Research Article

Positive Solutions for a System of Fourth-Order p -Laplacian Boundary Value Problems

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We investigate the existence of positive solutions for the system of fourth-order p -Laplacian boundary value problems $(|u''|^{p-1}u'')'' = f_1(t, u, v)$, $(|v''|^{q-1}v'')'' = f_2(t, u, v)$, $u^{(2i)}(0) = u^{(2i)}(1) = 0$, $i = 0, 1$, $v^{(2i)}(0) = v^{(2i)}(1) = 0$, $i = 0, 1$, where $p, q > 0$ and $f_1, f_2 \in C([0, 1] \times \mathbb{R}_+^2, \mathbb{R}_+)$ ($\mathbb{R}_+ := [0, \infty)$). Based on a priori estimates achieved by utilizing Jensen's integral inequalities and nonnegative matrices, we use fixed point index theory to establish our main results.

1. Introduction

This paper is concerned with the existence of positive solutions for the system of fourth-order p -Laplacian boundary value problems

$$\begin{aligned} (|u''|^{p-1}u'')'' &= f_1(t, u, v), \\ (|v''|^{q-1}v'')'' &= f_2(t, u, v), \\ u^{(2i)}(0) &= u^{(2i)}(1) = 0, \quad i = 0, 1, \\ v^{(2i)}(0) &= v^{(2i)}(1) = 0, \quad i = 0, 1, \end{aligned} \quad (1)$$

where $p, q > 0$ and $f_1, f_2 \in C([0, 1] \times \mathbb{R}_+^2, \mathbb{R}_+)$ ($\mathbb{R}_+ := [0, \infty)$).

In recent years, boundary value problems for fourth-order nonlinear ordinary differential equations have been extensively studied, with many papers on this direction published. See [1–4] and the references cited therein. The so-called p -Laplacian boundary value problems arise in non-Newtonian mechanics, nonlinear elasticity, glaciology, population biology, combustion theory, and nonlinear flow laws; see [5, 6]. That explains why many authors have extensively studied the existence of positive solutions for p -Laplacian

boundary value problems, by using topological degree theory, monotone iterative techniques, coincidence degree theory [7], and the Leggett-Williams fixed point theorem [8] or its variants; see [9–18] and the references therein.

To the best of our knowledge, problem (1) is a new topic in the existing literature. Closely related to our work here is [19] that studies the existence and multiplicity of positive solutions for the system of p -Laplacian boundary value problems

$$\begin{aligned} -\left((u'_i)^{p_i-1}\right)' &= f_i(t, u_1, \dots, u_n), \quad u_i(0) = u'_i(1) = 0, \\ i &= 1, \dots, n, \end{aligned} \quad (2)$$

where $n \geq 2$, $p_i > 1$, $f_i \in C([0, 1] \times \mathbb{R}_+^n, \mathbb{R}_+)$ ($i = 1, \dots, n$). Based on a priori estimates achieved by utilizing the Jensen integral inequalities and $\mathbb{R}_+^n$ -monotone matrices, the author used fixed point index theory to establish the existence and multiplicity of positive solutions for the previous problem. For more details of the recent progress in the boundary value problems for systems of nonlinear ordinary differential equations, we refer the reader to [19–27] and the references cited therein.

We observe that a close link exists between (1) and the problem below

$$\begin{aligned} u^{(4)} &= f_1(t, u, v), & v^{(4)} &= f_2(t, u, v), \\ u^{(2i)}(0) &= u^{(2i)}(1) = 0, & i &= 0, 1, \\ v^{(2i)}(0) &= v^{(2i)}(1) = 0, & i &= 0, 1, \end{aligned} \quad (3)$$

and this link can be established by Jensen's integral inequalities for concave functions and convex functions. In other words, (1) may be regarded as a perturbation of (3). With this perspective, a priori estimates of positive solutions for some problems associated with (1) can be derived by utilizing Jensen's integral inequalities. It is the a priori estimates that permit us to use fixed point index theory to establish our main results.

This paper is organized as follows. In Section 2, we provide some preliminary results. Our main results, namely, Theorems 6 and 7, are stated and proved in Section 3.

2. Preliminaries

We first make the following hypothesis throughout this paper:

$$(H1) \quad f_1, f_2 \in C([0, 1] \times \mathbb{R}_+^2, \mathbb{R}_+).$$

Let $E := C[0, 1]$ and

$$\begin{aligned} P &:= \{u \in E : u(t) \geq 0, \forall t \in [0, 1]\}, \\ \|u\| &:= \max_{t \in [0, 1]} |u(t)|; \end{aligned} \quad (4)$$

then $(E, \|\cdot\|)$ is a real Banach space and P is a cone in E . It is easy to see that (1) is equivalent to the system of nonlinear integral equations

$$\begin{aligned} u(t) &= \int_0^1 k(t, s) \left(\int_0^1 k(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right)^{1/p} ds, \\ v(t) &= \int_0^1 k(t, s) \left(\int_0^1 k(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right)^{1/q} ds, \end{aligned} \quad (5)$$

where

$$k(t, s) := \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \quad (6)$$

Define the operators A_i ($i = 1, 2$): $P \rightarrow P$ and $A : P \times P \rightarrow P \times P$ by

$$\begin{aligned} A_1(u, v)(t) &:= \int_0^1 k(t, s) \left(\int_0^1 k(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right)^{1/p} ds, \\ A_2(u, v)(t) &:= \int_0^1 k(t, s) \left(\int_0^1 k(s, \tau) f_2(\tau, u(\tau), v(\tau)) d\tau \right)^{1/q} ds, \\ A(u, v)(t) &:= (A_1, A_2)(u, v)(t). \end{aligned} \quad (7)$$

Obviously, under condition (H1), the operators A_i ($i = 1, 2$): $P \rightarrow P$ and $A : P \times P \rightarrow P \times P$ are completely continuous operators. In our setting, the existence of positive solutions for (1) is equivalent to that of positive fixed points of A .

Below are some elementary inequalities.

Lemma 1. Let $\varphi(t) := \sin \pi t$, then $\varphi \in P \setminus \{0\}$ and

$$\begin{aligned} \int_0^1 k(t, s) \varphi(t) dt &= \frac{1}{\pi^2} \varphi(s), \\ \int_0^1 k(t, s) \varphi(s) ds &= \frac{1}{\pi^2} \varphi(t). \end{aligned} \quad (8)$$

Lemma 2. Suppose that $\psi \in P$ is concave on $[0, 1]$. Then

$$\int_0^1 \psi(t) \sin \pi t dt \geq \frac{2}{\pi^2} \|\psi\|. \quad (9)$$

Lemma 3. Let $a \in \mathbb{R}_+$, $b \in \mathbb{R}_+$. If $\sigma \in (0, 1]$, then

$$(a + b)^\sigma \geq 2^{\sigma-1} (a^\sigma + b^\sigma). \quad (10)$$

If $\sigma \in [1, +\infty)$, then

$$(a + b)^\sigma \leq 2^{\sigma-1} (a^\sigma + b^\sigma). \quad (11)$$

Lemma 4 (see [28]). Let E be a real Banach space, and K is a cone in E . Suppose that $\Omega \subset E$ is a bounded open set and $T : \overline{\Omega} \cap K \rightarrow K$ is a completely continuous operator. If there exists $u_0 \in K \setminus \{0\}$ such that

$$u - Tu \neq \lambda u_0, \quad \forall \lambda > 0, u \in \partial\Omega \cap K, \quad (12)$$

then $i(T, \Omega \cap K, K) = 0$, where i indicates the fixed point index on K .

Lemma 5 (see [28]). Let E be a real Banach space, and K is a cone in E . Suppose that $\Omega \subset E$ is a bounded open set, $0 \in \Omega$, and $T : \overline{\Omega} \cap K \rightarrow K$ is a completely continuous operator. If

$$u - \lambda Tu \neq 0, \quad \forall \lambda \in [0, 1], u \in \partial\Omega \cap K, \quad (13)$$

then $i(T, \Omega \cap K, K) = 1$.

3. Main Results

Let $p^* := \max\{1, p\}$, $p_* := \min\{1, p\}$, $q^* := \max\{1, q\}$, $q_* := \min\{1, q\}$. Then the previous constants satisfy the simple relations

$$\begin{aligned} 0 < p_* \leq 1 \leq p^*, \quad 0 < q_* \leq 1 \leq q^*, \\ p_* p^* = p, \quad q_* q^* = q. \end{aligned} \quad (14)$$

We now list our hypotheses on f_1 and f_2 .

(H2) There exist five constants $a_{i1}, b_{i1} \geq 0, c > 0$ such that

$$\begin{aligned} f_1(t, x, y) &\geq a_{11}x^{p^*q_*} + b_{11}y^{p^*q_*} - c, \\ f_2(t, x, y) &\geq a_{21}x^{q^*p_*} + b_{21}y^{q^*p_*} - c \end{aligned} \quad (15)$$

for all $x, y \in \mathbb{R}^+$, $t \in [0, 1]$, and the matrix $A_1 := \begin{pmatrix} a_{11} & b_{11} \\ a_{21} & b_{21} \end{pmatrix}$ is invertible with A_1^{-1} nonnegative, where

$$\begin{aligned} a_1 &:= 2^{p_*/p-1} a_{11}^{p_*/p} \pi^{-4}, \\ b_1 &:= 2^{p_*/p-1} b_{11}^{p_*/p} \pi^{-4}, \\ a_2 &:= 2^{q_*/q-1} a_{21}^{q_*/q} \pi^{-4}, \\ b_2 &:= 2^{q_*/q-1} b_{21}^{q_*/q} \pi^{-4}. \end{aligned} \quad (16)$$

(H3) There exist five constants $c_{i1}, d_{i1} \geq 0, r > 0$ such that

$$\begin{aligned} f_1(t, x, y) &\leq c_{11}x^{p_*p^*} + d_{11}y^{p_*q^*}, \\ f_2(t, x, y) &\leq c_{21}x^{q_*p^*} + d_{21}y^{q_*q^*} \end{aligned} \quad (17)$$

for all $x, y \in [0, r]$, $t \in [0, 1]$, and the matrix $A_2 := \begin{pmatrix} 1-c_1 & -d_1 \\ -c_2 & 1-d_2 \end{pmatrix}$ is invertible with A_2^{-1} nonnegative, where

$$\begin{aligned} c_1 &:= 2^{p_*/p-1} c_{11}^{p_*/p} \pi^{-4}, \\ d_1 &:= 2^{p_*/p-1} d_{11}^{p_*/p} \pi^{-4}, \\ c_2 &:= 2^{q_*/q-1} c_{21}^{q_*/q} \pi^{-4}, \\ d_2 &:= 2^{q_*/q-1} d_{21}^{q_*/q} \pi^{-4}. \end{aligned} \quad (18)$$

(H4) There exist five constants $l_{i1}, m_{i1} \geq 0$ and, a real number $r > 0$ such that

$$\begin{aligned} f_1(t, x, y) &\geq l_{11}x^{p_*p^*} + m_{11}y^{p_*q^*}, \\ f_2(t, x, y) &\geq l_{21}x^{q_*p^*} + m_{21}y^{q_*q^*} \end{aligned} \quad (19)$$

for all $x, y \in [0, r]$, $t \in [0, 1]$, and the matrix $A_3 := \begin{pmatrix} 1-l_1 & -m_1 \\ -l_2 & 1-m_2 \end{pmatrix}$ is invertible with A_3^{-1} nonnegative, where

$$\begin{aligned} l_1 &:= 2^{p_*/p-1} l_{11}^{p_*/p} \pi^{-4}, \\ m_1 &:= 2^{p_*/p-1} m_{11}^{p_*/p} \pi^{-4}, \end{aligned}$$

$$l_2 := 2^{q_*/q-1} l_{21}^{q_*/q} \pi^{-4},$$

$$m_2 := 2^{q_*/q-1} m_{21}^{q_*/q} \pi^{-4}.$$

(20)

(H5) There exist five constants $p_{i1}, q_{i1} \geq 0, c > 0$ such that

$$f_1(t, x, y) \leq p_{11}x^{p_*p^*} + q_{11}y^{p_*q^*} + c, \quad (21)$$

$$f_2(t, x, y) \leq p_{21}x^{q_*p^*} + q_{21}y^{q_*q^*} + c$$

for all $x, y \in \mathbb{R}^+$, $t \in [0, 1]$, and the matrix $A_4 := \begin{pmatrix} p_{11} & q_{11} \\ p_{21} & q_{21} \end{pmatrix}$ is invertible with A_4^{-1} nonnegative, where

$$\begin{aligned} p_1 &:= 4^{p_*/p-1} p_{11}^{p_*/p} \pi^{-4}, \\ q_1 &:= 4^{p_*/p-1} q_{11}^{p_*/p} \pi^{-4}, \\ p_2 &:= 4^{q_*/q-1} p_{21}^{q_*/q} \pi^{-4}, \\ q_2 &:= 4^{q_*/q-1} q_{21}^{q_*/q} \pi^{-4}. \end{aligned} \quad (22)$$

In the sequel, we set $\Omega_\rho := \{(u, v) \in E^2 : \|(u, v)\| < \rho\}$ for all $\rho > 0$.

Theorem 6. If (H1)–(H3) hold, then (1) has at least one positive solution.

Proof. Let

$$\mathcal{M}_1 := \{(u, v) \in P \times P : (u, v) = A(u, v) + \lambda(\varphi, \varphi), \lambda \geq 0\}, \quad (23)$$

where $\varphi \in P \setminus \{0\}$ is defined by $\varphi(t) := \sin \pi t$. We want to show that $\mathcal{M}_1$ is bounded. Indeed, if $(u, v) \in \mathcal{M}_1$, then there exists $\lambda \geq 0$ such that

$$(u, v) = A(u, v) + \lambda(\varphi, \varphi), \quad (24)$$

and thus $(u, v) \geq A(u, v)$. Note that $u, v \in \mathcal{M}_1$ implies that u, v are concave. So are $u^{p_*}, v^{q_*} \in P$ with $p_*, q_* \in (0, 1]$. Now Lemma 2 implies that

$$\|u^{p_*}\| \leq \frac{\pi^2}{2} \int_0^1 u^{p_*}(t) \sin \pi t \, dt, \quad (25)$$

$$\|v^{q_*}\| \leq \frac{\pi^2}{2} \int_0^1 v^{q_*}(t) \sin \pi t \, dt$$

hold for every $(u, v) \in \mathcal{M}_1$. Note that $k(t, s) \in [0, 1]$ for all $t, s \in [0, 1]$ and $p_*, p_*/p \in (0, 1]$. By (H2), Jensen's integral inequalities, and Lemma 3, we obtain

$$\begin{aligned}
 u^{p_*}(t) &\geq \left(\int_0^1 k(t, s) \left(\int_0^1 k(s, \tau) f_1(\tau, u(\tau), v(\tau)) d\tau \right)^{1/p} ds \right)^{p_*} \\
 &\geq \iint_0^1 k(t, s) k(s, \tau) f_1^{p_*/p}(\tau, u(\tau), v(\tau)) d\tau ds \\
 &\geq \iint_0^1 k(t, s) k(s, \tau) \\
 &\quad \times \left(\left(a_{11} u^{p_* p}(\tau) + b_{11} v^{p_* q_*}(\tau) \right)^{p_*/p} - c^{p_*/p} \right) d\tau ds \\
 &\geq 2^{p_*/p-1} \iint_0^1 k(t, s) k(s, \tau) \\
 &\quad \times \left(a_{11}^{p_*/p} u^{p_*}(\tau) + b_{11}^{p_*/p} v^{q_*}(\tau) - c^{p_*/p} \right) d\tau ds
 \end{aligned} \tag{26}$$

for all $u, v \in \mathcal{M}_1, t \in [0, 1]$. Note $q_*, q_*/q \in (0, 1]$. The same argument for the previous inequality can be used to obtain

$$\begin{aligned}
 v^{q_*}(t) &\geq 2^{q_*/q-1} \iint_0^1 k(t, s) k(s, \tau) \\
 &\quad \times \left(a_{21}^{q_*/q} u^{p_*}(\tau) + b_{21}^{q_*/q} v^{q_*}(\tau) - c^{q_*/q} \right) d\tau ds
 \end{aligned} \tag{27}$$

for all $u, v \in \mathcal{M}_1, t \in [0, 1]$. Multiply the last two inequalities by $\varphi(t)$, integrate over $[0, 1]$, and use Lemma 1 and (H2) to obtain

$$\begin{aligned}
 \int_0^1 u^{p_*}(t) \sin \pi t dt &\geq a_1 \int_0^1 u^{p_*}(t) \sin \pi t dt \\
 &\quad + b_1 \int_0^1 v^{q_*}(t) \sin \pi t dt - h_1, \\
 \int_0^1 v^{q_*}(t) \sin \pi t dt &\geq a_2 \int_0^1 u^{p_*}(t) \sin \pi t dt \\
 &\quad + b_2 \int_0^1 v^{q_*}(t) \sin \pi t dt - h_2,
 \end{aligned} \tag{28}$$

for every $(u, v) \in \mathcal{M}_1$, where $h_1 := 2^{p_*/p} c^{p_*/p} \pi^{-5}$, $h_2 := 2^{q_*/q} c^{q_*/q} \pi^{-5}$. The last two inequalities can be written as

$$\begin{aligned}
 &\begin{pmatrix} a_1 - 1 & b_1 \\ a_2 & b_2 - 1 \end{pmatrix} \begin{pmatrix} \int_0^1 u^{p_*}(t) \sin \pi t dt \\ \int_0^1 v^{q_*}(t) \sin \pi t dt \end{pmatrix} \\
 &= A_1 \begin{pmatrix} \int_0^1 u^{p_*}(t) \sin \pi t dt \\ \int_0^1 v^{q_*}(t) \sin \pi t dt \end{pmatrix} \leq \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.
 \end{aligned} \tag{29}$$

Now (H2) implies

$$\begin{pmatrix} \int_0^1 u^{p_*}(t) \sin \pi t dt \\ \int_0^1 v^{q_*}(t) \sin \pi t dt \end{pmatrix} \leq A_1^{-1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}. \tag{30}$$

Consequently, there exists a constant $M > 0$ such that

$$\int_0^1 u^{p_*}(t) \sin \pi t dt \leq M, \quad \int_0^1 v^{q_*}(t) \sin \pi t dt \leq M. \tag{31}$$

Recalling (25), we obtain

$$\|u^{p_*}\| \leq \frac{\pi^2 M}{2}, \quad \|v^{q_*}\| \leq \frac{\pi^2 M}{2}, \quad \forall (u, v) \in \mathcal{M}_1. \tag{32}$$

Therefore, $\mathcal{M}_1$ is bounded, as required. Taking $R > \sup\{\|(u, v)\| : (u, v) \in \mathcal{M}_1\}$, we have

$$\begin{aligned}
 (u, v) &\neq A(u, v) + \lambda(\varphi, \varphi), \quad \forall (u, v) \in \partial Q_R \cap (P \times P), \\
 \lambda &\geq 0.
 \end{aligned} \tag{33}$$

Now Lemma 4 yields

$$i(A, Q_R \cap (P \times P), P \times P) = 0. \tag{34}$$

Let

$$\mathcal{M}_2 := \{(u, v) \in \overline{Q_r} \cap (P \times P) : (u, v) = \lambda A(u, v), \lambda \in [0, 1]\}. \tag{35}$$

Now we want to show that $\mathcal{M}_2 = \{0\}$. Indeed, if $(u, v) \in \mathcal{M}_2$, then there exists $\lambda \in [0, 1]$ such that $(u, v) = \lambda A(u, v)$, and thus $(u, v) \leq A(u, v)$ for all $(u, v) \in \mathcal{M}_2$. Note that

$k(t, s) \in [0, 1]$ for all $t, s \in [0, 1]$ and $p^*, p^*/p \geq 1$. By (H3), Jensen's integral inequalities, and Lemma 3, we obtain

$$\begin{aligned}
 u^{p^*}(t) &\leq \left(\int_0^1 k(t, s) \left(\int_0^1 k(s, \tau) \right. \right. \\
 &\quad \left. \left. \times f_1(\tau, u(\tau), v(\tau)) d\tau \right)^{1/p} ds \right)^{p^*} \\
 &\leq \iint_0^1 k(t, s) k(s, \tau) f_1^{p^*/p}(\tau, u(\tau), v(\tau)) d\tau ds \\
 &\leq \iint_0^1 k(t, s) k(s, \tau) \\
 &\quad \times \left(c_{11} u^{p^*/p}(\tau) + d_{11} v^{p^*/p}(\tau) \right)^{p^*/p} d\tau ds \\
 &\leq 2^{p^*/p-1} \iint_0^1 k(t, s) k(s, \tau) \\
 &\quad \times \left(c_{11}^{p^*/p} u^{p^*}(\tau) + d_{11}^{p^*/p} v^{p^*}(\tau) \right) d\tau ds \quad (36)
 \end{aligned}$$

for all $(u, v) \in \mathcal{M}_2, t \in [0, 1]$. Note that $q^*, q^*/q \geq 1$. The same argument for the previous inequality can be used to obtain

$$\begin{aligned}
 v^{q^*}(t) &\leq 2^{q^*/q-1} \iint_0^1 k(t, s) k(s, \tau) \\
 &\quad \times \left(c_{21}^{q^*/q} u^{q^*}(\tau) + d_{21}^{q^*/q} v^{q^*}(\tau) \right) d\tau ds \quad (37)
 \end{aligned}$$

for all $(u, v) \in \mathcal{M}_2, t \in [0, 1]$. Multiply the last two inequalities by $\varphi(t)$, integrate over $[0, 1]$, and use Lemma 1 and (H3) to obtain

$$\begin{aligned}
 \int_0^1 u^{p^*}(t) \sin \pi t dt &\leq c_1 \int_0^1 u^{p^*}(t) \sin \pi t dt \\
 &\quad + d_1 \int_0^1 v^{q^*}(t) \sin \pi t dt, \\
 \int_0^1 v^{q^*}(t) \sin \pi t dt &\leq c_2 \int_0^1 u^{p^*}(t) \sin \pi t dt \\
 &\quad + d_2 \int_0^1 v^{q^*}(t) \sin \pi t dt \quad (38)
 \end{aligned}$$

for all $(u, v) \in \mathcal{M}_2$. The last two inequalities can be written as

$$\begin{aligned}
 &\begin{pmatrix} 1-c_1 & -d_1 \\ -c_2 & 1-d_2 \end{pmatrix} \begin{pmatrix} \int_0^1 u^{p^*}(t) \sin \pi t dt \\ \int_0^1 v^{q^*}(t) \sin \pi t dt \end{pmatrix} \\
 &= A_2 \begin{pmatrix} \int_0^1 u^{p^*}(t) \sin \pi t dt \\ \int_0^1 v^{q^*}(t) \sin \pi t dt \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (39)
 \end{aligned}$$

Now (H3) implies

$$\begin{pmatrix} \int_0^1 u^{p^*}(t) \sin \pi t dt \\ \int_0^1 v^{q^*}(t) \sin \pi t dt \end{pmatrix} \leq A_2^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (40)$$

so that

$$\int_0^1 u^{p^*}(t) \sin \pi t dt = 0, \quad \int_0^1 v^{q^*}(t) \sin \pi t dt = 0, \quad (41)$$

whence $u \equiv 0, v \equiv 0$, and $\mathcal{M}_2 = \{0\}$, as required. Thus we have

$$(u, v) \neq \lambda A(u, v), \quad \forall (u, v) \in \partial Q_r \cap (P \times P), \lambda \in [0, 1]. \quad (42)$$

Now Lemma 5 yields

$$i(A, \overline{Q_r} \cap (P \times P), P \times P) = 1. \quad (43)$$

Combining (34) and (43) gives

$$i(A, (Q_R \setminus \overline{Q_r}) \cap (P \times P), P \times P) = 0 - 1 = -1. \quad (44)$$

Consequently, A has at least one fixed point on $(Q_R \setminus \overline{Q_r}) \cap (P \times P)$. Thus (1) has at least one positive solution. This completes the proof. $\square$

Theorem 7. If (H1), (H4), and (H5) hold, then (1) has at least one positive solution.

Proof. Let

$$\begin{aligned}
 \mathcal{M}_3 &:= \{(u, v) \in \overline{Q_r} \cap (P \times P) : \\
 &\quad (u, v) = A(u, v) + \lambda(\varphi, \varphi), \lambda \geq 0\}. \quad (45)
 \end{aligned}$$

We now assert that $\mathcal{M}_3 \subset \{0\}$. Indeed, if $(u, v) \in \mathcal{M}_3$, then there exists $\lambda \in [0, 1]$ such that $(u, v) = A(u, v) + \lambda(\varphi, \varphi)$. Thus $(u, v) \geq A(u, v)$. Note that $k(t, s) \in [0, 1]$ for all $t, s \in [0, 1]$ and

$p_*, p_*/p \in (0, 1]$. By (H4), Jensen's integral inequalities, and Lemma 3, we obtain

$$\begin{aligned}
 u^{p_*}(t) &\geq \left(\int_0^1 k(t, s) \left(\int_0^1 k(s, \tau) \right. \right. \\
 &\quad \left. \left. \times f_1(\tau, u(\tau), v(\tau)) d\tau \right)^{1/p} ds \right)^{p_*} \\
 &\geq \iint_0^1 k(t, s) k(s, \tau) f_1^{p_*/p}(\tau, u(\tau), v(\tau)) d\tau ds \\
 &\geq \iint_0^1 k(t, s) k(s, \tau) \\
 &\quad \times \left(l_{11} u^{p_* p}(\tau) + m_{11} v^{p_* q}(\tau) \right)^{p_*/p} d\tau ds \\
 &\geq 2^{p_*/p-1} \iint_0^1 k(t, s) k(s, \tau) \\
 &\quad \times \left(l_{11}^{p_*/p} u^{p_*}(\tau) + m_{11}^{p_*/p} v^{q_*}(\tau) \right) d\tau ds \quad (46)
 \end{aligned}$$

for all $(u, v) \in \mathcal{M}_3$, $t \in [0, 1]$. Note that q_* , $q_*/q \in (0, 1]$. The same argument for the previous inequality can be used to obtain

$$\begin{aligned}
 v^{q_*}(t) &\geq 2^{q_*/q-1} \iint_0^1 k(t, s) k(s, \tau) \\
 &\quad \times \left(l_{21}^{q_*/q} u^{p_*}(\tau) + m_{21}^{q_*/q} v^{q_*}(\tau) \right) d\tau ds \quad (47)
 \end{aligned}$$

for all $(u, v) \in \mathcal{M}_3$, $t \in [0, 1]$. Multiply the last two inequalities by $\varphi(t)$, integrate over $[0, 1]$, and use Lemma 1 and (H4) to obtain

$$\begin{aligned}
 \int_0^1 u^{p_*}(t) \sin \pi t dt &\geq l_1 \int_0^1 u^{p_*}(t) \sin \pi t dt \\
 &\quad + m_1 \int_0^1 v^{q_*}(t) \sin \pi t dt, \\
 \int_0^1 v^{q_*}(t) \sin \pi t dt &\geq l_2 \int_0^1 u^{p_*}(t) \sin \pi t dt \\
 &\quad + m_2 \int_0^1 v^{q_*}(t) \sin \pi t dt \quad (48)
 \end{aligned}$$

for all $(u, v) \in \mathcal{M}_3$. The last two inequalities can be written as

$$\begin{aligned}
 &\begin{pmatrix} l_1 - 1 & m_1 \\ l_2 & m_2 - 1 \end{pmatrix} \begin{pmatrix} \int_0^1 u^{p_*}(t) \sin \pi t dt \\ \int_0^1 v^{q_*}(t) \sin \pi t dt \end{pmatrix} \\
 &= A_3 \begin{pmatrix} \int_0^1 u^{p_*}(t) \sin \pi t dt \\ \int_0^1 v^{q_*}(t) \sin \pi t dt \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (49)
 \end{aligned}$$

Now (H4) implies

$$\begin{pmatrix} \int_0^1 u^{p_*}(t) \sin \pi t dt \\ \int_0^1 v^{q_*}(t) \sin \pi t dt \end{pmatrix} \leq A_3^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (50)$$

so that

$$\int_0^1 u^{p_*}(t) \sin \pi t dt = 0, \quad \int_0^1 v^{q_*}(t) \sin \pi t dt = 0, \quad (51)$$

whence $u \equiv 0$, $v \equiv 0$, and $\mathcal{M}_3 \subset \{0\}$, as asserted. As a result of this, we obtain

$$\begin{aligned}
 (u, v) &\neq A(u, v) + \lambda(\varphi, \varphi), \quad \forall (u, v) \in \partial Q_r \cap (P \times P), \\
 \lambda &\geq 0. \quad (52)
 \end{aligned}$$

By Lemma 4, we have

$$i(A, Q_r \cap (P \times P), P \times P) = 0. \quad (53)$$

Let

$$\mathcal{M}_4 := \{(u, v) \in P \times P : (u, v) = \lambda A(u, v), \lambda \in [0, 1]\}. \quad (54)$$

Now we want to show that $\mathcal{M}_4$ is bounded. Indeed, if $(u, v) \in \mathcal{M}_4$, then there exists $\lambda \in [0, 1]$ such that $(u, v) = \lambda A(u, v)$. This implies $(u, v) \leq A(u, v)$. Note that $k(t, s) \in [0, 1]$ for all $t, s \in [0, 1]$ and p^* , $p^*/p \geq 1$. By (H5), Jensen's integral inequalities, and Lemma 3, we obtain

$$\begin{aligned}
 u^{p^*}(t) &\leq \iint_0^1 k(t, s) k(s, \tau) f_1^{p^*/p} \\
 &\quad \times (\tau, u(\tau), v(\tau)) d\tau ds \\
 &\leq \iint_0^1 k(t, s) k(s, \tau) \left(p_{11} u^{p^* p}(\tau) + q_{11} v^{p^* q}(\tau) \right. \\
 &\quad \left. + c \right)^{p^*/p} d\tau ds \\
 &\leq 2^{p^*/p-1} \iint_0^1 k(t, s) k(s, \tau) \\
 &\quad \times \left(\left(p_{11} u^{p^* p}(\tau) + q_{11} v^{p^* q}(\tau) \right)^{p^*/p} \right. \\
 &\quad \left. + c^{p^*/p} \right) d\tau ds
 \end{aligned}$$

$$\begin{aligned}
&\leq 2^{p^*/p-1} \iint_0^1 k(t, s) k(s, \tau) \\
&\quad \times \left(2^{p^*/p-1} \left(p_{11}^{p^*/p} u^{p^*}(\tau) \right. \right. \\
&\quad \left. \left. + q_{11}^{p^*/p} v^{q^*}(\tau) \right) \right. \\
&\quad \left. + c^{p^*/p} \right) d\tau ds \\
&= 4^{p^*/p-1} \iint_0^1 k(t, s) k(s, \tau) \\
&\quad \times \left(p_{11}^{p^*/p} u^{p^*} + q_{11}^{p^*/p} v^{q^*} \right) d\tau ds \\
&\quad + 2^{p^*/p-1} \iint_0^1 k(t, s) k(s, \tau) c^{p^*/p} d\tau ds
\end{aligned} \tag{55}$$

for all $(u, v) \in \mathcal{M}_4$, $t \in [0, 1]$. Note that q^* , $q^*/q \geq 1$. The same argument for the previous inequality can be used to obtain

$$\begin{aligned}
v^{q^*}(t) &\leq 4^{q^*/q-1} \iint_0^1 k(t, s) k(s, \tau) \\
&\quad \times \left(p_{21}^{q^*/q} u^{p^*} + q_{21}^{q^*/q} v^{q^*} \right) d\tau ds \tag{56} \\
&\quad + 2^{q^*/q-1} \iint_0^1 k(t, s) k(s, \tau) c^{q^*/q} d\tau ds
\end{aligned}$$

for all $(u, v) \in \mathcal{M}_4$, $t \in [0, 1]$. Multiply the last two inequalities by $\varphi(t)$, integrate over $[0, 1]$, and use Lemma 1 and (H5) to obtain

$$\begin{aligned}
\int_0^1 u^{p^*}(t) \sin \pi t dt &\leq p_1 \int_0^1 u^{p^*}(t) \sin \pi t dt \\
&\quad + q_1 \int_0^1 v^{q^*}(t) \sin \pi t dt + h_3, \\
\int_0^1 v^{q^*}(t) \sin \pi t dt &\leq p_2 \int_0^1 u^{p^*}(t) \sin \pi t dt \\
&\quad + q_2 \int_0^1 v^{q^*}(t) \sin \pi t dt + h_4
\end{aligned} \tag{57}$$

for all $(u, v) \in \mathcal{M}_4$, where $h_3 := 2^{p^*/p} c^{p^*/p} \pi^{-5}$, $h_4 := 2^{q^*/q} c^{q^*/q} \pi^{-5}$. The last two inequalities can be written as

$$\begin{aligned}
&\begin{pmatrix} 1-p_1 & -q_1 \\ -p_2 & 1-q_2 \end{pmatrix} \begin{pmatrix} \int_0^1 u^{p^*}(t) \sin \pi t dt \\ \int_0^1 v^{q^*}(t) \sin \pi t dt \end{pmatrix} \\
&= A_4 \begin{pmatrix} \int_0^1 u^{p^*}(t) \sin \pi t dt \\ \int_0^1 v^{q^*}(t) \sin \pi t dt \end{pmatrix} \leq \begin{pmatrix} h_3 \\ h_4 \end{pmatrix}.
\end{aligned} \tag{58}$$

Now (H3) implies

$$\begin{pmatrix} \int_0^1 u^{p^*}(t) \sin \pi t dt \\ \int_0^1 v^{q^*}(t) \sin \pi t dt \end{pmatrix} \leq A_4^{-1} \begin{pmatrix} h_3 \\ h_4 \end{pmatrix}, \tag{59}$$

and thus there exists a constant $M > 0$ such that

$$\int_0^1 u^{p^*}(t) \sin \pi t dt \leq M, \quad \int_0^1 v^{q^*}(t) \sin \pi t dt \leq M \tag{60}$$

for every $(u, v) \in \mathcal{M}_4$. The concavity of u, v and Lemma 2 imply

$$\|u\| \leq \frac{\pi^2}{2} \int_0^1 u(t) \sin \pi t dt, \quad \|v\| \leq \frac{\pi^2}{2} \int_0^1 v(t) \sin \pi t dt. \tag{61}$$

The fact that $p^*, q^* \geq 1$, together with Jensen's inequalities, implies

$$\begin{aligned}
\left(\int_0^1 u(t) \sin \pi t dt \right)^{p^*} &\leq \int_0^1 u^{p^*}(t) (\sin \pi t)^{p^*} dt \\
&\leq \int_0^1 u^{p^*}(t) \sin \pi t dt \leq M, \\
\left(\int_0^1 v(t) \sin \pi t dt \right)^{q^*} &\leq \int_0^1 v^{q^*}(t) (\sin \pi t)^{q^*} dt \\
&\leq \int_0^1 v^{q^*}(t) \sin \pi t dt \leq M,
\end{aligned} \tag{62}$$

and, in turn,

$$\int_0^1 u(t) \sin \pi t dt \leq M^{1/p^*}, \quad \int_0^1 v(t) \sin \pi t dt \leq M^{1/q^*}. \tag{63}$$

Recalling (61), we obtain

$$\|u\| \leq \frac{\pi^2 M^{1/p^*}}{2}, \quad \|v\| \leq \frac{\pi^2 M^{1/q^*}}{2} \tag{64}$$

for all $(u, v) \in \mathcal{M}_4$. This proves the boundedness of $\mathcal{M}_4$. Taking $R > \sup\{\|(u, v)\| : (u, v) \in \mathcal{M}_4\}$, we have

$$(u, v) \neq \lambda A(u, v), \quad \forall (u, v) \in \partial Q_R \cap (P \times P), \quad \lambda \in [0, 1]. \tag{65}$$

Now Lemma 3 yields

$$i(A, Q_R \cap (P \times P), P \times P) = 1. \tag{66}$$

Combining (53) with (66) gives

$$i(A, (Q_R \setminus \overline{Q_r}) \cap (P \times P), P \times P) = 1 - 0 = 1. \tag{67}$$

Hence, A has at least one fixed point on $(Q_R \setminus \overline{Q_r}) \cap (P \times P)$. Thus (1) has at least one positive solution. This completes the proof. $\square$

References

- [1] C. Pang, W. Dong, and Z. Wei, "Multiple solutions for fourth-order boundary value problem," *Journal of Mathematical Analysis and Applications*, vol. 314, no. 2, pp. 464–476, 2006.
- [2] Z. Wei and C. Pang, "The method of lower and upper solutions for fourth order singular m -point boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 322, no. 2, pp. 675–692, 2006.
- [3] Z. Wei, "A class of fourth order singular boundary value problems," *Applied Mathematics and Computation*, vol. 153, no. 3, pp. 865–884, 2004.
- [4] Z. Wei and C. Pang, "Multiple sign-changing solutions for fourth order m -point boundary value problems," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 66, no. 4, pp. 839–855, 2007.
- [5] J. I. Diaz and F. de Thélin, "On a nonlinear parabolic problem arising in some models related to turbulent flows," *SIAM Journal on Mathematical Analysis*, vol. 25, no. 4, pp. 1085–1111, 1994.
- [6] R. Glowinski and J. Rappaz, "Approximation of a nonlinear elliptic problem arising in a non-Newtonian fluid flow model in glaciology," *Mathematical Modelling and Numerical Analysis*, vol. 37, no. 1, pp. 175–186, 2003.
- [7] R. E. Gaines and J. L. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Springer, 1977.
- [8] R. W. Leggett and L. R. Williams, "Multiple positive fixed points of a nonlinear operators on ordered Banach spaces," *Indiana University Mathematics Journal*, vol. 28, no. 4, pp. 673–688, 1979.
- [9] V. Anuradha, D. D. Hai, and R. Shivaji, "Existence results for superlinear semipositone BVP'S," *Proceedings of the American Mathematical Society*, vol. 124, no. 3, pp. 757–763, 1996.
- [10] R. Avery and J. Henderson, "Existence of three positive pseudo-symmetric solutions for a one dimensional p -Laplacian," *Journal of Mathematical Analysis and Applications*, vol. 277, no. 2, pp. 395–404, 2003.
- [11] C. Bai and J. Fang, "Existence of multiple positive solutions for nonlinear m -point boundary value problems," *Journal of Mathematical Analysis and Applications*, vol. 140, no. 2-3, pp. 297–305, 2003.
- [12] A. Ben-Naoum and C. Decoster, "On the p -Laplacian separated boundary value problem," *Differential Integral Equations*, vol. 10, no. 6, pp. 1093–1112, 1997.
- [13] Z. Guo, J. Yin, and Y. Ke, "Multiplicity of positive solutions for a fourth-order quasilinear singular differential equation," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 27, pp. 1–15, 2010.
- [14] M. del Pino, M. Elgueta, and R. Manasevich, "A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t, u) = 0, u(0) = u(T) = 0, p > 1$," *Journal of Differential Equations*, vol. 80, no. 1, pp. 1–13, 1989.
- [15] J. Xu and Z. Yang, "Positive solutions for a fourth order p -Laplacian boundary value problem," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 74, no. 7, pp. 2612–2623, 2011.
- [16] X. Zhang and L. Liu, "Positive solutions of fourth-order four-point boundary value problems with p -Laplacian operator," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 2, pp. 1414–1423, 2007.
- [17] X. Zhang and L. Liu, "A necessary and sufficient condition for positive solutions for fourth-order multi-point boundary value problems with p -Laplacian," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 68, no. 10, pp. 3127–3137, 2008.
- [18] Z. Yang and D. O'Regan, "Positive solutions of a focal problem for one-dimensional p -Laplacian equations," *Mathematical and Computer Modelling*, vol. 55, no. 7-8, pp. 1942–1950, 2012.
- [19] Z. Yang, "Positive solutions for a system of p -Laplacian boundary value problems," *Computers and Mathematics with Applications*, vol. 62, no. 12, pp. 4429–4438, 2011.
- [20] Z. Yang and J. Sun, "Positive solutions of nonlinear second-order ordinary differential equations boundary value problems," *Acta Mathematica Sinica*, vol. 47, no. 1, pp. 111–118, 2004.
- [21] Z. Yang and L. Kong, "Positive solutions of a system of second order boundary value problems involving first order derivatives via $\mathbb{R}_+^n$ -monotone matrices," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 75, no. 4, pp. 2037–2046, 2012.
- [22] Z. Yang, "Positive solutions for a system of nonlinear Hammerstein integral equations and applications," *Applied Mathematics and Computation*, vol. 218, no. 22, pp. 11138–11150, 2012.
- [23] Z. Yang and Z. Zhang, "Positive solutions for a system of nonlinear singular Hammerstein integral equations via nonnegative matrices and applications," *Positivity*, vol. 16, no. 4, pp. 738–800, 2012.
- [24] K. Q. Lan and W. Lin, "Multiple positive solutions of systems of Hammerstein integral equations with applications to fractional differential equations," *Journal of the London Mathematical Society*, vol. 83, no. 2, pp. 449–469, 2011.
- [25] Z. Yang, "Existence of positive solutions for a system of generalized Lidstone problems," *Computers and Mathematics with Applications*, vol. 60, no. 3, pp. 501–510, 2010.
- [26] Z. Yang and D. O'Regan, "Positive solvability of systems of nonlinear Hammerstein integral equations," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 600–614, 2005.
- [27] P. Jebelean and R. Precup, "Solvability of p, q -laplacian systems with potential boundary conditions," *Applicable Analysis*, vol. 89, no. 2, pp. 221–228, 2010.
- [28] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, Mass, USA, 1988.