Hindawi Publishing Corporation
ISRN Astronomy and Astrophysics
Volume 2013, Article ID 910354, 8 pages
http://dx.doi.org/10.1155/2013/910354

Research Article

Robe's Restricted Three-Body Problem with Variable Masses and Perturbing Forces

Jagadish Singh[1] and Oni Leke[2]

[1] *Department of Mathematics, Faculty of Science, Ahmadu Bello University, Zaria, Nigeria*
[2] *Department of Mathematics, College of Science, University of Agriculture, PMB 2373, Makurdi, Nigeria*

Correspondence should be addressed to Oni Leke; lekkyonix4ree@yahoo.com

Received 24 December 2012; Accepted 20 February 2013

Academic Editors: P. P. Avelino, A. Ferrari, P. A. Hughes, and V. Pierrard

The linear stability of equilibrium points of a test particle of infinitesimal mass in the framework of Robe's circular restricted three-body problem, as in Hallan and Rana, together with effect of variation in masses of the primaries with time according to the combined Meshcherskii law, is investigated. It is seen that, due to a small perturbation in the centrifugal force and an arbitrary constant κ of a particular integral of the Gylden-Meshcherskii problem, every point on the line joining the centers of the primaries is an equilibrium point provided they lie within the shell. Further, a number of pairs of equilibrium points lying on the $\xi\zeta$-plane and forming triangles with the centers of the shell and the second primary exist, for some values of $\kappa > 1$. The points collinear with the center of the shell are found to be stable under some conditions and the range of stability depends on the small perturbations and κ, while the triangular points are unstable. Illustrative numerical exploration is given to indicate significant improvement of the problem in Hallan and Rana.

1. Introduction

In general, the classical restricted three-body problem (RTBP) assumes that the masses of the participating bodies are constant and do not change with time during mechanical motion. The study of two bodies with variable mass has received attention over the years. One of such equation that describes a particular mass variation law is the Gylden-Meshcherskii problem [1, 2]. This problem extends the classical two-body problem of Newton and Kepler by considering time varying gravitating masses, an important extension of the two body problem for modeling cometary motion and cosmological phenomena and also for describing the evolution of binary stars during secular mass loss owning to photon and corpuscular activity. The slow loss of isotropic mass acts like a perturbation and the problem could be dealt with using a typical method of the theory of perturbations.

Meshcherskii [3] studied the mechanics of the bodies of variable mass. Gelf'gat [4] examined the restricted problem of three-body of variable masses in which the primary bodies move within the framework of the Gylden-Meshcherskii problem (GMP) and established the existence of five libration points analogous to the classical libration points. Singh and Leke [5] studied the stability of the equilibrium points when the luminous primaries move within the framework of the GMP and vary their masses in accordance with the unified Meshcherskii law.

The classical RTBP assumes that the infinitesimal mass moves under only the mutual gravitational force of the primaries, but in practice Coriolis and centrifugal forces are effective there, and small perturbations affect these forces. Recent interesting studies of this kind of the problem that have been carried out include Hallan and Mangang [6], Singh et al. [7], and Singh and Leke [5].

Robe [8] considered a new kind of restricted three-body problem, in which one of the primaries of mass m_1, is a rigid spherical shell, filled with homogenous, incompressible fluid of density ρ_1 and the other one is a point mass m_2 outside the shell and moving around the first primary in Keplerian orbit. A third body of infinitesimal mass m_3 is a small solid sphere of density ρ_3 that moves inside the shell. He discussed the linear stability of an equilibrium point of the problem obtained in two cases. In the first case, m_2 moves around m_1 in circular

orbits; in the second case the orbit is elliptic, but the shell is empty or densities of m_1 and m_3 are equal. Robe's problem can be used to study the small oscillation of the Earth inner core taking into account the Moon's attraction, the stability of the Earth's center, and the motion of artificial satellites in the Earth atmosphere under the action of floating force of the atmosphere. Given its significance and applications, further studies under different assumptions have been carried out since then.

Shrivastava and Garain [9] investigated the effect of small perturbation in the Coriolis and centrifugal forces on the location of libration point in the Robe [8] circular restricted problem of three bodies when the shell is empty. Hallan and Rana [10] studied the linear stability of this same problem and showed that the equilibrium point is linearly stable under some conditions. Later, its nonlinear stability was investigated by Hallan and Mangang [6].

In this paper, we extend the work of Hallan and Rana [10] assuming that the masses of the primaries vary with time in accordance with the unified Meshcherskii [3] law and their motion determined by the GMP. The emergence of more equilibrium points near the center of the shell and infinite number of triangular equilibrium points is observed.

2. Equations of Motion

The relative motion of the two bodies m_1 and m_2 whose masses vary with time according to Gylden [1] and Meshcherskii [2] can be described in polar coordinates as

$$\ddot{r} - \frac{C^2}{r^3} + \frac{\mu}{r^2} = 0, \tag{1}$$

where

$$C = r^2\dot{\theta}, \tag{2}$$

$\mu(t) = \mu_1 + \mu_2$, $\mu_1 = Gm_1$, $\mu_2 = Gm_2$, $C \neq 0$ is a constant of the area integral; $\dot{\theta} = \omega(t)$ is the angular velocity of revolution of the bodies; G is the gravitational constant while r is the distance between the primaries. Equation (1) has the particular solutions of the type

$$\begin{gathered} \omega(t) = \frac{\omega_0}{R^2(t)}, \qquad r = \rho_{12}R(t), \\ x_1 = \xi_1 R(t), \qquad x_2 = \xi_2 R(t), \end{gathered} \tag{3}$$

where ξ_1 and ξ_2 are constants given by

$$\begin{gathered} \xi_1 = -\frac{\mu_{20}}{\mu_0}\rho_{12}, \quad \xi_2 = \frac{\mu_{10}}{\mu_0}\rho_{12}, \quad \omega_0, \rho_{12} = \text{constant}; \\ R(t) = \sqrt{\sigma t^2 + 2\delta t + \gamma} : \delta, \gamma, \mu_0, \mu_{10}, \mu_{20} \text{ are constants.} \end{gathered} \tag{4}$$

Also, (1) has a particular integral [4]

$$r\mu = \kappa C^2 : 0 < \kappa < \infty. \tag{5}$$

Now, let $m_1^*(t)$, $m_2(t)$, and m_3 be the masses of the first primary (which is a shell and a fluid of density ρ_1); second primary and the infinitesimal body, respectively. The masses of the primaries are assumed to vary with time isotropically and the first one contains the infinitesimal mass of density ρ_3, while the second one is a point mass located outside the first and describes a circular orbit around it. Let us suppose that R is the radius of the shell; M_1, M_2, and M_3 are the centers of the shell, $m_2(t)$ and m_3, respectively. Let the positions vector between the center of the mass of the first and second primary be $\vec{r}$ and those between the infinitesimal body and the primaries $\vec{r}_{13}$ and $\vec{r}_{23}$, respectively.

Following the terminologies of Robe [8], taking into account the gravitational and buoyancy forces exerted by the fluid, and the attraction of $m_2(t)$, the equations of motion of the third body of density ρ_3 in a rotating coordinate system under the effects of small perturbations in the Coriolis and centrifugal forces have the forms

$$\begin{gathered} \ddot{x} - 2\alpha\omega\dot{y} = \omega^2\beta x + \dot{\omega}\alpha y - \frac{\mu_2(x - x_2)}{r_{23}^3} - \frac{K(x - x_1)}{r_{13}^3}, \\ \ddot{y} + 2\alpha\omega\dot{x} = \omega^2\beta y - \dot{\omega}\alpha x - \frac{\mu_2 y}{r_{23}^3} - \frac{Ky}{r_{13}^3}, \\ \ddot{z} = -\frac{\mu_2 z}{r_{23}^3} - \frac{Kz}{r_{13}^3}, \end{gathered} \tag{6}$$

where $K = (4/3)R^3G\pi\rho_1(1 - (\rho_1/\rho_3))$, $r_{13}^2 = (x - x_1)^2 + y^2 + z^2$, $r_{23}^2 = (x - x_2) + y^2 + z^2$, $\alpha = 1 + \epsilon$, $|\epsilon| \ll 1$, $\beta = 1 + \epsilon'$, $|\epsilon'| \ll 1$.

Now, (6) does not fully autonomize to a system of equations with constant coefficients when the masses vary according to the combined Meshcherskii law and their motion determined by the GMP, except when the density parameter is zero (i.e., $K = 0$). In this regard, the equations of motion of the autonomized system after choosing units of measurements have the forms

$$\xi'' - 2\alpha\eta' = \frac{\partial U}{\partial \xi}, \qquad \eta'' + 2\alpha\xi' = \frac{\partial U}{\partial \eta}, \qquad \zeta'' = \frac{\partial U}{\partial \zeta}, \tag{7}$$

where

$$\begin{gathered} U = \frac{(\beta + \kappa - 1)(\xi^2 + \eta^2)}{2} + \frac{(\kappa - 1)\zeta^2}{2} + \frac{\kappa\upsilon}{\rho_{32}}, \\ \rho_{13} = \left[(\xi + \upsilon)^2 + \eta^2 + \zeta^2\right]^{1/2}, \\ \rho_{32} = \left[(\xi + \upsilon - 1)^2 + \eta^2 + \zeta^2\right]^{1/2}, \end{gathered} \tag{8}$$

where υ is the mass parameter and is such that $0 < \upsilon < 1$ while κ physically coincides with the sum of the masses of the primaries.

3. Equilibrium Points

The equilibrium points are found by solving the equations

$$U_\xi = U_\eta = U_\zeta = 0. \tag{9}$$

Equations (9) have the solutions of the type

$$(\beta+\kappa-1)\xi-\frac{\kappa\upsilon(\xi+\upsilon-1)}{\left[(\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{3/2}}=0,\quad \eta=0,\ \zeta=0, \tag{10}$$

$$(\beta+\kappa-1)\xi-\frac{\kappa\upsilon(\xi+\upsilon-1)}{\left[(\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{3/2}}=0,$$

$$\frac{\kappa-1}{\kappa}-\frac{\upsilon}{\left[(\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{3/2}}=0,\quad \eta=0,\ \zeta\neq 0, \tag{11}$$

$$(\beta+\kappa-1)\xi-\frac{\kappa\upsilon(\xi+\upsilon-1)}{\left[(\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{3/2}}=0,$$

$$(\beta+\kappa-1)-\frac{\kappa\upsilon}{\left[(\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{3/2}}=0,\quad \eta\neq 0,\ \zeta=0. \tag{12}$$

3.1. Points Near the Center. Now, substituting $\eta=\zeta=0$ in (10), and then denoting it by $f(\xi)$, we have

$$f(\xi)=(\beta+\kappa-1)\xi+\frac{\kappa\upsilon}{(\xi+\upsilon-1)^2}=0. \tag{13}$$

Now, $f'(\xi)>0$ for $\xi<1-\upsilon$; therefore, $f(\xi)$ is increasing in the open interval $(-\infty,1-\upsilon)$.

As $\xi\to-\infty$, $f(\xi)\to-\infty$ and as $\xi\to(1-\upsilon)$, $f(\xi)\to+\infty$; consequently, $f(\xi)$ vanishes only once in the interval $(-\infty,1-\upsilon)$ and indicates that (13) has only one root in this interval.

Now, when there is no perturbation in the centrifugal force, the only solution of (13) is $\xi=-\upsilon$. When a small perturbation is given, the solution of (13) can be assumed as

$$\xi_p=-\upsilon+\chi,\quad \chi\ll 1. \tag{14}$$

Substituting (14) in (10), solving and ignoring second- and higher-order term in χ, we get

$$\xi_p=-\upsilon+q(\beta-1), \tag{15}$$

where

$$q=\frac{\upsilon}{\kappa(1+2\upsilon)}. \tag{16}$$

These points lie to the right or left of the center of the shell depending on whether the perturbation is positive or negative. Further, due to the centrifugal force and the arbitrary constant κ $(0<\kappa<\infty)$, every point on the line joining the centers of the primaries is an equilibrium point, provided these points lie within the shell.

3.2. Triangular Points. The triangular points of the autonomized systems are the solutions of (11) when $\eta=0$, $\zeta\neq 0$, and $\kappa\neq 1$. Solving, we get

$$\xi=-(\kappa-1)(1-\upsilon)\left[1-(\beta-1)\right],$$
$$\zeta=\pm\left[\left(\frac{\kappa\upsilon}{\kappa-1}\right)^{2/3}+\kappa(1-\upsilon)^2\left\{2(\beta-1)(\kappa-1)-\kappa\right\}\right]^{1/2}. \tag{17}$$

This gives the position of a pair of equilibrium points which lie in the $\xi\zeta$-plane, forming triangles with the center of the shell and the second primary. These solutions depend on the mass ratio, small perturbation in the centrifugal force, and κ and exist only when $\kappa>1$. Numerically, it is observed that these equilibrium points do not exist in the Earth-Moon system when $\kappa<1.0036$ and $\kappa>1.01$ for any value of the perturbation in the centrifugal force chosen very small. We note that these points do not exist in any study of the Robe [8] when the density parameter is zero.

Equations (12) have no solution, since $0<\upsilon<1$. Hence, we conclude that all the equilibrium points perturbed by small perturbations in the Coriolis and centrifugal forces lie on the $\xi\zeta$-plane.

The equilibrium point near the center of the shell and the triangular equilibrium points of the nonautonomous system of (6), when $K=0$, are sought using the Meshcherskii's [3] transformation, in the forms

$$x^{(1)}(t)=-\upsilon q^{*}R(t),\qquad x^{(2)}(t)=\xi^{(2)}R(t),$$
$$z^{(2,3)}(t)=\zeta^{(2,3)}R(t), \tag{18}$$

where $q^{*}=[1-(\epsilon'/\kappa(1+2\upsilon))]$; $\xi^{(2)}$ and $\zeta^{(2,3)}$ are the triangular points of the autonomized systems.

4. Stability of Equilibrium Points

The test for stability of the equilibrium points of the nonautonomous system using the definition of a Lyapunov stable solution [11] on the points near the center of the shell yields

$$\lim_{t\to\infty}x(t)=-\upsilon\left[1-\frac{\epsilon'}{\kappa(1+2\upsilon)}\right]\lim_{t\to\infty}R(t)=-\infty. \tag{19}$$

This at once proves that these points are unstable according to Lyapunov's theorem. The same applies to the triangular equilibrium points.

In order to examine the linear stability of equilibrium points of the autonomized system, we displace the infinitesimal mass from its position (ξ_0,η_0,ζ_0) by applying small displacements u,v,w to the position (ξ,η,ζ) and then linearize (7) to obtain the variational equations

$$u''-2\alpha v'=\left(U^0_{\xi\xi}\right)u+\left(U^0_{\xi\eta}\right)v+\left(U^0_{\xi\zeta}\right)w,$$
$$v''+2\alpha u'=\left(U^0_{\xi\eta}\right)u+\left(U^0_{\eta\eta}\right)v+\left(U^0_{\eta\zeta}\right)w, \tag{20}$$
$$w''=\left(U^0_{\xi\zeta}\right)u+\left(U^0_{\eta\zeta}\right)v+\left(U^0_{\zeta\zeta}\right)w,$$

where the partial derivatives are computed at the equilibrium point under consideration.

4.1. Equilibrium Points Near the Center of the Shell. The values of the partial derivatives computed at the equilibrium point $(\xi_p, 0, 0)$ are

$$U^0_{\xi\xi} = \beta + \kappa - 1 + \frac{2\kappa\upsilon}{\left|\left[q(\beta-1)-1\right]^2\right|^{3/2}},$$
$$U^0_{\eta\eta} = \beta + \kappa - 1 - \frac{\kappa\upsilon}{\left|\left[q(\beta-1)-1\right]^2\right|^{3/2}},$$
$$U^0_{\zeta\zeta} = \kappa - 1 - \frac{\kappa\upsilon}{\left|\left[q(\beta-1)-1\right]^2\right|^{3/2}}, \qquad U^0_{\xi\zeta} = U^0_{\eta\zeta} = U^0_{\xi\eta} = 0. \tag{21}$$

Substituting (21) in (20), knowing that $\alpha = 1+\epsilon$, $\beta = 1+\epsilon'$ at once yields

$$u'' - 2\alpha v' = \left[\kappa(1+2\upsilon) + (1+6\kappa\upsilon q)\epsilon'\right]u,$$
$$v'' + 2\alpha u' = \left[\kappa(1-\upsilon) + (1-3\kappa\upsilon q)\epsilon'\right]v, \tag{22}$$
$$w'' = \left[\kappa - 1 - \kappa\upsilon\left(1+3q\epsilon'\right)\right]w.$$

The third equation of (22) does not depend on the other two and it shows that the motion parallel to the ζ-axis is stable when $\kappa < 1/(1-\upsilon(1+3q\epsilon'))$ and unstable when the reverse holds.

Now, the characteristic equation corresponding to the first two equations of (22) is given by

$$\lambda^4 + P\lambda^2 + Q = 0, \tag{23}$$

where

$$P = 4 + 8\epsilon - \left(2 + \frac{3\upsilon^2}{1+2\upsilon}\right)\epsilon' - \kappa(2+\upsilon),$$
$$Q = \kappa^2(1-\upsilon)(1+2\upsilon) + \kappa\epsilon'\left[(2+\upsilon) + \frac{3\upsilon^2}{1+2\upsilon}(1-4\upsilon)\right]. \tag{24}$$

Here, $Q > 0$, $P <=> 0$ when $\kappa <=> (4(1+2\upsilon) + 8\epsilon(1+2\upsilon) - (3\upsilon^2+4\upsilon+2)\epsilon')/(1+2\upsilon)(2+\upsilon)$, respectively.

The roots of (22) are

$$\lambda^2_{1,2} = \frac{-\left[4 + 8\epsilon - \kappa(2+\upsilon) - (2+3\kappa\upsilon q)\epsilon'\right] \pm \sqrt{D}}{2}, \tag{25}$$

where

$$D = -16(\kappa-1) + \kappa\upsilon(9\kappa\upsilon - 8) + 16(4 - 2\kappa - \kappa\upsilon)\epsilon + \frac{2\epsilon'}{1+2\upsilon}\left[27\kappa\upsilon^3 - 12\kappa\upsilon^2 - 16\upsilon - 8\right]. \tag{26}$$

Now, in the case $\epsilon = \epsilon' = 0$, (26) is zero only when $\upsilon_{C\kappa} = (4/9\kappa) + (4\sqrt{9\kappa-8}/9\kappa)$. Therefore, when the small perturbations are present, $D = 0$ when

$$\upsilon_{C\kappa p} = \frac{4}{9\kappa}\left[1 + \sqrt{a} + \frac{\epsilon}{\sqrt{a}}\left(9\kappa - 16 + 2\sqrt{a}\right) + \frac{3\epsilon'\left(27\kappa^2 - 120\kappa + 128 - 48\kappa\sqrt{a} + 56\sqrt{a}\right)}{2\kappa\sqrt{a}\left(9\kappa + 8 + 8\sqrt{a}\right)}\right], \tag{27}$$

where $a = 9\kappa - 8$.

Equation (27) gives the various values of υ, which exist for different values of κ and also depends on the small perturbations given. Thus, we denote them by $\upsilon_{C\kappa p}$ (see Table 1). These are the critical mass parameters and describe the joint effect of the involved parameters on the stability of motion around the equilibrium point $(\xi_p, 0, 0)$. When $\epsilon = \epsilon' = 0$, $\kappa = 1$, it becomes the same as that obtained by Robe [8].

Table 1 gives the critical mass values for $0 < \kappa < \infty$, and $|\epsilon| = 0.001$, $|\epsilon'| = 0.002$ when the perturbations are considered as positive or negative. We observe that (see Table 1) when $a \le 0$ the critical mass does not exist whether $\epsilon < 0$, $\epsilon > 0$, $\epsilon' > 0$, $\epsilon' < 0$ but exists for $a > 0$ and tends to zero as κ approaches infinity. Therefore, the values of $\upsilon_{C\kappa p}$ increases or decreases with increase in κ and whether the perturbations are positive or negative. Also, for $8/9 < \kappa \le 1.33$, the values increase with increase in κ, become unity at the peak, and begin to decrease to zero. Hence, stability holds in all the regions of κ except for $\kappa = 1.33$ when $\epsilon < 0$, $\epsilon' > 0$ and $\kappa = 1.33$ when $\epsilon > 0$, $\epsilon' < 0$ because in this case $\upsilon < 1 < \upsilon_{c\kappa p}$.

Figures 1(a)–1(f) give the graphs of $\upsilon_{C\kappa p}$ for $\epsilon = 0.001$, $\epsilon' = 0.002$ as κ approaches infinity. These graphs indicate that the region of stability is increasing with increase in κ.

We note from (27) that $dD/d\upsilon > 0$ whenever $\upsilon > 4/9\kappa[1 + 2\epsilon - (4\epsilon'(9\kappa+8)/27\kappa^3)(1-(16/9\kappa)-(64/81\kappa^2))]$ and negative when the converse holds. When $\upsilon = 0$, (27) reduces to $D = -16(\kappa-1) + 32(2-\kappa)\epsilon - 16\epsilon'$, decreases further, and becomes negative for any $0 < \kappa < \infty$ as υ increases from zero to $\upsilon = 4/9\kappa[1 + 2\epsilon - (4\epsilon'(9\kappa+8)/27\kappa^3)(1-(16/9\kappa)-(64/81\kappa^2))]$. Further, as υ increases from this value to $\upsilon_{c\kappa p}$, the discriminant vanishes. Finally, as $\upsilon_{c\kappa p}$ increases to unity, the discriminant increases to

$$D = -16(\kappa-1) + \kappa(9\kappa-8) + 2(4-3\kappa)\left(8\epsilon - 3\epsilon'\right). \tag{28}$$

Equation (28) is positive for $0 < \kappa < \infty$, depending on whether $(8\epsilon - 3\epsilon')$ is positive or negative.

When $\kappa = 1$, $\epsilon = \epsilon' = 0$, υ increases from 4/9 to 8/9 and to unity; the discriminant increases from $-(16/9)$ to 0 and to 1, respectively.

Now, the nature of the roots in (25) depends on the discriminant, small perturbations in the Coriolis and centrifugal forces, mass ratio, and the constant κ of a particular integral of the GMP. In view of this, we consider the three regions

Table 1: Numerical computation of the critical mass values $\upsilon_{скр}$.

κ	$\upsilon_{скр}(\epsilon < 0, \epsilon' > 0)$	$\upsilon_{скр}(\epsilon > 0, \epsilon' > 0)$	$\upsilon_{скр}(\epsilon < 0, \epsilon' < 0)$	$\upsilon_{скр}(\epsilon > 0, \epsilon' < 0)$
$0 < \kappa < 8/9$	Complex quantity	Complex quantity	Complex quantity	Complex quantity
8/9	Infinity	Infinity	Infinity	Infinity
0.889	0.782608	0.531688	0.499806	0.248886
0.9	0.674168	0.65147	0.648508	0.62581
0.95	0.82264	0.815112	0.814474	0.806946
1	0.893404	0.88896	0.888818	0.884373
1.3	0.999643	0.999482	0.999519	0.999358
1.33	1.00001	0.999993	0.999997	0.999982
1.34	0.999956	0.999985	0.999978	1.00001
1.35	0.999823	0.999896	0.999877	0.999949
2	0.924097	0.925267	0.924634	0.925804
3	0.79306	0.7944	0.793422	0.794762
4	0.698297	0.699581	0.69853	0.699815
5	0.628898	0.630101	0.629056	0.630259
6	0.57585	0.576976	0.575962	0.577088
7	0.533791	0.53485	0.533874	0.534932
8	0.499469	0.500469	0.499531	0.500531
9	0.47081	0.471759	0.470859	0.471807
10	0.446435	0.447339	0.446473	0.447377
50	0.195566	0.195969	0.195565	0.195968
1000	0.0425464	0.0426324	0.0425464	0.0426324
1000000000	0.000042122	0.0000422063	0.000042122	0.0000422063
10^{50}	1.332×10^{-25}	1.335×10^{-25}	1.332×10^{-25}	1.335×10^{-25}
$\kappa \to \infty$	$\upsilon_{скр} \to 0$	$\upsilon_{скр} \to 0$	$\upsilon_{скр} \to 0$	$\upsilon_{скр} \to 0$

of D together with the changes in P. When $\upsilon < \upsilon_{скр}$, the discriminant is negative; therefore, for $P > 0$ or $P < 0$, the real parts of the two of the roots are positive and equal in both cases and the equilibrium points are unstable. When $\upsilon = \upsilon_{скр}$ and $P < 0$, two of the roots are real and equal, while the other two are negative and equal as well, and so the equilibrium points are unstable. In the case when $P = 0$ all the roots are zero and the points are unstable. Also, when $P > 0$, all four roots are imaginary, with two positive and equal and the other two negative and equal. In this case we have positive stable resonance. Lastly, when $\upsilon_{скр} < \upsilon$, the discriminant is positive and if $P > 0$, then the roots are distinct and imaginary and the equilibrium points are stable. In the situation $P < 0$, the roots are real and distinct and the equilibrium points are unstable due to a positive root. Thus, we conclude that the equilibrium points near the center of the shell perturbed by the small perturbation in the centrifugal force and κ are unstable for $\upsilon < \upsilon_{скр}$ and $\upsilon_{скр} \leq \upsilon < 1$ and stable for $\upsilon_{скр} \leq \upsilon < 1$, depending on the arbitrary constant κ of the GMP, the mass ratio υ, and small perturbations given in the Coriolis and centrifugal forces.

Now, we consider the straight line $AO'B$ (Figure 2) whose equation in (ϵ, ϵ') plane is

$$p = \frac{2\epsilon}{3}\left(9\kappa - 16 + 2\sqrt{a}\right) + \frac{\epsilon'\left(27\kappa^2 - 120\kappa + 128 - 48\kappa\sqrt{a} + 56\sqrt{a}\right)}{\kappa\left(9\kappa + 8 + 8\sqrt{a}\right)} = 0 \tag{29}$$

and divides the plane into two parts giving four regions I, II, III, and IV.

When $\kappa = 1$, in (29), we get $(43/25)\epsilon' - (10/3)\epsilon = 0$, which is the same as found by Hallan and Rana [10].

Now, firstly for the points in the region I, (29) is positive only when $8/9 < \kappa < 1.34$ and negative for $1.34 \leq \kappa < \infty$. Therefore, for the point $(-\epsilon, \epsilon')$ in the region I and $8/9 < \kappa < 1.34$, $\upsilon_{Cкр} > (4/9\kappa)(1 + \sqrt{9\kappa - 8})$ and so the range of stability is decreasing as κ is increasing. Conversely, in the same region I, when $1.34 \leq \kappa < \infty$ then $\upsilon_{Cкр} < (4/9\kappa)(1 + \sqrt{9\kappa - 8})$ and so the range of stability is increasing as κ is increasing. Secondly, for the point (ϵ, ϵ') in the region II, (29) is always positive for

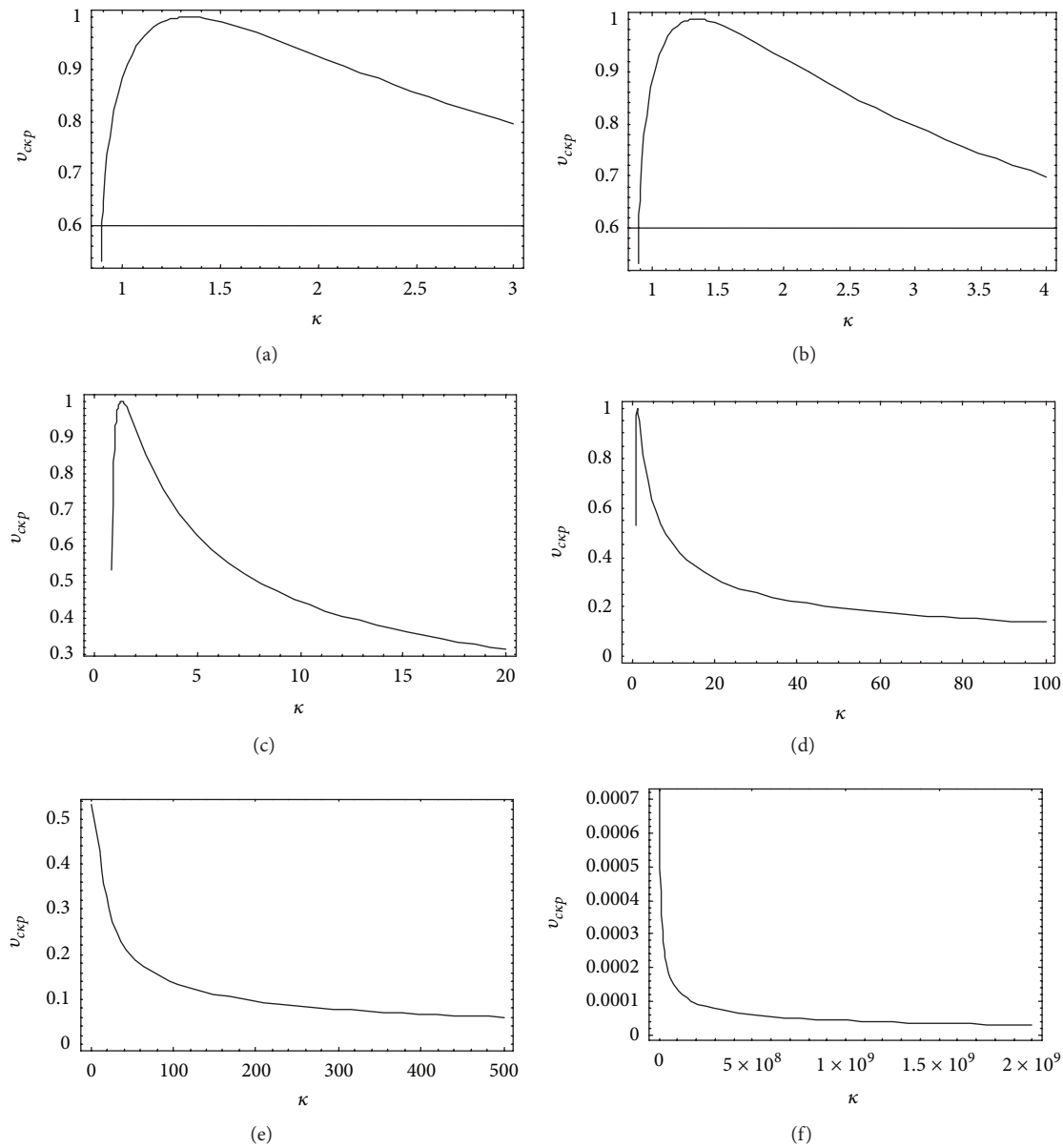

Figure 1: Region of stable motion.

$8/9 < \kappa < \infty$ and the range of stability is decreasing in this region. In the third region III, (29) is negative when $8/9 < \kappa < 1.34$ and positive when $1.34 \leq \kappa < \infty$. Consequently, for the point $(\epsilon, -\epsilon')$ the range of stability is increasing and decreasing as κ is increasing, respectively. Fourthly, for the region IV containing the point $(-\epsilon, -\epsilon')$, here, (29) is negative for $8/9 < \kappa < 1$, positive for $1 < \kappa < 1.34$, and again negative when $1.34 \leq \kappa < \infty$. Therefore, in this region, the range of stability increases, decreases, and finally increases as κ is increasing. Lastly, for the points on the line $AO'B$, the range of stability remains unaltered.

Now, for the point lying on the ϵ-axis, $\epsilon' = 0$; therefore, when $\epsilon > 0$, (29) is negative when $8/9 < \kappa < 1.34$, and positive for $\kappa \geq 1.34$; in these cases, the range of stability is increasing and decreasing, respectively, as κ is increasing. Thus, in our case, the Coriolis force is both a stabilizing and destabilizing force when the centrifugal force is kept constant, because of the presence of the constant κ of a particular of the GMP. Conversely, for the point on the ϵ-axis, $\epsilon' = 0$, we see that, when $\epsilon < 0$, (29) is positive when $8/9 < \kappa < 1.34$, and negative for $\kappa \geq 1.34$; in these cases, the range of stability is decreasing and increasing respectively with increase in κ, and so the Coriolis force is again both a stabilizing and destabilizing force when the centrifugal force is kept constant.

For the point lying on the ϵ'-axis, $\epsilon = 0$; hence, when $\epsilon' > 0$, (29) is positive throughout the physically possible interval $8/9 < \kappa < \infty$. Here, the range of stability decreases as κ is increases, establishing that the centrifugal force is a destabilizing force when the Coriolis force is absent. Similarly, for $\epsilon' < 0$, the range of stability increases for $8/9 < \kappa < 1.34$,

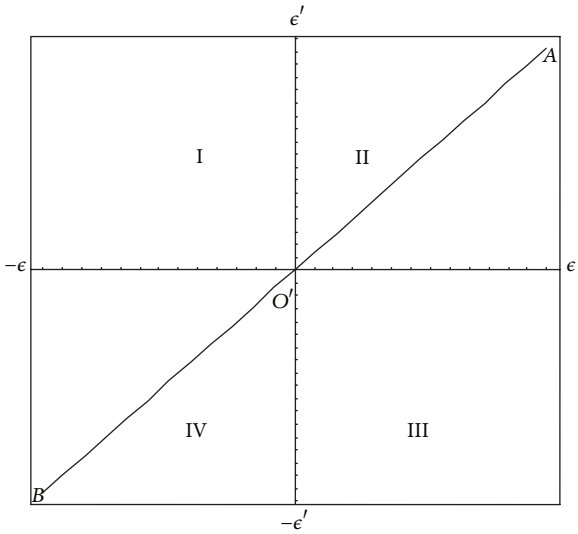

Figure 2: Four regions of the (ϵ, ϵ') plane.

decreases for $1.34 \leq \kappa < 33.9$, and afterwards increases for $33.9 \leq \kappa < \infty$. Hence, the centrifugal force has both stabilizing and destabilizing behavior when the Coriolis force is kept constant.

4.2. Stability of the Triangular Points. To determine the stability of the triangular equilibrium points, we consider the following partial derivatives:

$$U^0_{\xi\xi} = \beta + \frac{3\kappa(1-\upsilon)^2(\kappa-1)^{5/3}\left(\kappa+2\epsilon'\right)}{(\kappa\upsilon)^{2/3}},$$

$$U^0_{\xi\zeta} = \frac{-3\zeta\left(\kappa-\kappa\epsilon'+\epsilon'\right)(1-\upsilon)(\kappa-1)^{5/3}}{(\kappa\upsilon)^{2/3}},$$

$$U^0_{\zeta\zeta} = 3(\kappa-1)\left[1-\frac{\kappa(1-\upsilon)^2(\kappa-1)^{2/3}\left(\kappa-2\kappa\epsilon'+2\epsilon'\right)}{(\kappa\upsilon)^{2/3}}\right]$$

$$U^0_{\eta\eta} = \beta, \qquad U^0_{\xi\eta} = U^0_{\eta\zeta} = 0. \tag{30}$$

Substituting (30) in the variational equations (20), we obtain the characteristic equation of the triangular points expressed as

$$\lambda^6 - a_1\lambda^4 + a_2\lambda^2 - a_3 = 0, \tag{31}$$

where

$$\begin{aligned}
a_1 &= 3\kappa - 5 - 8\epsilon + 2\epsilon'\left[1+\frac{3\kappa^2(1-\upsilon)^2(\kappa-1)^{5/3}}{(\kappa\upsilon)^{2/3}}\right],\\
a_2 &= 1 + 2\epsilon' + \frac{3\kappa\epsilon'(\kappa+1-\upsilon)(1-\upsilon)(\kappa-1)^{5/3}}{(\kappa\upsilon)^{2/3}}\\
&\quad + \frac{18\kappa\epsilon'(1-\upsilon)^2(\kappa-1)^{11/3}}{(\kappa\upsilon)^{2/3}}\\
&\quad + \frac{36\kappa^4\epsilon'(1-\upsilon)^2(\kappa-1)^{13/3}}{(\kappa\upsilon)^{4/3}}\\
&\quad - \frac{18\kappa^3(1-\upsilon)^4(\kappa-1)^{13/3}}{(\kappa\upsilon)^{4/3}}\\
&\quad + \frac{3\kappa^2(1-\upsilon)^2(\kappa-1)^{5/3}}{(\kappa\upsilon)^{4/3}}\\
&\quad + \frac{18\kappa^2(1-\upsilon)^2(\kappa-1)^{8/3}}{(\kappa\upsilon)^{2/3}}\\
&\quad + \frac{18\kappa^4(1-\upsilon)^4(\kappa-1)^{10/3}}{(\kappa\upsilon)^{4/3}},\\
a_3 &= 7 - 6\kappa - \frac{3\kappa^2(\kappa-1)^{5/3}}{(\kappa\upsilon)^{2/3}}\\
&\quad \times\left[\frac{3\zeta^2(\kappa-1)^{5/3}}{(\kappa\upsilon)^{2/3}} - 2(1-\upsilon)^2\right.\\
&\qquad \left. -3(\kappa-1)(1-\upsilon)^2 + \frac{3\kappa^2(1-\upsilon)^4(\kappa-1)^{5/3}}{(\kappa\upsilon)^{2/3}}\right]\\
&\quad - 24\epsilon\left[\kappa-1+\frac{\kappa^2(1-\upsilon)^2(\kappa-1)^{5/3}}{(\kappa\upsilon)^{2/3}}\right]\\
&\quad + \epsilon'\left[\frac{3\kappa^2(1-\upsilon)^{5/3}}{(\kappa\upsilon)^{2/3}} + 3\kappa - 4\right.\\
&\qquad - \frac{3(5-6\kappa)(1-\upsilon)^2(\kappa-1)^{5/3}}{(\kappa\upsilon)^{2/3}}\\
&\qquad + \frac{18\kappa(1-\upsilon)^2(\kappa-1)^{8/3}}{(\kappa\upsilon)^{2/3}}\\
&\qquad + \frac{18\kappa^4(1-\upsilon)^4(\kappa-1)^{10/3}}{(\kappa\upsilon)^{4/3}}\\
&\qquad - \frac{36\kappa^3(1-\upsilon)^4(\kappa-1)^{10/3}}{(\kappa\upsilon)^{4/3}}\\
&\qquad - \frac{18\kappa\zeta^2(\kappa-1)^{10/3}}{(\kappa\upsilon)^{4/3}}\\
&\qquad \left. + \frac{18\kappa^2\zeta^2(\kappa-1)^{10/3}}{(\kappa\upsilon)^{4/3}}\right].
\end{aligned} \tag{32}$$

The linear stability of the triangular points is determined by the roots of the characteristic equation (31). We explored the equilibrium points, the partial derivatives, and values of the coefficients of equation (31) numerically using the software package *Mathematica*, for $\kappa > 1$, $\alpha = 1.001$, $\beta = 1.002$, and $0 < \upsilon < 1$ and observed that the signs of the quantities a_1, a_2, and a_3 could be negative or positive depending on the interval where κ lies. The following cases arise:

(i) $a_1 < 0$, $a_2 > 0$, and $a_3 < 0$; there is one change in sign, which implies there is exactly one positive root according to the Descartes rule of sign,

(ii) $a_1 > 0$, $a_2 < 0$, and $a_3 < 0$; there is also one change in sign and at least one positive root,

(iii) $a_1 > 0$, $a_2 < 0$, and $a_3 > 0$; there are two changes in sign indicating there two positive, two negative, and two complex roots.

A positive root and positive real part of the complex roots induce instability at the triangular point. Hence, we conclude that the triangular points of Robe's problem of two variable mass bodies when the shell is empty are unstable.

5. Conclusion

We have established the equations of motion of an infinitesimal body in the frame of Robe's [8] circular restricted three-body problem under effects of small perturbations in the Coriolis and centrifugal forces, when the density parameter is zero and the masses of the primaries vary isotropically with time in accordance with the Meshcherskii [3] combined law. The loss or gain of mass to the masses of the primaries acts like a perturbation and the problem is dealt with using the typical perturbation method. The equations of motion (7) of the autonomized system is different from those in Shrivastava and Garain [9] and Hallan and Rana [10].

We also investigated the possible equilibrium points and found that the equilibrium point at the center of the shell found by Robe [8] is shifted due to a small perturbation in the centrifugal force and the parameter κ to the left or right of the center of the shell accordingly as whether the perturbation is positive or negative. Also, because of the small perturbation in the centrifugal force, which allows the appearance of the parameter κ, every point inside the shell on the line joining the centers of the primaries is an equilibrium point and contrary to those of Shrivastava and Garain [9] and Hallan and Rana [10], where there is only one. Further, for some values of the parameter $\kappa > 1$, a number of pair of equilibrium points forming triangles with the center of the shell and the second primary exists. When $\kappa = 1$, these points do not exist and the only one equilibrium point is that found by Shrivastava and Garain [9]. When there is no perturbation in the centrifugal force, this point fully coincides with that of Robe [8].

The linear stability of the equilibrium point near the center of the shell is unstable for $\upsilon < \upsilon_{скр}$ and $\upsilon_{скр} \leq \upsilon < 1$ and stable for $\upsilon_{скр} \leq \upsilon < 1$, depending on the arbitrary constant κ of the GMP, the mass ratio υ, and small perturbations given in the Coriolis and centrifugal forces. When $\kappa = 1$, the range of stable motion is accordant with Hallan and Rana [10]. The increase, decrease, or unchanged range of stability depends on the constant κ of the GMP and whether these perturbations are positive or negative. When there are no perturbations the range of stable motion reduces to that of Robe [8]. The triangular points are unstable due to a positive root and positive real part of the complex roots of the characteristic equation of sixth degree.

We have assumed that the density parameter is zero, not because previous authors [8–10] have done so, but because the equations of motion with variable coefficients does not fully autonomize to that with constant coefficients when the masses vary isotropically and their motion is governed by the GMP.

Modern concept about the formation of celestial bodies and their evolution lead to the necessity of investigating dynamics problems in celestial mechanics in which the motion of the bodies is determined by nonsteady gravitational and nonreactive forces. This study may be used to investigate the dynamic problem in the Earth-Moon system since the distance between them is changing and the masses are also changing due to meteoric activities, space dust, outgassing, and so forth.

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