Research Article

Some Properties of the Sequence Space $E_n^q(\mathcal{M}, p, s, \|\cdot, \dots, \cdot\|)$

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We introduce the sequence space $E_n^q(\mathcal{M}, p, s, \|\cdot, \dots, \cdot\|)$ defined by a Musielak-Orlicz function $\mathcal{M} = (M_k)$. We also study some topological properties and prove some inclusion relations involving this space.

1. Introduction and Preliminaries

The concept of 2-normed spaces was initially developed by Gähler [1] in the mid-1960s, while one can see that of n -normed spaces in Misiak [2]. Since then, many others have studied this concept and obtained various results; see Gunawan [3, 4] and Gunawan and Mashadi [5]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension d , where $d \geq n \geq 2$. A real valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four conditions:

- (1) $\|x_1, x_2, \dots, x_n\| = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent in X ;
- (2) $\|x_1, x_2, \dots, x_n\|$ is invariant under permutation;
- (3) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbb{K}$; and
- (4) $\|x + x', x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$

is called an n -norm on X , and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n -normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the n -norm $\|x_1, x_2, \dots, x_n\|_E =$ the volume of the n -dimensional parallelepiped spanned by the vectors $x_1, x_2, \dots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \dots, x_n\|_E = |\det(x_{ij})|, \quad (1)$$

where $x_i = (x_{i1}, x_{i2}, \dots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \dots, n$. Let $(X, \|\cdot, \dots, \cdot\|)$ be an n -normed space of dimension $d \geq n \geq 2$

and $\{a_1, a_2, \dots, a_n\}$ be linearly independent set in X . Then the following function $\|\cdot, \dots, \cdot\|_\infty$ on X^{n-1} defined by

$$\begin{aligned} \|x_1, x_2, \dots, x_n\|_\infty \\ = \max \{\|x_1, x_2, \dots, x_{n-1}, a_i\| : i = 1, 2, \dots, n\} \end{aligned} \quad (2)$$

defines an $(n-1)$ -norm on X with respect to $\{a_1, a_2, \dots, a_n\}$.

A sequence (x_k) in an n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k \rightarrow \infty} \|x_k - L, z_1, \dots, z_{n-1}\| = 0 \quad \text{for every } z_1, \dots, z_{n-1} \in X. \quad (3)$$

A sequence (x_k) in an n -normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to be Cauchy if

$$\lim_{\substack{k \rightarrow \infty \\ p \rightarrow \infty}} \|x_k - x_p, z_1, \dots, z_{n-1}\| = 0 \quad (4)$$

for every $z_1, \dots, z_{n-1} \in X$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n -norm. Any complete n -normed space is said to be n -Banach space.

An Orlicz function M is a function which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \rightarrow \infty$ as $x \rightarrow \infty$.

Lindenstrauss and Tzafriri [6] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$; then

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}, \quad (5)$$

which is called an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}. \quad (6)$$

It is shown in [6] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p ($p \geq 1$). The Δ_2 -condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$ and for $L > 1$. A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function; see [7, 8]. A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup \{ |v|u - (M_k) : u \geq 0 \}, \quad k = 1, 2, \dots, \quad (7)$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$\begin{aligned} t_{\mathcal{M}} &= \{ x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0 \}, \\ h_{\mathcal{M}} &= \{ x \in w : I_{\mathcal{M}}(cx) < \infty \forall c > 0 \}, \end{aligned} \quad (8)$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_{\mathcal{M}}. \quad (9)$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf \left\{ k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1 \right\} \quad (10)$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf \left\{ \frac{1}{k} (1 + I_{\mathcal{M}}(kx)) : k > 0 \right\}. \quad (11)$$

Let X be a linear metric space. A function $p : X \rightarrow \mathbb{R}$ is called paranorm if

- (1) $p(x) \geq 0$ for all $x \in X$,
- (2) $p(-x) = p(x)$ for all $x \in X$,
- (3) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$,
- (4) if (λ_n) is a sequence of scalars with $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$, and (x_n) is a sequence of vectors with $p(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$; then $p(\lambda_n x_n - \lambda x) \rightarrow 0$ as $n \rightarrow \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm, and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [9,

Theorem 10.4.2, p. 183]). For more details about sequence spaces, see [10–23] and the references therein.

Let $\sum_{k=0}^{\infty} a_k$ be an infinite series with sequence of partial sums (s_k) and q be any positive real number. The Euler transform (E, q) s of the sequence $s = (s_n)$ is defined by

$$E_n^q(s) = \frac{1}{(1+q)^n} \sum_{v=0}^n \binom{n}{v} q^{n-v} s_v. \quad (12)$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable (E, q) to the number s if

$$E_n^q(s) = \frac{1}{(1+q)^n} \sum_{v=0}^n \binom{n}{v} q^{n-v} s_v \rightarrow s \quad \text{as } n \rightarrow \infty \quad (13)$$

and is said to be absolutely summable (E, q) or summable $|E, q|$ if

$$\sum_k |E_k^q(s) - E_{k-1}^q(s)| < \infty. \quad (14)$$

Let $x = (x_n)$ be a sequence of scalars; we denote $N_n(x) = E_n^q(x) - E_{n-1}^q(x)$, where $E_n^q(x)$ is defined by (12). After applications of Abel's transform, we have

$$\begin{aligned} N_n(x) &= -\frac{1}{(1+q)^{n-1}} \sum_{k=0}^{n-2} x_{k+1} A_k \\ &\quad + \frac{s_{n-1} A_{n-1}}{(1+q)^{n-1}} + \frac{s_n}{(1+q)^n} - \frac{q^{n-1}}{(1+q)^n} s_0, \end{aligned} \quad (15)$$

where

$$A_k = \sum_{i=0}^k \left[\frac{q}{1+q} \binom{n}{i} - \binom{n-1}{i} \right] q^{n-i-1}. \quad (16)$$

Note that, for any sequences $x = (x_n)$, $y = (y_n)$ and scalar λ , we have $N_n(x + y) = N_n(x) + N_n(y)$ and $N_n(\lambda x) = \lambda N_n(x)$.

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. We define the following sequence space in the present paper:

$$\begin{aligned} E_n^q(\mathcal{M}, p, s, \|\cdot, \dots, \cdot\|) &= \left\{ x = (x_k) : \sum_{k=1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} < \infty, \right. \\ &\quad \left. s \geq 0, \text{ for some } \rho > 0 \right\}. \end{aligned} \quad (17)$$

If we take $p = (p_k) = 1$ for all $k \in \mathbb{N}$, we have

$$\begin{aligned} E_n^q(\mathcal{M}, s, \|\cdot, \dots, \cdot\|) &= \left\{ x = (x_k) : \sum_{k=1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right] < \infty, \right. \\ &\quad \left. s \geq 0, \text{ for some } \rho > 0 \right\}. \end{aligned} \quad (18)$$

If we take $s = 0$, we get

$$\begin{aligned} E_n^q(\mathcal{M}, p, \|\cdot, \dots, \cdot\|) \\ = \left\{ x = (x_k) : \sum_{k=1}^{\infty} \left[M_k \left(\left\| \frac{N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \right. \\ \left. < \infty, s \geq 0, \text{ for some } \rho > 0 \right\}. \end{aligned} \quad (19)$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$, then

$$|a_k + b_k|^{p_k} \leq K \{|a_k|^{p_k} + |b_k|^{p_k}\} \quad (20)$$

for all k and $a_k, b_k \in \mathbb{C}$. Also, $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In this paper, we study some topological properties and inclusion relations between the previously defined sequence spaces.

2. Main Results

Theorem 1. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers; the space $E_n^q(\mathcal{M}, p, s, \|\cdot, \dots, \cdot\|)$ is linear over the complex field $\mathbb{C}$.

Proof. Let $x = (x_k)$, $y = (y_k) \in E_n^q(\mathcal{M}, p, s, \|\cdot, \dots, \cdot\|)$, and $\alpha, \beta \in \mathbb{C}$. Then, there exist positive integers ρ_1 and ρ_2 such that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(x)}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} &< \infty, \\ \sum_{k=1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(y)}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} &< \infty. \end{aligned} \quad (21)$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since (M_k) is nondecreasing, convex function and by using inequality (20), we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(\alpha x + \beta y)}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ \leq \sum_{k=1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(\alpha x)}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right. \\ \left. + \left(\left\| \frac{N_k(\beta y)}{\rho_3}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ \leq K \sum_{k=1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(x)}{\rho_1}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ + K \sum_{k=1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(y)}{\rho_2}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ < \infty. \end{aligned} \quad (22)$$

Therefore, $\alpha x + \beta y \in E_n^q(\mathcal{M}, p, s, \|\cdot, \dots, \cdot\|)$. Hence, $E_n^q(\mathcal{M}, p, s, \|\cdot, \dots, \cdot\|)$ is a linear space. $\square$

Theorem 2. Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function and $p = (p_k)$ be a bounded sequence of positive real numbers. Then, the space $E_n^q(\mathcal{M}, p, s, \|\cdot, \dots, \cdot\|)$ is a paranormed space with

$$\begin{aligned} g(x) \\ = \inf \left\{ \rho^{p_n/H} : \left(\sum_{k=1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{1/H} \right. \\ \left. \leq 1, n = 1, 2, 3, \dots \right\}, \end{aligned} \quad (23)$$

where $H = \max(1, \sup_k p_k)$.

Proof. Clearly, $g(x) = g(-x)$ and $g(x + y) \leq g(x) + g(y)$. Since $M_k(0) = 0$, we get $\inf\{\rho^{p_n/H}\} = 0$ for $x = 0$. Finally, we prove that multiplication is continuous. Let λ be any number such that

$$\begin{aligned} g(\lambda x) \\ = \inf \left\{ \rho^{p_n/H} : \sum_{k=1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{\lambda N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \right. \\ \left. \leq 1, n = 1, 2, 3, \dots \right\}. \end{aligned} \quad (24)$$

Thus, we have

$$\begin{aligned} g(\lambda x) &= \inf \left\{ (\lambda s)^{p_n/H} : \sum_{k=1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(x)}{s}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \right. \\ &\quad \left. \leq 1, n = 1, 2, 3, \dots \right\}, \end{aligned} \quad (25)$$

where $s = \rho/|\lambda|$. Since $|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$, then $|\lambda|^{p_k/H} \leq (\max(1, |\lambda|^H))^{1/H}$. Hence,

$$\begin{aligned} g(\lambda x) \\ \leq (\max(1, |\lambda|^H))^{1/H} \\ \inf \left\{ (s)^{p_n/H} : \left(\sum_{k=1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{1/H} \right. \\ \left. \leq 1, n = 1, 2, \dots \right\}, \end{aligned} \quad (26)$$

and therefore $g(\lambda x)$ converges to zero when $g(x)$ converges to zero in $E_n^q(\mathcal{M}, p, s, \|\cdot, \dots, \cdot\|)$. Now, suppose that $\lambda_n \rightarrow 0$

as $n \rightarrow \infty$ and x in $E_n^q(\mathcal{M}, p, s, \|\cdot, \dots, \cdot\|)$. For arbitrary $\epsilon > 0$, let n_0 be a positive integer such that

$$\sum_{k=n_0+1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} < \frac{\epsilon}{2} \quad (27)$$

for some $\rho > 0$. This implies that

$$\left(\sum_{k=n_0+1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{1/H} \leq \frac{\epsilon}{2}. \quad (28)$$

Let $0 < |\lambda| < 1$; then, using convexity of (M_k) , we get

$$\begin{aligned} & \sum_{k=n_0+1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{\lambda N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & < |\lambda| \sum_{k=n_0+1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & < \left(\frac{\epsilon}{2} \right)^H. \end{aligned} \quad (29)$$

Since (M_k) is continuous everywhere in $[0, \infty)$, then

$$h(t) = \sum_{k=1}^{n_0} \frac{1}{k^s} \left[M_k \left(\left\| \frac{t N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \quad (30)$$

is continuous at 0. So there is $0 < \delta < 1$ such that $|h(t)| < \epsilon/2$ for $0 < t < \delta$. Let K be such that $|\lambda_n| < \delta$ for $n > K$; we have

$$\left(\sum_{k=1}^{n_0} \frac{1}{k^s} \left[M_k \left(\left\| \frac{\lambda_n N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{1/H} < \frac{\epsilon}{2}. \quad (31)$$

Thus,

$$\left(\sum_{k=1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{\lambda_n N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \right)^{1/H} < \epsilon, \quad (32)$$

for $n > K$.

Hence, $g(\lambda x) \rightarrow 0$ as $\lambda \rightarrow 0$. This completes the proof of the theorem. $\square$

Theorem 3. If $\mathcal{M}' = (M'_k)$ and $\mathcal{M}'' = (M''_k)$ are two Musielak-Orlicz functions and s, s_1 , and s_2 are nonnegative real numbers, then one has

- (i) $E_n^q(\mathcal{M}', p, s, \|\cdot, \dots, \cdot\|) \cap E_n^q(\mathcal{M}'', p, s, \|\cdot, \dots, \cdot\|) \subseteq E_n^q(\mathcal{M}' + \mathcal{M}'', p, s, \|\cdot, \dots, \cdot\|)$,
- (ii) If $s_1 \leq s_2$, then $E_n^q(\mathcal{M}', p, s_1, \|\cdot, \dots, \cdot\|) \subseteq E_n^q(\mathcal{M}', p, s_2, \|\cdot, \dots, \cdot\|)$,
- (iii) If $\mathcal{M}'$ and $\mathcal{M}''$ are equivalent, then $E_n^q(\mathcal{M}', p, s, \|\cdot, \dots, \cdot\|) = E_n^q(\mathcal{M}'', p, s, \|\cdot, \dots, \cdot\|)$.

Proof. It is obvious, so we omit the details. $\square$

Theorem 4. Suppose that $0 < r_k \leq p_k < \infty$ for each k . Then $E_n^q(\mathcal{M}, r, s, \|\cdot, \dots, \cdot\|) \subseteq E_n^q(\mathcal{M}, p, s, \|\cdot, \dots, \cdot\|)$.

Proof. Let $x \in E_n^q(\mathcal{M}, r, s, \|\cdot, \dots, \cdot\|)$. Then there exists some $\rho > 0$ such that

$$\sum_{k=1}^{\infty} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{r_k} < \infty. \quad (33)$$

This implies that $M_k(\|N_k(x)/\rho, z_1, \dots, z_{n-1}\|) \leq 1$ for sufficiently large value of k , say $k \geq k_0$, for some fixed $k_0 \in \mathbb{N}$. Since (M_k) is nondecreasing, we get

$$\begin{aligned} & \sum_{k \geq k_0} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \leq \sum_{k \geq k_0} \frac{1}{k^s} \left[M_k \left(\left\| \frac{N_k(x)}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{r_k} < \infty. \end{aligned} \quad (34)$$

Hence, $x \in E_n^q(\mathcal{M}, p, s, \|\cdot, \dots, \cdot\|)$. $\square$

Theorem 5. (i) If $0 < p_k \leq 1$ for each k , then $E_n^q(\mathcal{M}, p, s, \|\cdot, \dots, \cdot\|) \subseteq E_n^q(\mathcal{M}, s, \|\cdot, \dots, \cdot\|)$.

(ii) If $p_k \geq 1$ for all k , then $E_n^q(\mathcal{M}, s, \|\cdot, \dots, \cdot\|) \subseteq E_n^q(\mathcal{M}, p, s, \|\cdot, \dots, \cdot\|)$.

Proof. It is easy to prove by using Theorem 4; so we omit the details. $\square$

3. Statistical Convergence

The notion of statistical convergence was introduced by Fast [24] and Schoenberg [25] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [26], Connor [27], Šalát [28], Mursaleen [29], Işık [30], Savaş [31], Malkowsky and Savas [32], Kolk [33], Maddox [34], Mohiuddine and Aiyub [35], and many others. In the recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the Stone-Čech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability.

Let $\theta = (k_r)$ be the sequence of positive integers such that $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. Then θ is called a lacunary sequence. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and the ratio $k_r/(k_r - 1)$ will be denoted by q_r (see [36]).

Let $\theta = (k_r)$ be a lacunary sequence; then the sequence $x = (x_k)$ is said to be $N_k(u)$ -lacunary statistically convergent to the number l provided that, for every $\epsilon > 0$,

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\| \geq \epsilon \right\} \right| = 0. \quad (35)$$

In this case, we write $[N_k(u), S]_\theta - \lim x = l$ or $x_k \rightarrow l([N_k(u), S]_\theta)$.

Let $\theta = (k_r)$ be a lacunary sequence, $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, $u = (u_k)$ be a sequence of strictly positive real numbers, and $p = (p_k)$ be a bounded sequence of positive real numbers. A sequence $x = (x_k)$ is said to be strongly $N_k(u)$ -lacunary convergent to the number l with respect to the Musielak-Orlicz function $\mathcal{M}$ provided that

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left[M_k \left(\left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} = 0. \quad (36)$$

The set of all strongly $N_k(u)$ -lacunary convergent sequences to the number l with respect to the Musielak-Orlicz function $\mathcal{M}$ is denoted by $[N_k, u, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_\theta$. In this case, we write $x_k \rightarrow l([N_k, u, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_\theta)$. In the special case $\mathcal{M}(x) = x$, $p_k = p_0$ for all $k \in \mathbb{N}$, we shall write $[N_k, u, \|\cdot, \dots, \cdot\|]_\theta$ instead of $[N_k, u, \mathcal{M}, p, \|\cdot, \dots, \cdot\|]_\theta$.

In this section, we give some results about $N_k(u)$ -lacunary statistical convergence and give some relations between the set of $N_k(u)$ -lacunary statistical convergence sequences and other spaces which are defined with respect to Musielak-Orlicz function $\mathcal{M}$.

Theorem 6. Let $\theta = (k_r)$ be a lacunary sequence.

- (i) If a sequence $x = (x_k)$ is strongly $N_k(u)$ -lacunary convergent to l , then it is $N_k(u)$ -lacunary statistically convergent to l .
- (ii) If a bounded sequence $x = (x_k)$ is $N_k(u)$ -lacunary statistically convergent to l , then it is strongly $N_k(u)$ -lacunary convergent to l .

Proof. (i) Let $\epsilon > 0$ and $x_k \rightarrow l([N_k, u, \|\cdot, \dots, \cdot\|]_\theta)$. Then, we can write

$$\begin{aligned} & \sum_{k \in I_r} \left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\|^{p_0} \\ & \geq \sum_{k \in I_r} \left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\|^{p_0} \\ & \quad \left\| (u_k N_k(x) - l)/\rho, z_1, \dots, z_{n-1} \right\| \geq \epsilon \\ & \geq \epsilon^{p_0} \left| \left\{ k \in I_r : \left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\| \geq \epsilon \right\} \right|. \end{aligned} \quad (37)$$

Hence, $x_k \rightarrow l([N_k, u, S]_\theta)$.

(ii) Suppose that $x_k \rightarrow l([N_k, u, S]_\theta)$, and let $x \in l_\infty$. Let $\epsilon > 0$ be given and take N_ϵ such that

$$\begin{aligned} & \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\| \geq \left(\frac{\epsilon}{2} \right)^{(1/p_0)} \right\} \right| \\ & \leq \frac{\epsilon}{2K^{p_0}} \end{aligned} \quad (38)$$

for all $r > N_\epsilon$ and set $T_r = \{k \in I_r : \|(u_k N_k(x) - l)/\rho, z_1, \dots, z_{n-1}\| \geq (\epsilon/2)^{1/p_0}\}$, where $K = \sup_k |x_k| < \infty$. Now, for all $r > N_\epsilon$, we have

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} \left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\|^{p_0} \\ & = \frac{1}{h_r} \sum_{\substack{k \in I_r \\ k \in T_r}} \left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\|^{p_0} \\ & \quad + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ k \notin T_r}} \left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\|^{p_0} \\ & \leq \frac{1}{h_r} \left(\frac{h_r \epsilon}{2K^{p_0}} \right) K^{p_0} + \frac{\epsilon}{2h_r} h_r = \epsilon. \end{aligned} \quad (39)$$

Thus, $(x_k) \in [N_k, u, \|\cdot, \dots, \cdot\|]_\theta$. This completes the proof of the theorem. $\square$

Theorem 7. For any lacunary sequence θ , if $\lim_{r \rightarrow \infty} \inf q_r > 1$, then $[N_k, u, S] \subset [N_k, u, S]_\theta$.

Proof. If $\lim_{r \rightarrow \infty} \inf q_r > 1$, then there exists a $\delta > 0$ such that $1 + \delta \leq q_r$ for sufficiently large r . Since $h_r = k_r - k_{r-1}$, we have $k_r/h_r \leq (1 + \delta)/\delta$. Let $x_k \rightarrow l([N_k, u, S])$. Then, for every $\epsilon > 0$, we have

$$\begin{aligned} & \frac{1}{k_r} \left| \left\{ k \leq k_r : \left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\| \geq \epsilon \right\} \right| \\ & \geq \frac{1}{k_r} \left| \left\{ k \in I_r : \left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\| \geq \epsilon \right\} \right| \\ & \geq \frac{\delta}{1 + \delta} \frac{1}{k_r} \left| \left\{ k \in I_r : \left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\| \geq \epsilon \right\} \right|. \end{aligned} \quad (40)$$

Hence, $[N_k, u, S] \subset [N_k, u, S]_\theta$. $\square$

Theorem 8. Let $\theta = (k_r)$ be a lacunary sequence, $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H$. Then, $[N_k, u, \mathcal{M}, \|\cdot, \dots, \cdot\|]_\theta \subset [N_k, u, S]_\theta$.

Proof. Let $x \in [N_k, u, \mathcal{M}, \|\cdot, \dots, \cdot\|]_\theta$. Then there exists a number $\rho > 0$ such that

$$\frac{1}{h_r} \sum_{k \in I_r} \left[M_k \left(\left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \rightarrow 0, \quad (41)$$

as $r \rightarrow \infty$.

Then, given $\epsilon > 0$, we have

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} \left[M_k \left(\left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \geq \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \|(u_k N_k(x) - l)/\rho, z_1, \dots, z_{n-1}\| \geq \epsilon}} \left[M_k \left(\left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \geq \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \|(u_k N_k(x) - l)/\rho, z_1, \dots, z_{n-1}\| \geq \epsilon}} [M_k(\epsilon_1)]^{p_k}, \end{aligned} \quad (42)$$

where $\epsilon/\rho = \epsilon_1$

$$\begin{aligned} & \geq \frac{1}{h_r} \sum \min \left\{ [M_k(\epsilon_1)]^h, [M_k(\epsilon_1)]^H \right\} \\ & \geq \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\| \geq \epsilon \right\} \right| \\ & \quad \cdot \min \left\{ [M_k(\epsilon_1)]^h, [M_k(\epsilon_1)]^H \right\}. \end{aligned}$$

Hence, $x \in [N_k, u, S]_\theta$. This completes the proof of the theorem. $\square$

Theorem 9. Let $\theta = (k_r)$ be a lacunary sequence, $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function, and $p = (p_k)$ be a bounded sequence; then $[N_k, u, S]_\theta \subset [N_k, u, \mathcal{M}, \|\cdot, \dots, \cdot\|]_\theta$.

Proof. Let $x = (x_k) \in l_\infty$ with $x_k \rightarrow l([N_k, u, S]_\theta)$. Since $x \in l_\infty$, there is a constant $T > 0$ such that $\|(u_k N_k(x) - l)/\rho, z_1, \dots, z_{n-1}\| \leq T$, and given $\epsilon > 0$, we have

$$\begin{aligned} & \frac{1}{h_r} \sum_{k \in I_r} \left[M_k \left(\left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & = \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \|(u_k N_k(x) - l)/\rho, z_1, \dots, z_{n-1}\| \geq \epsilon}} \left[M_k \left(\left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \quad + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \|(u_k N_k(x) - l)/\rho, z_1, \dots, z_{n-1}\| < \epsilon}} \left[M_k \left(\left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\| \right) \right]^{p_k} \\ & \leq \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \|(u_k N_k(x) - l)/\rho, z_1, \dots, z_{n-1}\| \geq \epsilon}} \max \left\{ \left[M_k \left(\frac{T}{\rho} \right) \right]^h, \left[M_k \left(\frac{T}{\rho} \right) \right]^H \right\} \\ & \quad + \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \|(u_k N_k(x) - l)/\rho, z_1, \dots, z_{n-1}\| < \epsilon}} \left[M_k \left(\frac{\epsilon}{\rho} \right) \right]^{p_k} \end{aligned}$$

$$\begin{aligned} & \leq \max \left\{ [M_k(K)]^h, [M_k(K)]^H \right\} \\ & \quad \times \frac{1}{h_r} \left| \left\{ k \in I_r : \left\| \frac{u_k N_k(x) - l}{\rho}, z_1, \dots, z_{n-1} \right\| \geq \epsilon \right\} \right| \\ & \quad + \max \left\{ [M_k(\epsilon_1)]^h, [M_k(\epsilon_1)]^H \right\}, \\ & \quad \frac{T}{\rho} = K, \quad \frac{\epsilon}{\rho} = \epsilon_1. \end{aligned} \quad (43)$$

Hence, $x \in [N_k, u, \mathcal{M}, \|\cdot, \dots, \cdot\|]_\theta$. $\square$

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References

- [1] S. Gähler, "Lineare 2-normierte Räume," *Mathematische Nachrichten*, vol. 28, pp. 1–43, 1965.
- [2] A. Misiak, " n -inner product spaces," *Mathematische Nachrichten*, vol. 140, pp. 299–319, 1989.
- [3] H. Gunawan, "On n -inner products, n -norms, and the Cauchy-Schwarz inequality," *Scientiae Mathematicae Japonicae*, vol. 5, pp. 47–54, 2001.
- [4] H. Gunawan, "The space of p -summable sequences and its natural n -norm," *Bulletin of the Australian Mathematical Society*, vol. 64, no. 1, pp. 137–147, 2001.
- [5] H. Gunawan and M. Mashadi, "On n -normed spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 27, no. 10, pp. 631–639, 2001.
- [6] J. Lindenstrauss and L. Tzafriri, "On Orlicz sequence spaces," *Israel Journal of Mathematics*, vol. 10, pp. 379–390, 1971.
- [7] L. Maligranda, *Orlicz Spaces and Interpolation*, vol. 5 of *Seminários de Matemática*, Polish Academy of Science, Warszawa, Poland, 1989.
- [8] J. Musielak, *Orlicz Spaces and Modular Spaces*, vol. 1034 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1983.
- [9] A. Wilansky, *Summability through Functional Analysis*, vol. 85 of *North-Holland Mathematics Studies*, North-Holland, Amsterdam, The Netherlands, 1984.
- [10] F. Başar, *Summability Theory and Its Applications*, Monographs, Bentham Science Publishers, E-Books, Istanbul, Turkey, 2012.
- [11] F. Başar, B. Altay, and M. Mursaleen, "Some generalizations of the space $bv(p)$ of p -bounded variation sequences," *Nonlinear Analysis: Theory, Methods and Applications A*, vol. 68, no. 2, pp. 273–287, 2008.
- [12] C. Belen and S. A. Mohiuddine, "Generalized weighted statistical convergence and application," *Applied Mathematics and Computation*, vol. 219, no. 18, pp. 9821–9826, 2013.
- [13] T. Bilgin, "Some new difference sequences spaces defined by an Orlicz function," *Filomat*, vol. 17, pp. 1–8, 2003.
- [14] N. L. Braha and M. Et, "The sequence space $E_n^q(M, p, s)$ and N_k -lacunary statistical convergence," *Banach Journal of Mathematical Analysis*, vol. 7, no. 1, pp. 88–96, 2013.
- [15] R. Çolak, B. C. Tripathy, and M. Et, "Lacunary strongly summable sequences and q -lacunary almost statistical convergence," *Vietnam Journal of Mathematics*, vol. 34, no. 2, pp. 129–138, 2006.

- [16] A. M. Jarrah and E. Malkowsky, "The space $bv(p)$, its β -dual and matrix transformations," *Collectanea Mathematica*, vol. 55, no. 2, pp. 151–162, 2004.
- [17] I. J. Maddox, "Statistical convergence in a locally convex space," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 104, no. 1, pp. 141–145, 1988.
- [18] M. Mursaleen, "Generalized spaces of difference sequences," *Journal of Mathematical Analysis and Applications*, vol. 203, no. 3, pp. 738–745, 1996.
- [19] M. Mursaleen, "Matrix transformations between some new sequence spaces," *Houston Journal of Mathematics*, vol. 9, no. 4, pp. 505–509, 1983.
- [20] M. Mursaleen, "On some new invariant matrix methods of summability," *The Quarterly Journal of Mathematics*, vol. 34, no. 133, pp. 77–86, 1983.
- [21] K. Raj and S. K. Sharma, "Some sequence spaces in 2-normed spaces defined by Musielak-Orlicz function," *Acta Universitatis Sapientiae. Mathematica*, vol. 3, no. 1, pp. 97–109, 2011.
- [22] K. Raj and S. K. Sharma, "Some generalized difference double sequence spaces defined by a sequence of Orlicz-functions," *Cubo*, vol. 14, no. 3, pp. 167–189, 2012.
- [23] K. Raj and S. K. Sharma, "Some multiplier sequence spaces defined by a Musielak-Orlicz function in n -normed spaces," *New Zealand Journal of Mathematics*, vol. 42, pp. 45–56, 2012.
- [24] H. Fast, "Sur la convergence statistique," *Colloquium Mathematicum*, vol. 2, pp. 241–244, 1951.
- [25] I. J. Schoenberg, "The integrability of certain functions and related summability methods," *The American Mathematical Monthly*, vol. 66, pp. 361–375, 1959.
- [26] J. A. Fridy, "On statistical convergence," *Analysis*, vol. 5, no. 4, pp. 301–303, 1985.
- [27] J. Connor, "A topological and functional analytic approach to statistical convergence," in *Analysis of Divergence*, Applied and Numerical Harmonic Analysis, pp. 403–413, Birkhäuser, Boston, Mass, USA, 1999.
- [28] T. Šalát, "On statistically convergent sequences of real numbers," *Mathematica Slovaca*, vol. 30, no. 2, pp. 139–150, 1980.
- [29] M. Mursaleen, " λ -statistical convergence," *Mathematica Slovaca*, vol. 50, no. 1, pp. 111–115, 2000.
- [30] M. Işık, "On statistical convergence of generalized difference sequences," *Soochow Journal of Mathematics*, vol. 30, no. 2, pp. 197–205, 2004.
- [31] E. Savaş, "Strong almost convergence and almost λ -statistical convergence," *Hokkaido Mathematical Journal*, vol. 29, no. 3, pp. 531–566, 2000.
- [32] E. Malkowsky and E. Savas, "Some λ -sequence spaces defined by a modulus," *Archivum Mathematicum*, vol. 36, no. 3, pp. 219–228, 2000.
- [33] E. Kolk, "The statistical convergence in Banach spaces," *Acta et Commentationes Universitatis Tartuensis*, vol. 928, pp. 41–52, 1991.
- [34] I. J. Maddox, "A new type of convergence," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 83, no. 1, pp. 61–64, 1978.
- [35] S. A. Mohiuddine and M. Aiyub, "Lacunary statistical convergence in random 2-normed spaces," *Applied Mathematics & Information Sciences*, vol. 6, no. 3, pp. 581–585, 2012.
- [36] A. R. Freedman, J. J. Sember, and M. Raphael, "Some Cesàro-type summability spaces," *Proceedings of the London Mathematical Society*, vol. 37, no. 3, pp. 508–520, 1978.