Research Article
Sunlet Decomposition of Certain Equipartite Graphs

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Let \( L_{2m} \) stand for the sunlet graph which is a graph that consists of a cycle and an edge terminating in a vertex of degree one attached to each vertex of cycle \( C_n \). The necessary condition for the equipartite graph \( K_n + I \ast \overline{K}_m \) to be decomposed into \( L_{2m} \) for \( n \geq 2 \) is that the order of \( L_{2m} \) must divide \( n^2m^2/2 \), the order of \( K_n + I \ast \overline{K}_m \). In this work, we show that this condition is sufficient for the decomposition. The proofs are constructive using graph theory techniques.

1. Introduction

Let \( C_r, K_n, \overline{K}_m \) denote cycle of length \( r \), complete graph on \( n \) vertices, and complement of complete graph on \( m \) vertices. For \( n \) even, \( K_n + I \) denotes the multigraph obtained by adding the edges of a 1-factor to \( K_n \), thus duplicating \( n/2 \) edges. The total number of edges in \( K_n + I \) is \( n^2/2 \). The lexicographic product, \( G \ast H \), of graphs \( G \) and \( H \), is the graph obtained by replacing every vertex of \( G \) by a copy of \( H \) and every edge of \( G \) by the complete bipartite graph \( K_{|H|,|H|} \).

For a graph \( H \), an \( H \)-decomposition of a graph \( G, H \mid G \), is a set of subgraphs of \( G \), each isomorphic to \( H \), whose edge set partitions the edge set of \( G \). Note that for any graph \( G \) and \( H \) and any positive integer \( m \), if \( H \mid G \) then \( (H \ast \overline{K}_m) \mid (G \ast \overline{K}_m) \).

Let \( G \) be a graph of order \( n \) and \( H \) any graph. The corona (crown) of \( G \) with \( H \), denoted by \( G \circ H \), is the graph obtained by taking one copy of \( G \) and \( n \) copies of \( H \) and joining the \( i \)th vertex of \( G \) with every vertex in the \( i \)th copy of \( H \). A special corona graph is \( C_2 \circ K_2 \), that is, a cycle with pendant points which has \( 2n \) vertices. This is called sunlet graph and denoted by \( L_n, q = 2n \).

Obvious necessary condition for the existence of a \( k \)-cycle decomposition of a simple connected graph \( G \) is that \( G \) has at least \( k \) vertices (or trivially, just one vertex), the degree of every vertex in \( G \) is even, and the total number of edges in \( G \) is a multiple of the cycle length \( k \). These conditions have been shown to be sufficient in the case that \( G \) is the complete graph \( K_n \), the complete graph minus a 1-factor \( K_n - I \) [1, 2], and the complete graph plus a 1-factor \( K_n + I \) [3].

The study of cycle decomposition of \( K_n \ast \overline{K}_m \) was initiated by Hoffman et al. [4]. The necessary and sufficient conditions for the existence of a \( C_p \)-decomposition of \( K_n \ast \overline{K}_m \), where \( p \geq 5 \) (\( p \) is prime) that (i) \( m(n - 1) \) is even and (ii) \( p \) divides \( m(n - 1)m^2 \), were obtained by Manikandan and Paulraj [5, 6]. Similarly, when \( p \geq 3 \) is a prime, the necessary and sufficient conditions for the existence of a \( C_{2p} \)-decomposition of \( K_n \ast \overline{K}_m \) were given by Smith [7]. For a prime number \( p \geq 3 \), Smith [8] showed that \( C_{3p} \)-decomposition of \( K_n \ast \overline{K}_m \) exists if the obvious necessary conditions are satisfied. In [9], Anitha and Lekshmi proved that the complete graph \( K_n \) and the complete bipartite graph \( K_{nn} \) for \( n \) even have decompositions into sunlet graph \( L_n \). Similarly, in [10], it was shown that the complete equipartite graph \( K_n \ast \overline{K}_m \) has a decomposition into sunlet graph of length \( 2p \), for a prime \( p \).

We extend these results by considering the decomposition of \( K_n + I \ast \overline{K}_m \) into sunlet graphs and prove the following result.

Let \( m \geq 2, n > 2 \), and \( q \geq 6 \) be even integers. The graph \( K_n + I \ast \overline{K}_m \) can be decomposed into sunlet graph of length \( q \) if and only if \( q \) divides \( n^2m^2/2 \), the number of edges in \( K_n + I \ast \overline{K}_m \).
2. Proof of the Result

To prove the result, we need the following.

Lemma 1 (see [10]). For \( r \geq 3 \), \( L_{2r} \) decomposes \( C_r \ast K_2 \).

Lemma 2. For any integer \( r > 2 \) and a positive even integer \( m \), the graph \( C_r \ast K_m \) has a decomposition into sunlet graph \( L_\eta \), for \( \eta = rm \).

Proof

Case 1 \((r \text{ is even})\). First observe that \( C_r \ast K_2 \) can be decomposed into 2 sunlet graphs with \( 2r \) vertices. Now, set \( m = 2r \) and decompose \( C_r \ast K_2 \) into cycles \( C_n \). To decompose \( C_r \ast K_2 \) into \( t \)-cycles \( C_{\eta} \), denote vertices in ith part of \( C_r \ast K_2 \) by \( x_{i,j} \) for \( j = 1, \ldots, t \) and create \( t \) base cycles \( x_1, x_2, \ldots, x_{t-1}, x_t \). Next, combine these base cycles into one cycle \( C_n \) by replacing each edge \( x_{i,j}x_{i,j+1} \) with \( x_{i,j}x_{i,j+1} \). To create the remaining cycles \( C_{\eta} \), we apply mappings \( \phi_i \) for \( s = 0, 1, \ldots, t - 1 \) defined on the vertices as follows.

Subcase 1.1 \((i \text{ odd})\). Consider

\[ \phi_i(x_{i,j}) = x_{i,j}. \]  

This is the desired decomposition into cycles \( C_{\eta} \).

Subcase 1.2 \((i \text{ even})\). Consider

\[ \phi_i(x_{i,j}) = x_{i,j+1}. \]  

This is the desired decomposition into cycles \( C_{\eta} \).

Now take each cycle \( C_{\eta} \), and make it back into \( C_r \ast K_2 \). Each \( C_{\eta} \ast K_2 \) decomposes into 2 sunlet graphs \( L_{2r} \) (by Lemma 1), and we have \( C_r \ast K_m \) decomposing into sunlet graphs with length \( rm \) for \( r \) even. Note that

\[ C_r \ast K_2 = \left( C_r \ast K_1 \right) \ast K_2. \]  

Case 2 \((r \text{ odd})\)

Subcase 2.1 \((m \equiv 2 \pmod{4})\). Set \( m = 2r \). First create cycles \( C_{1,1} \), \( C_{r-1,1} \) in \( C_r \ast K_2 \) as in Case 1. Then, take complete tripartite graph \( K_{1,1} \) with partite sets \( X_i = \{ x_{i,j} \} \) for \( i = 0, r, 1 \), \( j = 1, \ldots, t \) and decompose it into triangles using well-known construction via Latin square, that is, construct \( t \times t \) Latin square and consider each element in the form \((a, b, c)\) where \( a \) denotes the row, \( b \) denotes the column, and \( c \) denotes the entry with \( 1 \leq a, n, c \leq t \). Each cycle is of the form \( x_{1,a_r}x_{r-1,b}x_{r,c} \). Then, for every triangle \( x_{1,a}x_{r-1,b}x_{r,c} \), replace the edge \( x_{1,a}x_{r-1,b} \) in each \( C_{1,1} \), by the edges \( x_{1,a}x_{r-1,b}x_{r,c} \). This is the desired decomposition into cycles \( C_{\eta} \). Therefore, \( C_r \ast K_2 \) now take each cycle \( C_{\eta} \), make it into \( C_r \ast K_2 \), and by Lemma 1, \( C_r \ast K_2 \) has a decomposition into sunlet graphs \( L_{2r} \) for \( \eta = rm \).

Subcase 2.2 \((m \equiv 0 \pmod{4})\). Set \( m = 2t \). The graph \( C_r \ast K_2 \) decomposes into Hamilton cycle \( C_{2t} \), by [3]. Next, make each cycle \( C_r \) into \( C_r \ast K_2 \). Each graph \( C_r \ast K_2 \) decomposes into sunlet graph \( L_{2r} \) by Lemma 1.

Theorem 3. Let \( r, m \) be positive integers satisfying \( r, m \equiv 0 \pmod{4} \), then \( L_r \) decomposes \( C_r \ast K_m \).

Proof. Let the partite sets (layers) of the \( r \)-partite graph \( C_r \ast K_m \) be \( U_1, U_2, \ldots, U_r \). Set \( m = 2t \). Obtain a new graph from \( C_r \ast K_m \) as follows.

Identify the subsets of vertices \( \{ x_{i,j} \} \) for \( 1 \leq i \leq r \) and \( 1 \leq j \leq m/2 \) into new vertices \( x_{i,j}^1 \), and identify the subset of vertices \( \{ x_{i,j} \} \) for \( 1 \leq i \leq r \) and \( m/2 + 1 \leq j \leq m \) into new vertices \( x_{i,j}^2 \) and two of these vertices \( x_{i,j}^1 \), where \( k = 1, 2 \), are adjacent if and only if the corresponding subsets of vertices in \( C_r \ast K_m \) induce \( K_{t/2} \). The resulting graph is isomorphic to \( C_r \ast K_2 \). Next, decompose \( C_r \ast K_2 \) into cycles \( C_{\eta} \) as follows:

\[ x_{k,1}, x_{k,1}, \ldots, x_{k,1}, \ldots, x_{k,2}, x_{k,2}, \ldots, x_{k,2}, \ldots, x_{k,2}, x_{k,2} \]

where \( k, d \) are calculated modulo 4.

To construct the remaining cycles, apply mapping \( \phi \) defined on the vertices.

Subcase 1.1 \((i \text{ odd in each cycle})\). Consider

\[ \phi(x_{i,j}) = x_{i,j+1}. \]  

This is the desired decomposition of \( C_r \ast K_2 \) into cycles \( C_{\eta} \).

Subcase 1.2 \((i \text{ even in each cycle})\). Consider

\[ \phi(x_{i,j}) = x_{i,j}. \]  

This is the desired decomposition of \( C_r \ast K_2 \) into cycles \( C_{\eta} \).

By lifting back these cycles \( C_{\eta} \) of \( C_r \ast K_2 \) to \( C_r \ast K_2 \), we get edge-disjoint subgraphs isomorphic to \( C_r \ast K_2 \). Obtain a new graph again from \( C_{\eta} \ast K_2 \).

For each \( j, 1 \leq j \leq t/2 \), identify the subsets of vertices \( \{ x_{2,2j-1}, x_{2,2j} \} \), where \( 1 \leq i \leq r/2 \) into new vertices \( x_{i,j}^1 \), and two of these vertices \( x_{i,j}^1 \) are adjacent if and only if the corresponding subsets of vertices in \( C_{\eta} \ast K_2 \) induce \( K_{t/2} \). The resulting graph is isomorphic to \( C_{\eta} \ast K_{t/2} \). Then, decompose \( C_{\eta} \ast K_{t/2} \) into cycles \( C_{\eta} \). Each \( C_{\eta} \ast K_{t/2} \) decomposes into cycles \( C_{\eta} \) by [12]. By lifting back these cycles \( C_{\eta} \ast K_{t/2} \) to \( C_{\eta} \ast K_m \), we get edge-disjoint subgraph isomorphic to \( C_{\eta} \ast K_2 \). Finally, each \( C_{\eta} \ast K_2 \) decomposes into two sunlet graphs \( L_r \) (by Lemma 1), and we have \( C_r \ast K_m \) decomposing into sunlet graphs \( L_r \), as required.

Theorem 4 (see [12]). The cycle \( C_m \) decomposes \( C_k \ast K_m \) for every even \( m > 3 \).

Theorem 5 (see [12]). If \( m \) and \( k \geq 3 \) are odd integers, then \( C_m \) decomposes \( C_k \ast K_m \).
Theorem 6. The sunlet graph \( L_m \) decomposes \( C_r \ast K_m \) if and only if either one of the following conditions is satisfied.

(1) \( r \) is a positive odd integer, and \( m \) is a positive even integer.

(2) \( r, m \) are positive even integers with \( m \equiv 0 \pmod{4} \).

Proof. (1) Set \( m = 2t \), where \( t \) is a positive integer. Let the partite sets (layers) of the \( r \)-partite graph \( C_r \ast K_m \) be \( U_1, U_2, \ldots, U_r \). For each \( j \), where \( 1 \leq j \leq t \), identify the subsets of vertices \( \{x_i, 1-j, x_{i,j}\} \), for \( 1 \leq i \leq r \) into new vertices \( x_i^j \), and two of these vertices \( x_i^j \) are adjacent if and only if the corresponding subsets of vertices in \( C_r \ast K_m \) induce \( K_{2,2} \). The resulting graph is isomorphic to \( C_r \ast K_t \). Then, decompose \( C_r \ast K_t \) into cycles \( C_t \), where \( t \) is a positive integer.

Now, \( C_t \ast C_r \ast K_t \) by Theorems 4 and 5.

By lifting back these \( t \)-cycles of \( C_r \ast K_t \), to \( C_r \ast K_{2t} \), we get edge-disjoint subgraphs isomorphic to \( C_t \ast K_{2t} \). Each copy of \( C_t \ast K_{2t} \) decomposes into sunlet graphs of length \( 2t \) (by Lemma 1), and we have \( C_t \ast K_t \) decomposing into sunlet graphs of length \( m \) as required.

(2) \( m = 2t \), where \( t \) is an even integer since \( m \equiv 0 \pmod{4} \).

Obtain a new graph \( C_t \ast K_{2t} \) from the graph \( C_r \ast K_m \) as in Case 1. By Theorem 4, \( C_t \ast C_r \ast K_t \). By lifting back these \( t \)-cycles of \( C_r \ast K_t \), to \( C_r \ast K_{2t} \), we get edge-disjoint subgraphs isomorphic to \( C_t \ast K_{2t} \). Each copy of \( C_t \ast K_{2t} \) decomposes into sunlet graph of length \( 2t \) (by Lemma 1). Therefore, \( L_m \) decomposes into \( C_t \ast K_t \) as required. \( \Box \)

Remark 7. In [10], it was shown that

\[
L_{2t} \ast K_1 \text{ can be decomposed into } 2 \text{ copies of } L_{2t}.
\] (7)

This, coupled with Lemma 1, gives the following.

Theorem 8 (see [10]). The graph \( C_r \ast K_{2t} \) decomposes into sunlet graphs \( L_{2t} \) for any positive integer \( l \).

Lemma 9 (see [3]). Let \( n \geq 4 \) be an even integer. Then, \( K_n + I \) is \( C_n \)-decomposable.

Lemma 10 (see [3]). Let \( m \) and \( n \) be integers with \( m \) odd, \( n \equiv 2 \pmod{4} \), \( 3 \leq m \leq n < 2m \), and \( n^2 \equiv 0 \pmod{2m} \). Then, \( K_n + I \) is \( C_m \)-decomposable.

Lemma 11 (see [3]). Let \( m \) and \( n \) be integers with \( m \) odd, \( n \equiv 0 \pmod{4} \), \( 3 \leq m \leq n < 2m \), and \( n^2 \equiv 0 \pmod{2m} \). Then, \( K_n + I \) is \( C_m \)-decomposable.

We can now prove the major result.

Theorem 12. For any even integers \( m \geq 2 \), \( n > 2 \), and \( q \geq 6 \), the sunlet graph \( L_q \) decomposes \( K_n + I \ast K_m \) if and only if \( n^2 / 2 \equiv 0 \pmod{m} \).

Proof. The necessity of the condition is obvious, and so we need only to prove its sufficiency. We split the problem into the following two cases.

Case 1 \((q \mid n)\)

Subcase 1.1 \((n > q)\). Cycle \( C_n \) decomposes \( K_n + I \) by Lemma 9, and we have

\[
C_n \ast K_m \mid K_n + I \ast K_m.
\] (8)

Each graph \( C_n \ast K_m \) decomposes into sunlet graph \( L_q \), where \( q = mn \) by Lemma 2, and we have \( K_n + I \ast K_m \) decomposing into sunlet graph \( L_q \), where \( q > n \).

Subcase 1.2 \((q = n)\). First, consider \( n \equiv 0 \pmod{4} \).

Cycle \( C_q \) decomposes \( K_q + I \) by Lemma 9, and we have

\[
C_q \ast K_m \mid K_q + I \ast K_m.
\] (9)

Now, sunlet graph \( L_q \mid (C_q \ast K_m) \) by Theorem 3, and hence sunlet graph \( L_q \) decomposes \( K_q + I \ast K_m \).

Also, consider \( n \equiv 2 \pmod{4} \).

Suppose \( m = 2t \). Cycle \( C_{q/2} \) decomposes \( K_q + I \) by Lemma 10, and we have

\[
C_{q/2} \ast K_{2t} \mid K_q + I \ast K_{2t}.
\] (10)

Now, sunlet graph \( L_q \mid (C_{q/2} \ast K_{2t}) \) by Theorem 8, and we have \( K_n + I \ast K_m \) decomposing into sunlet graph of length \( q \).

Case 2 \((q \mid m)\)

Subcase 2.1 \((m \equiv 0 \pmod{4})\). Suppose \( m = q \), and by Lemma 9, cycle \( C_n \) decomposes \( K_n + I \), and we have

\[
C_n \ast K_q \mid K_n + I \ast K_q.
\] (11)

Also, sunlet graph \( L_q \) decomposes each \( C_n \ast K_q \) by Theorem 6, and we have sunlet graph \( L_q \) decomposing \( K_n + I \ast K_m \).

Subcase 2.2 \((m \equiv 2 \pmod{4})\). Let \( m = q \) and \( r \leq n \) an odd integer. Cycle \( C_r \) decomposes \( K_n + I \), by Lemmas 9, 10, and 11, and we have

\[
C_r \ast K_q \mid K_n + I \ast K_q.
\] (12)

Now, each \( C_r \ast K_q \) decomposes into sunlet graph \( L_q \) by Theorem 6, and we have \( K_n + I \ast K_m \) decomposing into sunlet graph \( L_q \) as required.

Subcase 2.3 \((m > q)\). Set \( m = wq \), where \( w \) is any positive integer, then by Subcases 2.1 and 2.2, we have

\[
L_q \ast K_w \mid (K_n + I \ast K_q) \ast K_w.
\] (13)

Each graph \( L_q \ast K_w \) decomposes into sunlet graph \( L_q \) by Remark 7, and we have \( K_n + I \ast K_m \) decomposing into sunlet graph \( L_q \). \( \square \)
References


