

$$\begin{aligned}
& \int_0^\infty \frac{K_1(x)}{x} \left(\frac{1}{K_1(x)} \int_0^\infty \frac{(y/x)^{-1/q} (f_1(y))^p}{x+y} \Delta y \right)^{1/p} \\
& \times \left(\frac{1}{K_1(x)} \int_0^\infty \frac{(y/x)^{-1/q} (f_2(y))^{p'}}{x+y} \Delta y \right)^{1/p'} \Delta x \\
& \geq \int_0^\infty \left(\int_0^\infty \frac{(y/x)^{-1/q} f_1(y) f_2(y)}{x+y} \Delta y \right) \frac{\Delta x}{x} \quad (43) \\
& = \int_0^\infty f_1(y) f_2(y) \left(\int_0^\infty \frac{(y/x)^{-1/q} \Delta x}{x+y} \right) \Delta y \\
& = \int_0^\infty \frac{f_1(y) f_2(y)}{y} \left(\int_0^\infty \frac{(y/x)^{1-1/q} \Delta x}{x+y} \right) \Delta y \\
& = \int_0^\infty K_2(y) f_1(y) f_2(y) \frac{\Delta y}{y}.
\end{aligned}$$

Example 18. It is known that

$$\int_0^\infty \frac{(y/x)^{-1/q}}{x+y} dy = \int_0^\infty \frac{(y/x)^{1-1/q}}{x+y} dx = \frac{\pi}{\sin(\pi/q)}, \quad (44)$$

for all $x, y \in \mathbb{R}_+ = (0, \infty)$ with $q > 1$. If $\mathbb{T} = \mathbb{R}$; then from (39), we obtain

$$\begin{aligned}
& \int_0^\infty \left(\int_0^\infty \frac{(y/x)^{1-1/q} (f_1(y))^p}{x+y} dy \right)^{1/p} \\
& \times \left(\int_0^\infty \frac{(y/x)^{1-1/q} (f_2(y))^{p'}}{x+y} dy \right)^{1/p'} dx \quad (45) \\
& \geq \frac{\pi}{\sin(\pi/q)} \int_0^\infty f_1(y) f_2(y) dy.
\end{aligned}$$

In the rest of paper, we take $n = 1$, $a \geq 0$, $b = \infty$ in **H4**.

Theorem 19. If (38) is satisfied, then

$$\begin{aligned}
& \int_0^\infty \left(\left(\int_0^\infty \frac{(y/x)^{1-1/q} (f_1(y))^p}{x+y} \Delta y \right)^{1/p} \right. \\
& \left. + \left(\int_0^\infty \frac{(y/x)^{1-1/q} (f_2(y))^p}{x+y} \Delta y \right)^{1/p} \right)^p \Delta x \quad (46) \\
& \geq \int_0^\infty K_2(y) (f_1(y) + f_2(y))^p \Delta y
\end{aligned}$$

holds for all λ_Δ -integrable $f_j : \Lambda \rightarrow \mathbb{R}_+$, where $j \in \{1, 2\}$.

Proof. Use $\xi(x) = K_1(x)/x$ and $k(x, y) = \begin{cases} (y/x)^{-1/q}/(x+y), & x \neq 0, y \neq 0, x+y \neq 0, \\ 0, & \text{otherwise} \end{cases}$ in Corollary 7 to obtain

$$\begin{aligned}
& \int_0^\infty \left(\left(\int_0^\infty \frac{(y/x)^{-1/q} (f_1(y))^p}{x+y} \Delta y \right)^{1/p} \right. \\
& \left. + \left(\int_0^\infty \frac{(y/x)^{-1/q} (f_2(y))^p}{x+y} \Delta y \right)^{1/p} \right)^p \frac{\Delta x}{x} \quad (47) \\
& \geq \int_0^\infty K_2(y) (f_1(y) + f_2(y))^p \frac{\Delta y}{y}.
\end{aligned}$$

Now, if we replace $f_1(y)$ with $f_1(y)y^{1/p}$ and $f_2(y)$ with $f_2(y)y^{1/p}$, we obtain (46). $\square$

Remark 20. (a) We can give another proof of (47) using Minkowski's inequality on time scale [25, Theorem 7.2].

(b) If $p < 1$, then we have reverse inequalities.

Now we consider some generalizations of the Pólya-Knopp type inequalities.

Corollary 21. If (29) holds, and furthermore, if $U \subset \mathbb{R}^m$ is a closed convex set such that $\Phi : U \rightarrow \mathbb{R}$ is convex and continuous, then

$$\begin{aligned}
& \int_a^\infty \xi(x) \Phi \left(\frac{1}{\sigma(x) - a} \int_a^{\sigma(x)} \mathbf{f}(y) \Delta y \right) \Delta x \\
& \leq \int_a^\infty w(y) \Phi(\mathbf{f}(y)) \Delta y \quad (48)
\end{aligned}$$

holds for all λ_Δ -integrable $\mathbf{f} : \Lambda \rightarrow \mathbb{R}^m$ such that $\mathbf{f}(\Lambda) \subset U$.

Proof. The statement follows from Theorem 12 using $n = 1$. $\square$

Corollary 22. Assume (29). Then

$$\begin{aligned}
& \int_a^\infty \xi(x) \left(\frac{1}{\sigma(x) - a} \int_a^{\sigma(x)} (f_1(y))^p \Delta y \right)^{1/p} \\
& \times \left(\frac{1}{\sigma(x) - a} \int_a^{\sigma(x)} (f_1(y))^{p'} \Delta y \right)^{1/p'} \Delta x \quad (49) \\
& \geq \int_a^\infty w(y) f_1(y) f_2(y) \Delta y
\end{aligned}$$

holds for all λ_Δ -integrable $f_j : \Lambda \rightarrow \mathbb{R}_+$, where $j \in \{1, 2\}$.

Proof. Statement follows from Corollary 7 using $m = 2$. $\square$

Corollary 23. Assume (29). Then

$$\begin{aligned} \int_a^\infty \xi(x) & \left(\left(\frac{1}{\sigma(x) - a} \int_a^{\sigma(x)} (f_1(y))^p \Delta y \right)^{1/p} \right. \\ & \left. + \left(\frac{1}{\sigma(x) - a} \int_a^{\sigma(x)} (f_2(y))^p \Delta y \right)^{1/p} \right)^p \Delta x \quad (50) \\ & \geq \int_a^\infty w(y) (f_1(y) + f_2(y))^p \Delta y \end{aligned}$$

holds for all λ_Δ -integrable $f_j : \Lambda \rightarrow \mathbb{R}_+$, where $j \in \{1, 2\}$.

Proof. Statement follows from Corollary 8 using $m = 2$. $\square$

Example 24. When $\mathbb{T}$ consists of isolated points, then from Corollary 22, we have

$$\begin{aligned} \sum_{x \in [a, \infty)} \frac{\xi(x) (\sigma(x) - x)}{\sigma(x) - a} \\ \times \left(\sum_{y \in [a, \sigma(x))} (f_1(y))^p (\sigma(y) - y) \right)^{1/p} \\ \times \left(\sum_{y \in [a, \sigma(x))} (f_2(y))^{p'} (\sigma(y) - y) \right)^{1/p'} \\ \geq \sum_{y \in [a, \infty)} w(y) f_1(y) f_2(y) (\sigma(y) - y), \end{aligned} \quad (51)$$

where $w(y) = \sum_{x \in [y, \infty)} (\xi(x) (\sigma(x) - x) / (\sigma(x) - a))$.

Example 25. When $\mathbb{T}$ consists of isolated points, then from Corollary 23, we have

$$\begin{aligned} \sum_{x \in [a, \infty)} \frac{\xi(x) (\sigma(x) - x)}{\sigma(x) - a} \\ \times \left(\left(\sum_{y \in [a, \sigma(x))} (f_1(y))^p (\sigma(y) - y) \right)^{1/p} \right. \\ \left. + \left(\sum_{y \in [a, \sigma(x))} (f_2(y))^p (\sigma(y) - y) \right)^{1/p} \right)^p \\ \geq \sum_{y \in [a, \infty)} w(y) (f_1(y) + f_2(y))^p (\sigma(y) - y), \end{aligned} \quad (52)$$

where $w(y)$ is the same as in Example 24.

Example 26. For $\mathbb{T} = h\mathbb{N} = \{hn : n \in \mathbb{N}\}$ with $h > 0$, $a = 1$, and $\xi(x) = 1/\sigma(x)$, (51) takes the form

$$\begin{aligned} \sum_{n=1}^\infty \frac{1}{n(n+1)} & \left(\sum_{k=1}^n (f_1(kh))^p \right)^{1/p} \\ & \times \left(\sum_{k=1}^n (f_2(kh))^{p'} \right)^{1/p'} \\ & \geq \sum_{n=1}^\infty \frac{1}{n} f_1(nh) f_2(nh). \end{aligned} \quad (53)$$

Inequality (52) takes the form

$$\begin{aligned} \sum_{n=1}^\infty \frac{1}{n(n+1)} & \left(\left(\sum_{k=1}^n (f_1(kh))^p \right)^{1/p} \right. \\ & \left. + \left(\sum_{k=1}^n (f_2(kh))^p \right)^{1/p} \right)^p \\ & \geq \sum_{n=1}^\infty \frac{1}{n} (f_1(nh) + f_2(nh))^p. \end{aligned} \quad (54)$$

Example 27. For $\mathbb{T} = \mathbb{N}^2 = \{n^2 : n \in \mathbb{N}\}$, $a = 1$, and

$$\xi(x) = \frac{2(\sigma(x) - 1)}{(\sigma(x) - x)^2 (2\sqrt{x} + 3)}, \quad (55)$$

inequality (51) takes the form

$$\begin{aligned} \sum_{n=1}^\infty \frac{2}{(2n+1)(2n+3)} \\ \times \left(\sum_{k=1}^n (2k+1) (f_1(k^2))^p \right)^{1/p} \\ \times \left(\sum_{k=1}^n (2k+1) (f_2(k^2))^{p'} \right)^{1/p'} \\ \geq \sum_{k=1}^\infty f_1(k^2) f_2(k^2). \end{aligned} \quad (56)$$

Inequality (52) takes the form

$$\begin{aligned} \sum_{n=1}^\infty \frac{2}{(2n+1)(2n+3)} \\ \times \left(\left(\sum_{k=1}^n (2k+1) (f_1(k^2))^p \right)^{1/p} \right. \\ \left. + \left(\sum_{k=1}^n (2k+1) (f_2(k^2))^p \right)^{1/p} \right)^p \\ \geq \sum_{k=1}^\infty (f_1(k^2) + f_2(k^2))^p. \end{aligned} \quad (57)$$

If we take

$$\xi(x) = \frac{\sigma(x) - 1}{x\sigma(x)}, \quad (58)$$

Inequality (51) takes the form

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(2n+1)}{n^2(n+1)^2} \left(\sum_{k=1}^n (2k+1)(f_1(k^2))^p \right)^{1/p} \\ & \times \left(\sum_{k=1}^n (2k+1)(f_2(k^2))^{p'} \right)^{1/p'} \\ & \geq \sum_{k=1}^{\infty} \frac{(2k+1)}{k^2} f_1(k^2) f_2(k^2). \end{aligned} \quad (59)$$

Inequality (52) takes the form

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(2n+1)}{n^2(n+1)^2} \\ & \times \left(\left(\sum_{k=1}^n (2k+1)(f_1(k^2))^p \right)^{1/p} \right. \\ & \left. + \left(\sum_{k=1}^n (2k+1)(f_2(k^2))^p \right)^{1/p} \right)^p \\ & \geq \sum_{k=1}^{\infty} \frac{(2k+1)}{k^2} (f_1(k^2) + f_2(k^2))^p. \end{aligned} \quad (60)$$

By replacing $f_1(k^2) = (k^2/(2k+1))^{1/p} f_1(k^2)$, $f_2(k^2) = (k^2/(2k+1))^{1/p'} f_2(k^2)$ in (59) and in (60), we have

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(2n+1)}{n^2(n+1)^2} \left(\sum_{k=1}^n k^2 (f_1(k^2))^p \right)^{1/p} \\ & \times \left(\sum_{k=1}^n k^2 (f_2(k^2))^{p'} \right)^{1/p'} \\ & \geq \sum_{k=1}^{\infty} f_1(k^2) f_2(k^2), \\ & \sum_{n=1}^{\infty} \frac{(2n+1)}{n^2(n+1)^2} \left(\left(\sum_{k=1}^n k^2 (f_1(k^2))^p \right)^{1/p} \right. \\ & \left. + \left(\sum_{k=1}^n k^2 (f_2(k^2))^p \right)^{1/p} \right)^p \\ & \geq \sum_{k=1}^{\infty} (f_1(k^2) + f_2(k^2))^p, \end{aligned} \quad (61)$$

respectively.

Example 28. For $\mathbb{T} = q^{\mathbb{N}} = \{q^n : n \in \{\mathbb{N}\}, q > 1\}$, $a = q$ and

$$\xi(x) = \frac{\sigma(x) - a}{\sigma(x)(\sigma(x) - x)}, \quad (62)$$

The inequality (51) takes the form

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(q-1)}{q^n} \left(\sum_{k=1}^n q^{k-1} (f_1(q^k))^p \right)^{1/p} \\ & \times \left(\sum_{k=1}^n q^{k-1} (f_2(q^k))^{p'} \right)^{1/p'} \geq \sum_{n=1}^{\infty} f_1(q^n) f_2(q^n). \end{aligned} \quad (63)$$

Inequality (52) takes the form

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{(q-1)}{q^n} \\ & \times \left(\left(\sum_{k=1}^n q^{k-1} (f_1(q^k))^p \right)^{1/p} \right. \\ & \left. + \left(\sum_{k=1}^n q^{k-1} (f_2(q^k))^p \right)^{1/p} \right)^p \\ & \geq \sum_{n=1}^{\infty} (f_1(q^n) + f_2(q^n))^p. \end{aligned} \quad (64)$$

Remark 29. (a) In classical case for $h = 1$, inequalities (53) and (54) are the same as (1.7) and (1.9), also (63) and (64) are the same as (1.6) and (1.8) in [18, Corollary 1.3], respectively, while according to authors knowledge, (56), (57), (59), (60), and (61) are not existing in the literature.

(b) For $p < 1$, we get the reverse inequalities.

Remark 30. The results given in Section 5 can analogously be proved for $\Omega = \Lambda = [a_1, \infty)_{\mathbb{T}} \times \cdots \times [a_n, \infty)_{\mathbb{T}}$.

Remark 31. The results given in Corollaries 7 and 8 and in their given applications can also be obtained for finite value of $m > 2$ analogously.

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