Research Article

Conservative Intensional Extension of Tarski's Semantics

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We considered an extension of the first-order logic (FOL) by Bealer's intensional abstraction operator. Contemporary use of the term "intension" derives from the traditional logical Frege-Russell doctrine that an idea (logic formula) has both an extension and an intension. Although there is divergence in formulation, it is accepted that the "extension" of an idea consists of the subjects to which the idea applies, and the "intension" consists of the attributes implied by the idea. From the Montague's point of view, the meaning of an idea can be considered as particular extensions in different possible worlds. In the case of standard FOL, we obtain a commutative homomorphic diagram, which is valid in each given possible world of an intensional FOL: from a free algebra of the FOL syntax, into its intensional algebra of concepts, and, successively, into an extensional relational algebra (different from Cylindric algebras). Then we show that this composition corresponds to the Tarski's interpretation of the standard extensional FOL in this possible world.

1. Introduction

In "Über Sinn und Bedeutung," Frege concentrated mostly on the senses of names, holding that all names have a sense (meaning). It is natural to hold that the same considerations apply to any expression that has an extension. But two general terms can have the same extension and different cognitive significance; two predicates can have the same extension and different cognitive significance; two sentences can have the same extension and different cognitive significance. So, general terms, predicates, and sentences all have senses as well as extensions. The same goes for any expression that has an extension or is a candidate for extension.

The significant aspect of an expression's meaning is its extension. We can stipulate that the extension of a sentence is its truth-value, and that the extension of a singular term is its referent. The extension of other expressions can be seen as associated entities that contribute to the truth-value of a sentence in a manner broadly analogous to the way in which the referent of a singular term contributes to the truth-value of a sentence. In many cases, the extension of an expression will be what we intuitively think of as its referent, although this need not hold in all cases. While Frege himself is often interpreted as holding that a sentence's referent is its truth-value, this claim is counterintuitive and widely disputed. We

can avoid that issue in the present framework by using the technical term "extension." In this context, the claim that the extension of a sentence is its truth-value is a stipulation.

"Extensional" is most definitely a technical term. Say that the extension of a name is its denotation, the extension of a predicate is the set of things it applies to, and the extension of a sentence is its truth value. A logic is extensional if coextensional expressions can be substituted one for another in any sentence of the logic "salva veritate," that is, without a change in truth value. The intuitive idea behind this principle is that, in an extensional logic, the only logically significant notion of meaning that attaches to an expression is its extension. An intensional logic is exactly one in which substitutivity *salva veritate* fails for some of the sentences of the logic.

The first conception of intensional entities (or concepts) is built into the *possible-worlds* treatment of Properties, Relations, and Propositions (PRPs). This conception is commonly attributed to Leibniz and underlies Alonzo Church's alternative formulation of Frege's theory of senses ("A *formulation of the logic of sense and denotation*" in Henle, Kallen, and Langer, 3–24, and "Outline of a revised formulation of the logic of sense and denotation" in two parts, Nous, VII (1973), 24–33, and VIII, (1974), 135–156). This conception of PRPs is ideally suited for treating the *modalities* (necessity, possibility,

etc.) and to Montague's definition of intension of a given virtual predicate $\phi(x_1, \dots, x_k)$ (a FOL open-sentence with the tuple of free variables $(x_1, \dots, x_k)$), as a mapping from possible worlds into extensions of this virtual predicate. Among the possible worlds, we distinguish the *actual* possible world. For example, if we consider a set of predicates, of a given Database, and their extensions in different time-instances, then the actual possible world is identified by the current instance of the time.

The second conception of intensional entities is to be found in Russell's doctrine of logical atomism. In this doctrine, it is required that all complete definitions of intensional entities be finite as well as unique and noncircular: it offers an *algebraic* way for definition of complex intensional entities from simple (atomic) entities (i.e., algebra of concepts), conception also evident in Leibniz's remarks. In a predicate logics, predicates and open-sentences (with free variables) express classes (properties and relations), and sentences express propositions. Note that classes (intensional entities) are *reified*, that is, they belong to the same domain as individual objects (particulars). This endows the intensional logics with a great deal of uniformity, making it possible to manipulate classes and individual objects in the same language. In particular, when viewed as an individual object, a class can be a member of another class.

The distinction between intensions and extensions is important (as in lexicography [1]), considering that extensions can be notoriously difficult to handle in an efficient manner. The extensional equality theory of predicates and functions under higher-order semantics (e.g., for two predicates with the same set of attributes, $p = q$ is true iff these symbols are interpreted by the same relation), that is, the strong equational theory of intensions, is not decidable, in general. For example, the second-order predicate calculus and Church's simple theory of types, both under the standard semantics, are not even semi-decidable. Thus, separating intensions from extensions makes it possible to have an equational theory over predicate and function names (intensions) that is separate from the extensional equality of relations and functions.

Relevant recent work about the intension, and its relationship with FOL, has been presented in [2] in the consideration of rigid and *nonrigid* objects, with respect to the possible worlds, where the rigid objects, like "George Washington," are the same things from possible world to possible world. Nonrigid objects, like "the Secretary-General of United Nations," are varying from circumstance to circumstance and can be modeled semantically by functions from possible worlds to domain of rigid objects, like intensional entities. But in his approach, differently from that one, fitting changes also the syntax of the FOL, by introducing an "extension of" operator, $\downarrow$, in order to distinguish the intensional entity "gross-domestic-product-of-Denmark," and its use in "the gross domestic product of Denmark is currently greater than gross domestic product of Finland." In his approach, if x is an intensional variable, $\downarrow x$ is extensional, while $\downarrow$ is not applicable to extensional variables, differently from our where each variable (concept) has both intensional and extension. Moreover, in his approach the problem arises because the

action of letting x designate, that is, evaluating $\downarrow x$, and the action of passing to an alternative possible world, that is, of interpreting the existential modal operator $\Diamond$, are not actions that commute. To disambiguate this, one more piece of machinery is needed as well, which substantially and ad-hock changes the syntax and semantics of FOL, introduces the Higher-order Modal logics, and is not a conservative extension of Tarski's semantics.

In most recent work in [3, 4] it is given an intensional version of first-order *hybrid* logic, which is also a hybridized version of Fitting's intensional FOL, by a kind of generalized models, thus, different from our approaches to conservative extension of Tarski's semantics to intensional FOL.

Another recent relevant work is presented by I-logic in [5], which combines both approach to semantics of intensional objects of Montague and Fitting.

We recall that Intensional Logic Programming is a new form of logic programming based on intensional logic and possible worlds semantics and is a well-defined practice in using the intensional semantics [6]. Intensional logic allows us to use logic programming to specify nonterminating computations and to capture the dynamic aspects of certain problems in a natural and problem-oriented style. The meanings of formulas of an intensional first-order language are given according to intensional interpretations and to elements of a set of possible worlds. Neighborhood semantics is employed as an abstract formulation of the denotations of intensional operators. The model-theoretic and fixpoint semantics of intensional logic programs are developed in terms of least (minimum) intensional Herbrand models. Intensional logic programs with intensional operator definitions are regarded as metatheories.

In what follows, we denote by B^A the set of all functions from A to B , and by A^n an n -folded cartesian product $A \times \dots \times A$ for $n \geq 1$. By f, t we denote empty set $\emptyset$ and singleton set $\{\langle \rangle\}$, respectively (with the empty tuple $\langle \rangle$ i.e., the unique tuple of 0-ary relation), which may be thought of as falsity f and truth t , as those used in the relational algebra. For a given domain $\mathcal{D}$, we define that $\mathcal{D}^0$ is a singleton set $\{\langle \rangle\}$, so that $\{f, t\} = \mathcal{P}(\mathcal{D}^0)$, where $\mathcal{P}$ is the powerset operator.

2. Intensional FOL Language with Intensional Abstraction

Intensional entities are such concepts as propositions and properties. The term "intensional" means that they violate the principle of extensionality, the principle that extensional equivalence implies identity. All (or most) of these intensional entities have been classified at one time or another as kinds of Universals [7].

We consider a nonempty domain $\mathcal{D} = D_{-1} \cup D_I$, where a subdomain D_{-1} is made of particulars (extensional entities), and the rest $D_I = D_0 \cup D_1 \dots \cup D_n \dots$ is made of universals (D_0 for propositions (the 0-ary concepts)), and D_n , $n \geq 1$, for n -ary concepts.

The fundamental entities are *intensional abstracts* or so-called, that-clauses. We assume that they are singular terms; Intensional expressions like "believe," "mean," "assert,"

“know,” are standard two-place predicates that take “that”-clauses as arguments. Expressions like “is necessary,” “is true,” and “is possible” are one-place predicates that take “that”-clauses as arguments. For example, in the intensional sentence “it is necessary that ϕ ,” where ϕ is a proposition, the “that ϕ ” is denoted by the $\langle \phi \rangle$, where $\langle \rangle$ is the intensional abstraction operator, which transforms a logic formula into a *term*. Or, for example, “ x believes that ϕ ” is given by formula $p_i^2(x, \langle \phi \rangle)$ (p_i^2 is binary “believe” predicate).

Here we will present an intensional FOL with slightly different intensional abstraction than that originally presented in [8].

Definition 1. The syntax of the first-order logic language with intensional abstraction $\langle \rangle$, denoted by $\mathcal{L}$, is as follows:

logic operators ($\wedge, \neg, \exists$), predicate letters in P (functional letters is considered as particular case of predicate letters), variables $x, y, z, \dots$ in $\mathcal{V}$, abstraction $\langle _ \rangle$, and punctuation symbols (comma, parenthesis). With the following simultaneous inductive definition of *term* and *formula*,

- (1) all variables and constants (0-ary functional letters in P) are terms;
- (2) if $t_1, \dots, t_k$ are terms, then $p_i^k(t_1, \dots, t_k)$ is a formula ($p_i^k \in P$ is a k -ary predicate letter);
- (3) if ϕ and ψ are formulae, then $(\phi \wedge \psi)$, $\neg\phi$, and $(\exists x)\phi$ are formulae;
- (4) if $\phi(\mathbf{x})$ is a formula (virtual predicate) with a list of free variables in $\mathbf{x} = (x_1, \dots, x_n)$ (with ordering from-left-to-right of their appearance in ϕ), and α is its sublist of *distinct* variables, then $\langle \phi \rangle_\alpha^\beta$ is a term, where β is the remaining list of free variables preserving ordering in $\mathbf{x}$ as well. The externally quantifiable variables are the *free* variables not in α . When $n = 0$, $\langle \phi \rangle$ is a term that denotes a proposition, for $n \geq 1$ it denotes an n -ary concept.

An occurrence of a variable x_i in a formula (or a term) is *bound* (*free*) if and only if it lies (does not lie) within a formula of the form $(\exists x_i)\phi$ (or a term of the form $\langle \phi \rangle_\alpha^\beta$ with $x_i \in \alpha$). A variable is free (bound) in a formula (or term) if and only if it has (does not have) a free occurrence in that formula (or term).

A *sentence* is a formula having no free variables. The binary predicate letter p_1^2 for identity is singled out as a distinguished logical predicate, and formulae of the form $p_1^2(t_1, t_2)$ are to be rewritten in the form $t_1 \doteq t_2$. We denote by R_- the binary relation obtained by standard Tarski’s interpretation of this predicate p_1^2 . The logic operators $\forall, \vee, \Rightarrow$ are defined in terms of $(\wedge, \neg, \exists)$ in the usual way.

Remark 2. The k -ary functional symbols, for $k \geq 1$, in standard (extensional) FOL are considered as $(k+1)$ -ary predicate symbols p^{k+1} : the function $f : \mathcal{D}^k \rightarrow \mathcal{D}$ is considered as a relation obtained from its graph $R = \{(d_1, \dots, d_k, f(d_1, \dots, d_k)) \mid d_i \in \mathcal{D}\}$, represented by a predicate symbol p^{k+1} .

The universal quantifier is defined by $\forall = \neg\exists\neg$. Disjunction and implication are expressed by $\phi \vee \psi = \neg(\neg\phi \wedge \neg\psi)$

and $\phi \Rightarrow \psi = \neg\phi \vee \psi$. In FOL with the identity $\doteq$, the formula $(\exists_1 x)\phi(x)$ denotes the formula $(\exists x)\phi(x) \wedge (\forall x)(\forall y)(\phi(x) \wedge \phi(y) \Rightarrow (x \doteq y))$. We denote by R_- the Tarski’s interpretation of $\doteq$.

In what follows, any open-sentence, a formula ϕ with nonempty tuple of free variables $(x_1, \dots, x_m)$, will be called a m -ary *virtual predicate*, denoted also by $\phi(x_1, \dots, x_m)$. This definition contains the precise method of establishing the *ordering* of variables in this tuple: such a method that will be adopted here is the ordering of appearance, from left to right, of free variables in ϕ . This method of composing the tuple of free variables is the unique and canonical way of definition of the virtual predicate from a given formula.

An *intensional interpretation* of this intensional FOL is a mapping between the set $\mathcal{L}$ of formulae of the logic language and intensional entities in $\mathcal{D}$, $I : \mathcal{L} \rightarrow \mathcal{D}$, which is a kind of “conceptualization”, such that an open-sentence (virtual predicate) $\phi(x_1, \dots, x_k)$ with a tuple of all free variables $(x_1, \dots, x_k)$ is mapped into a k -ary *concept*, that is, an intensional entity $u = I(\phi(x_1, \dots, x_k)) \in D_k$, and (closed) sentence ψ into a proposition (i.e., *logic* concept) $v = I(\psi) \in D_0$ with $I(\top) = \text{Truth} \in D_0$ for a FOL tautology $\top$. A language constant c is mapped into a particular (an extensional entity) $a = I(c) \in D_{-1}$ if it is a proper name, otherwise in a correspondent concept in $\mathcal{D}$.

An assignment $g : \mathcal{V} \rightarrow \mathcal{D}$ for variables in $\mathcal{V}$ is applied only to free variables in terms and formulae. Such an assignment $g \in \mathcal{D}^{\mathcal{V}}$ can be recursively uniquely extended into the assignment $g^* : \mathcal{T} \rightarrow \mathcal{D}$, where $\mathcal{T}$ denotes the set of all terms (here I is an intensional interpretation of this FOL, as explained in what follows), by

- (1) $g^*(t) = g(x) \in \mathcal{D}$ if the term t is a variable $x \in \mathcal{V}$;
- (2) $g^*(t) = I(c) \in \mathcal{D}$ if the term t is a constant $c \in P$;
- (3) if t is an abstracted term $\langle \phi \rangle_\alpha^\beta$, then $g^*(\langle \phi \rangle_\alpha^\beta) = I(\phi[\beta/g(\beta)]) \in D_k$, $k = |\alpha|$ (i.e., the number of variables in α), where $g(\beta) = g(y_1, \dots, y_m) = (g(y_1), \dots, g(y_m))$ and $[\beta/g(\beta)]$ is a uniform replacement of each i th variable in the list β with the i th constant in the list $g(\beta)$. Notice that α is the list of all free variables in the formula $\phi[\beta/g(\beta)]$.

We denote by t/g (or ϕ/g) the ground term (or formula) without free variables, obtained by assignment g from a term t (or a formula ϕ), and by $\phi[x/t]$ the formula obtained by uniformly replacing x by a term t in ϕ .

The distinction between intensions and extensions is important especially because we are now able to have and *equational theory* over intensional entities (as $\langle \phi \rangle$), that is, predicate and function “names,” which is separate from the extensional equality of relations and functions. An *extensionalization function* h assigns to the intensional elements of $\mathcal{D}$ an appropriate extension as follows: for each proposition $u \in D_0$, $h(u) \in \{f, t\} \subseteq \mathcal{P}(D_{-1})$ is its extension (true or false value); for each n -ary concept $u \in D_n$, $h(u)$ is a subset of $\mathcal{D}^n$ (n th Cartesian product of $\mathcal{D}$); in the case of particulars $u \in D_{-1}$, $h(u) = u$.

The sets f, t are empty set $\{\}$ and set $\{\langle \rangle\}$ (with the empty tuple $\langle \rangle \in D_{-1}$, i.e., the unique tuple of 0-ary relation)

which may be thought of as falsity and truth, as those used in the Codd's relational-database algebra [9], respectively, while $Truth \in D_0$ is the concept (intension) of the tautology.

We define that $\mathcal{D}^0 = \{\langle \rangle\}$, so that $\{f, t\} = \mathcal{P}(\mathcal{D}^0)$, where $\mathcal{P}$ is the powerset operator. Thus we have (we denote the disjoint union by “+”):

$$h = \left(h_{-1} + \sum_{i \geq 0} h_i \right) : \sum_{i \geq -1} D_i \longrightarrow D_{-1} + \sum_{i \geq 0} \mathcal{P}(D^i), \quad (1)$$

where $h_{-1} = id : D_{-1} \rightarrow D_{-1}$ is identity mapping, the mapping $h_0 : D_0 \rightarrow \{f, t\}$ assigns the truth values in $\{f, t\}$ to all propositions, and the mappings $h_i : D_i \rightarrow \mathcal{P}(D^i)$, $i \geq 1$, assign an extension to all concepts. Thus, the intensions can be seen as *names* of abstract or concrete entities, while the extensions correspond to various rules that these entities play in different worlds.

Remark 3 (Tarski's constraints). This intensional semantics has to preserve standard Tarski's semantics of the FOL. That is, for any formula $\phi \in \mathcal{L}$ with a tuple of free variables $(x_1, \dots, x_k)$, and $h \in \mathcal{E}$, the following conservative conditions for all assignments $g, g' \in \mathcal{D}^{\mathcal{V}}$ have to be satisfied:

(T) $h(I(\phi/g)) = t$ if and only if $(g(x_1), \dots, g(x_k)) \in h(I(\phi))$ and if ϕ is a predicate letter p^k , $k \geq 2$ which represents a $(k-1)$ -ary functional symbol f^{k-1} in standard FOL,

(TF) $h(I(\phi/g)) = h(I(\phi/g')) = t$, and $\forall_{1 \leq i \leq k-1} (g'(x_i) = g(x_i))$ implies $g'(x_{k+1}) = g(x_{k+1})$.

Thus, intensional FOL has a simple Tarski first-order semantics, with a decidable unification problem, but we need also the actual world mapping which maps any intensional entity to its *actual world extension*. In what follows, we will identify a *possible world* by a particular mapping which assigns, in such a possible world, the extensions to intensional entities. This is a direct bridge between an intensional FOL and a possible worlds representation [10–15], where the intension (meaning) of a proposition is a *function*, from a set of possible $\mathcal{W}$ worlds into the set of truth values. Consequently, $\mathcal{E}$ denotes the set of possible *extensionalization functions* h satisfying the constraint (T). Each $h \in \mathcal{E}$ may be seen as a *possible world* (analogously to Montague's intensional semantics for natural language [12, 14]), as it has been demonstrated in [16, 17] and given by the bijection $is : \mathcal{W} \simeq \mathcal{E}$.

Now we are able to define formally this intensional semantics [15].

Definition 4. A two-step intensional semantics.

Let $\mathfrak{R} = \bigcup_{k \in \mathbb{N}} \mathcal{P}(\mathcal{D}^k) = \sum_{k \in \mathbb{N}} \mathcal{P}(D^k)$ be the set of all k -ary relations, where $k \in \mathbb{N} = \{0, 1, 2, \dots\}$. Notice that $\{f, t\} = \mathcal{P}(\mathcal{D}^0) \in \mathfrak{R}$, that is, the truth values are extensions in $\mathfrak{R}$.

The intensional semantics of the logic language with the set of formulae $\mathcal{L}$ can be represented by the mapping

$$\mathcal{L} \xrightarrow{I} \mathcal{D} \Rightarrow_{w \in \mathcal{W}} \mathfrak{R}, \quad (2)$$

where $\xrightarrow{I}$ is a *fixed intensional* interpretation $I : \mathcal{L} \rightarrow \mathcal{D}$ and $\Rightarrow_{w \in \mathcal{W}}$ is the set of all extensionalization functions $h = is(w) : \mathcal{D} \rightarrow \mathfrak{R}$ in $\mathcal{E}$, where $is : \mathcal{W} \rightarrow \mathcal{E}$ is the mapping from the set of possible worlds to the set of extensionalization functions.

We define the mapping $I_n : \mathcal{L}_{op} \rightarrow \mathfrak{R}^{\mathcal{W}}$, where $\mathcal{L}_{op}$ is the subset of formulae with free variables (virtual predicates), such that for any virtual predicate $\phi(x_1, \dots, x_k) \in \mathcal{L}_{op}$ the mapping $I_n(\phi(x_1, \dots, x_k)) : \mathcal{W} \rightarrow \mathfrak{R}$ is the Montague's meaning (i.e., *intension*) of this virtual predicate [10–14], that is, the mapping which returns with the extension of this (virtual) predicate in each possible world $w \in \mathcal{W}$.

We adopted this two-step intensional semantics, instead of well-known Montague's semantics (which lies in the construction of a compositional and recursive semantics that covers both intension and extension), because of a number of weakness of the second semantics:

Example 5. Let us consider the following two past participles: “bought” and “sold” (with unary predicates $p_1^1(x)$, “ x has been bought”, and $p_2^1(x)$, “ x has been sold”). These two different concepts in the Montague's semantics would have not only the same extension but also their intension, from the fact that their extensions are identical in every possible world.

Within the two-step formalism, we can avoid this problem by assigning two different concepts (meanings) $u = I(p_1^1(x))$ and $v = I(p_2^1(x))$ in D_1 . Note that we have the same problem in the Montague's semantics for two sentences with different meanings, which bear the same truth value across all possible worlds: in Montague's semantics, they will be forced to the *same* meaning.

Another relevant question with respect to this two-step interpretations of an intensional semantics is how in it the extensional identity relation $\doteq$ (binary predicate of the identity) of the FOL is managed. Here this extensional identity relation is mapped into the binary concept $Id = I(\doteq(x, y)) \in D_2$, such that $(\forall w \in \mathcal{W})(is(w)(Id) = R_-)$, where $\doteq(x, y)$ (i.e., $p_1^2(x, y)$) denotes an atom of the FOL of the binary predicate for identity in FOL, usually written by FOL formula $x \doteq y$.

Note that here we prefer to distinguish this *formal symbol* $\doteq \in P$ of the built-in identity binary predicate letter in the FOL, from the standard mathematical symbol “=” used in all mathematical definitions in this paper.

In what follows, we will use the function $f_{\langle \rangle} : \mathfrak{R} \rightarrow \mathfrak{R}$, such that for any relation $R \in \mathfrak{R}$, $f_{\langle \rangle}(R) = \{\langle \rangle\}$ if $R \neq \emptyset$; $\emptyset$ otherwise. Let us define the following set of algebraic operators for relations in $\mathfrak{R}$.

- (1) Binary operator $\bowtie_S : \mathfrak{R} \times \mathfrak{R} \rightarrow \mathfrak{R}$, such that for any two relations $R_1, R_2 \in \mathfrak{R}$, the $R_1 \bowtie_S R_2$ is equal to the relation obtained by natural join of these two relations if S is a nonempty set of pairs of joined columns of respective relations (where the first argument is the column index of the relation R_1 while the second argument is the column index of the joined column of

the relation R_2); otherwise it is equal to the cartesian product $R_1 \times R_2$.

For example, the logic formula $\phi(x_i, x_j, x_k, x_l, x_m) \wedge \psi(x_l, y_i, x_j, y_j)$ will be traduced by the algebraic expression $R_1 \bowtie_S R_2$ where $R_1 \in \mathcal{P}(\mathcal{D}^5)$, $R_2 \in \mathcal{P}(\mathcal{D}^4)$ are the extensions for a given Tarski's interpretation of the virtual predicate ϕ, ψ relatively, so that $S = \{(4, 1), (2, 3)\}$ and the resulting relation will have the following ordering of attributes: $(x_i, x_j, x_k, x_l, x_m, y_i, y_j)$.

- (2) Unary operator $\sim: \mathfrak{R} \rightarrow \mathfrak{R}$, such that for any k -ary (with $k \geq 0$) relation $R \in \mathcal{P}(\mathcal{D}^k) \subset \mathfrak{R}$, we have that $\sim(R) = \mathcal{D}^k \setminus R \in \mathcal{D}^k$, where “ $\setminus$ ” is the substraction of relations. For example, the logic formula $\neg\phi(x_i, x_j, x_k, x_l, x_m)$ will be traduced by the algebraic expression $\mathcal{D}^5 \setminus R$ where R is the extensions for a given Tarski's interpretation of the virtual predicate ϕ .

- (3) Unary operator $\pi_{-m}: \mathfrak{R} \rightarrow \mathfrak{R}$, such that for any k -ary (with $k \geq 0$) relation $R \in \mathcal{P}(\mathcal{D}^k) \subset \mathfrak{R}$, we have that $\pi_{-m}(R)$ is equal to the relation obtained by elimination of the m th column of the relation R if $1 \leq m \leq k$ and $k \geq 2$; equal to $f_{\langle i \rangle}(R)$ if $m = k = 1$; otherwise it is equal to R .

For example, the logic formula $(\exists x_k)\phi(x_i, x_j, x_k, x_l, x_m)$ will be traduced by the algebraic expression $\pi_{-3}(R)$ where R is the extensions for a given Tarski's interpretation of the virtual predicate ϕ and the resulting relation will have the following ordering of attributes: (x_i, x_j, x_l, x_m) .

Notice that the ordering of attributes of resulting relations corresponds to the method used for generating the ordering of variables in the tuples of free variables adopted for virtual predicates.

Analogously to Boolean algebras, which are extensional models of propositional logic, we introduce now an intensional algebra for this intensional FOL, as follows.

Definition 6. Intensional algebra for the intensional FOL in Definition 1 is a structure $\mathcal{A}_{\text{int}} = (\mathcal{D}, f, t, Id, Truth, \{\text{conj}_S\}_{S \in \mathcal{P}(\mathbb{N}^2)}, \text{neg}, \{\text{exists}_n\}_{n \in \mathbb{N}})$, with binary operations $\text{conj}_S: D_I \times D_I \rightarrow D_I$, unary operation $\text{neg}: D_I \rightarrow D_I$, unary operations $\text{exists}_n: D_I \rightarrow D_I$, such that for any extensionalization function $h \in \mathcal{E}$, and $u \in D_k, v \in D_j, k, j \geq 0$,

- (1) $h(Id) = R_{\perp}$ and $h(Truth) = \{\langle \rangle\}$.
- (2) $h(\text{conj}_S(u, v)) = h(u) \bowtie_S h(v)$, where $\bowtie_S$ is the natural join operation defined above and $\text{conj}_S(u, v) \in D_m$ where $m = k + j - |S|$ if for every pair $(i_1, i_2) \in S$ it holds that $1 \leq i_1 \leq k, 1 \leq i_2 \leq j$ (otherwise $\text{conj}_S(u, v) \in D_{k+j}$).
- (3) $h(\text{neg}(u)) = \sim(h(u)) = \mathcal{D}^k \setminus (h(u))$, where $\sim$ is the operation defined above and $\text{neg}(u) \in D_k$.
- (4) $h(\text{exists}_n(u)) = \pi_{-n}(h(u))$, where π_{-n} is the operation defined above and $\text{exists}_n(u) \in D_{k-1}$ if $1 \leq n \leq k$ (otherwise exists_n is the identity function).

Notice that for $u \in D_0$, $h(\text{neg}(u)) = \sim(h(u)) = \mathcal{D}^0 \setminus (h(u)) = \{\langle \rangle\} \setminus (h(u)) \in \{f, t\}$.

We define a derived operation union: $(\mathcal{P}(D_i) \setminus \emptyset) \rightarrow D_i$, $i \geq 0$, such that, for any $B = \{u_1, \dots, u_n\} \in \mathcal{P}(D_i)$ we have that $\text{union}(\{u_1, \dots, u_n\}) =_{\text{def}} u_1$ if $n = 1$; $\text{neg}(\text{conj}_S(\text{neg}(u_1), \text{conj}_S(\dots, \text{neg}(u_n)) \dots))$, where $S = \{(l, l) \mid 1 \leq l \leq i\}$, otherwise. Than we obtain that for $n \geq 2$,

$$\begin{aligned} h(\text{union}(B)) &= h(\text{neg}(\text{conj}_S(\text{neg}(u_1), \text{conj}_S(\dots, \text{neg}(u_n)) \dots))) \\ &= \mathcal{D}^i \setminus ((\mathcal{D}^i \setminus h(u_1)) \bowtie_S \dots \bowtie_S (\mathcal{D}^i \setminus h(u_n))) \\ &= \mathcal{D}^i \setminus ((\mathcal{D}^i \setminus h(u_1)) \cap \dots \cap (\mathcal{D}^i \setminus h(u_n))) \\ &= \bigcup \{h(u_j) \mid 1 \leq j \leq n\} = \bigcup \{h(u) \mid u \in B\}. \end{aligned} \quad (3)$$

Intensional interpretation $I: \mathcal{L} \rightarrow \mathcal{D}$ satisfies the following homomorphic extension.

- (1) The logic formula $\phi(x_i, x_j, x_k, x_l, x_m) \wedge \psi(x_l, y_i, x_j, y_j)$ will be intensionally interpreted by the concept $u_1 \in D_7$, obtained by the algebraic expression $\text{conj}_S(u, v)$ where $u = I(\phi(x_i, x_j, x_k, x_l, x_m)) \in D_5, v = I(\psi(x_l, y_i, x_j, y_j)) \in D_4$ are the concepts of the virtual predicates ϕ, ψ , relatively, and $S = \{(4, 1), (2, 3)\}$. Consequently, we have that for any two formulae $\phi, \psi \in \mathcal{L}$ and a particular operator conj_S uniquely determined by tuples of free variables in these two formulae, $I(\phi \wedge \psi) = \text{conj}_S(I(\phi), I(\psi))$.
- (2) The logic formula $\neg\phi(x_i, x_j, x_k, x_l, x_m)$ will be intensionally interpreted by the concept $u_1 \in D_5$, obtained by the algebraic expression $\text{neg}(u)$ where $u = I(\phi(x_i, x_j, x_k, x_l, x_m)) \in D_5$ is the concept of the virtual predicate ϕ . Consequently, we have that for any formula $\phi \in \mathcal{L}$, $I(\neg\phi) = \text{neg}(I(\phi))$.
- (3) The logic formula $(\exists x_k)\phi(x_i, x_j, x_k, x_l, x_m)$ will be intensionally interpreted by the concept $u_1 \in D_4$, obtained by the algebraic expression $\text{exists}_3(u)$ where $u = I(\phi(x_i, x_j, x_k, x_l, x_m)) \in D_5$ is the concept of the virtual predicate ϕ . Consequently, we have that for any formula $\phi \in \mathcal{L}$ and a particular operator exists_n uniquely determined by the position of the existentially quantified variable in the tuple of free variables in ϕ (otherwise $n = 0$ if this quantified variable is not a free variable in ϕ), $I((\exists x)\phi) = \text{exists}_n(I(\phi))$.

Once one has found a method for specifying the interpretations of singular terms of $\mathcal{L}$ (take in consideration the particularity of abstracted terms), the Tarski-style definitions of truth and validity for $\mathcal{L}$ may be given in the customary way. What is proposed specifically is a method for characterizing the intensional interpretations of singular terms of $\mathcal{L}$ in such a way that a given singular abstracted term $\langle \phi \rangle_{\alpha}^{\beta}$ will denote an appropriate property, relation, or proposition, depending on the value of $m = |\alpha|$. Thus, the mapping of intensional abstracts (terms) into $\mathcal{D}$ will be defined differently from that given in the version of Bealer [18], as follows.

Definition 7. An intensional interpretation I can be extended to abstracted terms as follows: for any abstracted term $\langle \phi \rangle_{\alpha}^{\beta}$, we define that

$$I(\langle \phi \rangle_{\alpha}^{\beta}) = \text{union} \left(\left\{ I \left(\phi \left[\frac{\beta}{g(\beta)} \right] \right) \mid g \in \mathcal{D}^{\bar{\beta}} \right\} \right), \quad (4)$$

where $\bar{\beta}$ denotes the set of elements in the list β , and the assignments in $\mathcal{D}^{\bar{\beta}}$ are limited only to the variables in $\bar{\beta}$.

Remark 8. Here we can make the question if there is a sense to extend the interpretation also to (abstracted) terms, because in Tarski's interpretation of FOL we do not have any interpretation for terms, but only the assignments for terms, as we defined previously by the mapping $g^* : \mathcal{T} \rightarrow \mathcal{D}$. The answer is positive, because the abstraction symbol $\langle _ \rangle_{\alpha}^{\beta}$ can be considered as a kind of the unary built-in functional symbol of intensional FOL, so that we can apply the Tarski's interpretation to this functional symbol into the fixed mapping $I(\langle _ \rangle_{\alpha}^{\beta}) : \mathcal{L} \rightarrow \mathcal{D}$, so that for any $\phi \in \mathcal{L}$ we have that $I(\langle \phi \rangle_{\alpha}^{\beta})$ is equal to the application of this function to the value ϕ , that is, to $I(\langle _ \rangle_{\alpha}^{\beta})(\phi)$. In such an approach, we would introduce also the typed variable X for the formulae in $\mathcal{L}$, so that the Tarski's assignment for this functional symbol with variable X , with $g(X) = \phi \in \mathcal{L}$, can be given by

$$\begin{aligned} g^* (\langle _ \rangle_{\alpha}^{\beta}(X)) &= I(\langle _ \rangle_{\alpha}^{\beta})(g(X)) = I(\langle _ \rangle_{\alpha}^{\beta})(\phi) \\ &= \begin{cases} \langle \rangle \in D_{-1}, \\ \text{if } \bar{\alpha} \cup \bar{\beta} \text{ is not equal to the set of free variables in } \phi; \\ = \text{union} \left(\left\{ I \left(\phi \left[\frac{\beta}{g'(\beta)} \right] \right) \mid g' \in \mathcal{D}^{\bar{\beta}} \right\} \right) \in D_{|\bar{\alpha}|}, \\ \text{otherwise.} \end{cases} \end{aligned} \quad (5)$$

Notice that if $\beta = \emptyset$ is the empty list, then $I(\langle \phi \rangle_{\alpha}^{\beta}) = I(\phi)$. Consequently, the denotation of $\langle \phi \rangle$ is equal to the meaning of a proposition ϕ , that is, $I(\langle \phi \rangle) = I(\phi) \in D_0$. In the case when ϕ is an atom $p_i^m(x_1, \dots, x_m)$, then $I(\langle p_i^m(x_1, \dots, x_m) \rangle_{x_1, \dots, x_m}^{\beta}) = I(p_i^m(x_1, \dots, x_m)) \in D_m$, while $I(\langle p_i^m(x_1, \dots, x_m) \rangle_{x_1, \dots, x_m}^{\beta}) = \text{union}(\{I(p_i^m(g(x_1), \dots, g(x_m))) \mid g \in \mathcal{D}^{\{x_1, \dots, x_m\}}\}) \in D_0$, with $h(I(\langle p_i^m(x_1, \dots, x_m) \rangle_{x_1, \dots, x_m}^{\beta})) = h(I((\exists x_1) \dots (\exists x_m) p_i^m(x_1, \dots, x_m))) \in \{f, t\}$.

For example,

$$\begin{aligned} h(I(\langle p_i^1(x_1) \wedge \neg p_i^1(x_1) \rangle_{x_1}^{\beta})) \\ = h(I((\exists x_1)(\langle p_i^1(x_1) \wedge \neg p_i^1(x_1) \rangle_{x_1}^{\beta}))) = f. \end{aligned} \quad (6)$$

The interpretation of a more complex abstract $\langle \phi \rangle_{\alpha}^{\beta}$ is defined in terms of the interpretations of the relevant syntactically simpler expressions, because the interpretation

of more complex formulae is defined in terms of the interpretation of the relevant syntactically simpler formulae, based on the intensional algebra above. For example, $I(p_i^1(x) \wedge p_k^1(x)) = \text{con}_{j_{\{(1,1)\}}}(I(p_i^1(x)), I(p_k^1(x)))$, $I(\neg \phi) = \text{neg}(I(\phi))$, $I(\exists x_i) \phi(x_i, x_j, x_i, x_k) = \text{exists}_3(I(\phi))$.

Consequently, based on the intensional algebra in Definition 6 and on intensional interpretations of abstracted terms in Definition 7, it holds that the interpretation of any formula in $\mathcal{L}$ (and any abstracted term) will be reduced to an algebraic expression over interpretations of primitive atoms in $\mathcal{L}$. This obtained expression is finite for any finite formula (or abstracted term) and represents the *meaning* of such finite formula (or abstracted term).

The *extension* of an abstracted term satisfy the following property.

Proposition 9. For any abstracted term $\langle \phi \rangle_{\alpha}^{\beta}$ with $|\alpha| \geq 1$, we have that

$$h(I(\langle \phi \rangle_{\alpha}^{\beta})) = \pi_{-\beta}(h(I(\phi))), \quad (7)$$

where $\pi_{-(y_1, \dots, y_k)} = \pi_{-y_1} \circ \dots \circ \pi_{-y_k}$, $\circ$ is the sequential composition of functions, and $\pi_{-\emptyset}$ is an identity.

Proof. Let $\mathbf{x}$ be a tuple of all free variables in ϕ , so that $\bar{\alpha} = \bar{\alpha} \cup \bar{\beta}$, $\alpha = (x_1, \dots, x_k)$, then we have that $h(I(\langle \phi \rangle_{\alpha}^{\beta})) = h(\text{union}(\{I(\phi[\beta/g(\beta)]) \mid g \in \mathcal{D}^{\bar{\beta}}\}))$, from Definition 7 = $\bigcup \{h(I(\phi[\beta/g(\beta)])) \mid g \in \mathcal{D}^{\bar{\beta}}\} = \bigcup \{(g_1(x_1), \dots, g_1(x_k)) \mid g_1 \in \mathcal{D}^{\bar{\alpha}} \text{ and } h(I(\phi[\beta/g(\beta)]))[\alpha/g_1(\alpha)]) = t\} \mid g \in \mathcal{D}^{\bar{\beta}}\} = \{g_1(\alpha) \mid g_1 \in \mathcal{D}^{\bar{\alpha} \cup \bar{\beta}} \text{ and } h(I(\phi/g_1)) = t\} = \pi_{-\beta}(\{g_1(\mathbf{x}) \mid g_1 \in \mathcal{D}^{\bar{\alpha}} \text{ and } h(I(\phi/g_1)) = t\}) = \pi_{-\beta}(\{g_1(\mathbf{x}) \mid g_1 \in \mathcal{D}^{\bar{\alpha}} \text{ and } h(I(\phi/g_1)) = t\}) = \pi_{-\beta}(h(I(\phi)))$, by (T) = $\pi_{-\beta}(h(I(\phi)))$. $\square$

We can correlate $\mathcal{E}$ with a possible-world semantics. Such a correspondence is a natural identification of intensional logics with modal Kripke-based logics.

Definition 10 (model). A model for intensional FOL with fixed intensional interpretation I , which expresses the two-step intensional semantics in Definition 4, is the Kripke structure $\mathcal{M}_{\text{int}} = (\mathcal{W}, \mathcal{D}, V)$, where $\mathcal{W} = \{is^{-1}(h) \mid h \in \mathcal{E}\}$, a mapping $V : \mathcal{W} \times P \rightarrow \bigcup_{n \in \omega} \{t, f\}^{\mathcal{D}^n}$, with P a set of predicate symbols of the language, such that for any world $w = is^{-1}(h) \in \mathcal{W}$, $p_i^n \in P$, and $(u_1, \dots, u_n) \in \mathcal{D}^n$ it holds that $V(w, p_i^n)(u_1, \dots, u_n) = h(I(p_i^n(u_1, \dots, u_n)))$. The satisfaction relation $\models_{w,g}$ for a given $w \in \mathcal{W}$ and assignment $g \in \mathcal{D}^{\mathcal{V}}$ is defined as follows:

- (1) $\mathcal{M} \models_{w,g} p_i^k(x_1, \dots, x_k)$ if and only if $V(w, p_i^k)(g(x_1), \dots, g(x_k)) = t$,
- (2) $\mathcal{M} \models_{w,g} \phi \wedge \psi$ if and only if $\mathcal{M} \models_{w,g} \phi$ and $\mathcal{M} \models_{w,g} \psi$,
- (3) $\mathcal{M} \models_{w,g} \neg \phi$ if and only if not $\mathcal{M} \models_{w,g} \phi$,
- (4) $\mathcal{M} \models_{w,g} (\exists x) \phi$ if and only if
 - (4.1) $\mathcal{M} \models_{w,g} \phi$, if x is not a free variable in ϕ ;
 - (4.2) exists $u \in \mathcal{D}$ such that $\mathcal{M} \models_{w,g} \phi[x/u]$, if x is a free variable in ϕ .

It is easy to show that the satisfaction relation $\models$ for this Kripke semantics in a world $w = is^{-1}(h)$ is defined by $\mathcal{M} \models_{w,g} \phi$ if and only if $h(I(\phi/g)) = t$.

We can enrich this intensional FOL by another modal operators, as, for example, the “necessity” universal logic operator $\Box$ with accessibility relation $\mathcal{R} = \mathcal{W} \times \mathcal{W}$, obtaining an S5 Kripke structure $\mathcal{M}_{\text{int}} = (\mathcal{W}, \mathcal{R}, \mathcal{D}, V)$. In this case, we are able to define the following equivalences between the abstracted terms without free variables $\phi \succ_{\alpha}^{\beta_1}/g$ and $\psi \succ_{\alpha}^{\beta_2}/g$, where all free variables (not in α) are instantiated by $g \in \mathcal{D}^{\mathcal{V}}$ (here $A \equiv B$ denotes the formula $(A \Rightarrow B) \wedge (B \Rightarrow A)$).

(i) (Strong) Intensional equivalence (or *equality*) “ $\simeq$ ” is defined by $\phi \succ_{\alpha}^{\beta_1}/g \simeq \psi \succ_{\alpha}^{\beta_2}/g$ if and only if $\Box(\phi[\beta_1/g(\beta_1)] \equiv \psi[\beta_2/g(\beta_2)])$, with $\mathcal{M} \models_{w,g'} \Box \phi$ if and only if for all $w' \in \mathcal{W}$, $(w, w') \in \mathcal{R}$ implies $\mathcal{M} \models_{w',g'} \phi$. From Example 5, we have that $\phi \succ_{\alpha}^{\beta_1}/g \simeq \psi \succ_{\alpha}^{\beta_2}/g$, that is, “ x has been bought” and “ x has been sold” are intensionally equivalent, but they have not the same meaning (the concept $I(p_1^1(x)) \in D_1$ is different from $I(p_2^1(x)) \in D_1$).

(ii) Weak intensional equivalence “ $\approx$ ” is defined by $\phi \succ_{\alpha}^{\beta_1}/g \approx \psi \succ_{\alpha}^{\beta_2}/g$ if and only if $\Diamond \phi[\beta_1/g(\beta_1)] \equiv \Diamond \psi[\beta_2/g(\beta_2)]$. The symbol $\Diamond = \neg \Box \neg$ is the correspondent existential modal operator. This weak equivalence is used for P2P database integration in a number of papers [16, 19–24].

Note that if we want to use the intensional equality in our language, then we need the correspondent operator in intensional algebra $\mathcal{A}_{\text{int}}$ for the “necessity” modal logic operator $\Box$.

This semantics is equivalent to the algebraic semantics for $\mathcal{L}$ in [8] for the case of the conception where intensional entities are considered to be *equal* if and only if they are *necessarily equivalent*. Intensional equality is much stronger than the standard *extensional equality* in the actual world, just because it requires the extensional equality in *all* possible worlds; in fact, if $\phi \succ_{\alpha}^{\beta_1}/g \simeq \psi \succ_{\alpha}^{\beta_2}/g$, then $h(I(\phi \succ_{\alpha}^{\beta_1}/g)) = h(I(\psi \succ_{\alpha}^{\beta_2}/g))$ for all extensionalization functions $h \in \mathcal{E}$ (i.e., possible worlds $is^{-1}(h) \in \widetilde{\mathcal{W}}$).

It is easy to verify that the intensional equality means that in every possible world $w \in \mathcal{W}$ the intensional entities u_1 and u_2 have the same extensions.

Let the logic modal formula $\Box \phi[\beta_1/g(\beta_1)]$, where the assignment g is applied only to free variables in β_1 of a formula ϕ not in the list of variables in $\alpha = (x_1, \dots, x_n)$, $n \geq 1$, represents an n -ary intensional concept such that $I(\Box \phi[\beta_1/g(\beta_1)]) \in D_n$ and $I(\phi[\beta_1/g(\beta_1)]) = I(\phi \succ_{\alpha}^{\beta_1}/g) \in D_n$. Then the extension of this n -ary concept is equal to (here

the mapping $\text{necess} : D_i \rightarrow D_i$ for each $i \geq 0$ is a new operation of the intensional algebra $\mathcal{A}_{\text{int}}$ in Definition 6)

$$\begin{aligned}
 & h \left(I \left(\Box \phi \left[\frac{\beta_1}{g(\beta_1)} \right] \right) \right) \\
 &= h \left(\text{necess} \left(I \left(\phi \left[\frac{\beta_1}{g(\beta_1)} \right] \right) \right) \right) \\
 &= \left\{ (g'(x_1), \dots, g'(x_n)) \mid \right. \\
 &\quad \left. \mathcal{M} \models_{w,g'} \Box \phi \left[\frac{\beta_1}{g(\beta_1)} \right], g' \in \mathcal{D}^{\mathcal{V}} \right\} \\
 &= \left\{ (g'(x_1), \dots, g'(x_n)) \mid g' \in \mathcal{D}^{\mathcal{V}}, \forall w_1 \right. \\
 &\quad \left. \left((w, w_1) \in \mathcal{R} \text{ implies } \mathcal{M} \models_{w_1,g'} \phi \left[\frac{\beta_1}{g(\beta_1)} \right] \right) \right\} \\
 &= \bigcap_{h_1 \in \mathcal{E}} h_1 \left(I \left(\phi \left[\frac{\beta_1}{g(\beta_1)} \right] \right) \right), \tag{8}
 \end{aligned}$$

while

$$\begin{aligned}
 & h \left(I \left(\Diamond \phi \left[\frac{\beta_1}{g(\beta_1)} \right] \right) \right) \\
 &= h \left(I \left(\neg \Box \neg \phi \left[\frac{\beta_1}{g(\beta_1)} \right] \right) \right) \\
 &= h \left(\text{neg} \left(\text{necess} \left(I \left(\neg \phi \left[\frac{\beta_1}{g(\beta_1)} \right] \right) \right) \right) \right) \\
 &= \mathcal{D}^n \setminus h \left(\text{necess} \left(I \left(\neg \phi \left[\frac{\beta_1}{g(\beta_1)} \right] \right) \right) \right) \\
 &= \mathcal{D}^n \setminus \left(\bigcap_{h_1 \in \mathcal{E}} h_1 \left(I \left(\neg \phi \left[\frac{\beta_1}{g(\beta_1)} \right] \right) \right) \right) \\
 &= \mathcal{D}^n \setminus \left(\bigcap_{h_1 \in \mathcal{E}} h_1 \left(\text{neg} \left(I \left(\phi \left[\frac{\beta_1}{g(\beta_1)} \right] \right) \right) \right) \right) \\
 &= \mathcal{D}^n \setminus \left(\bigcap_{h_1 \in \mathcal{E}} \mathcal{D}^n \setminus h_1 \left(I \left(\phi \left[\frac{\beta_1}{g(\beta_1)} \right] \right) \right) \right) \\
 &= \bigcup_{h_1 \in \mathcal{E}} h_1 \left(I \left(\phi \left[\frac{\beta_1}{g(\beta_1)} \right] \right) \right). \tag{9}
 \end{aligned}$$

Consequently, the concepts $\Box \phi[\beta_1/g(\beta_1)]$ and $\Diamond \phi[\beta_1/g(\beta_1)]$ are the *built-in* (or rigid) concept as well, whose extensions do not depend on possible worlds.

Thus, two concepts are *intensionally equal*, that is, $\langle \phi \rangle_{\alpha}^{\beta_1}/g \approx \langle \psi \rangle_{\alpha}^{\beta_2}/g$, if and only if $h(I(\phi[\beta_1/g(\beta_1)])) = h(I(\psi[\beta_2/g(\beta_2)]))$ for every h .

Analogously, two concepts are *weakly equivalent*, that is, $\langle \phi \rangle_{\alpha}^{\beta_1}/g \approx \langle \psi \rangle_{\alpha}^{\beta_2}/g$ if and only if $h(I(\Diamond\phi[\beta_1/g(\beta_1)])) = h(I(\Diamond\psi[\beta_2/g(\beta_2)]))$.

3. Application to the Intensional FOL without Abstraction Operator

In the case of the intensional FOL defined in Definition 1, without Bealer's intensional abstraction operator $\langle \cdot \rangle$, we obtain the syntax of the standard FOL but with intensional semantics as presented in [15].

Such a FOL has a well-known Tarski's interpretation, defined as follows.

An interpretation (Tarski) I_T consists in a nonempty domain $\mathcal{D}$ and a mapping that assigns to any predicate letter $p_i^k \in P$ a relation $R = I_T(p_i^k) \subseteq \mathcal{D}^k$, to any functional letter $f_i^k \in F$ a function $I_T(f_i^k) : \mathcal{D}^k \rightarrow \mathcal{D}$, or, equivalently, its graph relation $R = I_T(f_i^k) \subseteq \mathcal{D}^{k+1}$ where the $k + 1$ th column is the resulting function's value, and to each individual constant $c \in F$ one given element $I_T(c) \in \mathcal{D}$.

Consequently, from the intensional point of view, an interpretation of Tarski is a possible world in the Montague's intensional semantics, that is, $w = I_T \in \mathcal{W}$. The correspondent extensionalization function is $h = is(w) = is(I_T)$.

We define the satisfaction of a logic formulae in $\mathcal{L}$ for a given assignment $g : \mathcal{V} \rightarrow \mathcal{D}$ inductively, as follows.

If a formula ϕ is an atomic formula $p_i^k(t_1, \dots, t_k)$, then this assignment g satisfies ϕ if and only if $(g^*(t_1), \dots, g^*(t_k)) \in I_T(p_i^k)$; g satisfies $\neg\phi$ if and only if it does not satisfy ϕ ; g satisfies $\phi \wedge \psi$ iff g satisfies ϕ and g satisfies ψ ; g satisfies $(\exists x_i)\phi$ if and only if there exists an assignment $g' \in \mathcal{D}^{\mathcal{V}}$ that may differ from g only for the variable $x_i \in \mathcal{V}$, and g' satisfies ϕ .

A formula ϕ is true for a given interpretation I_T if and only if ϕ is satisfied by every assignment $g \in \mathcal{D}^{\mathcal{V}}$. A formula ϕ is valid (i.e., tautology) if and only if ϕ is true for every Tarski's interpretation $I_T \in \mathfrak{I}_T$. An interpretation I_T is a model of a set of formulae Γ if and only if every formula $\phi \in \Gamma$ is true in this interpretation. We denote by $\text{FOL}(\Gamma)$ the FOL with a set of assumptions Γ , and by $\mathfrak{I}_T(\Gamma)$ the subset of Tarski's interpretations that are models of Γ , with $\mathfrak{I}_T(\emptyset) = \mathfrak{I}_T$. A formula ϕ is said to be a *logical consequence* of Γ , denoted by $\Gamma \models \phi$, if and only if ϕ is true in all interpretations in $\mathfrak{I}_T(\Gamma)$. Thus, $\models \phi$ if and only if ϕ is a tautology.

The basic set of axioms of the FOL are that of the propositional logic with two additional axioms: (A1) $(\forall x)(\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow (\forall x)\psi)$ (x does not occur in ϕ

and it is not bound in ψ), and (A2) $(\forall x)\phi \Rightarrow \phi[x/t]$, (neither x nor any variable in t occurs bound in ϕ). For the FOL with identity, we need the *proper* axiom (A3) $x_1 \doteq x_2 \Rightarrow (x_1 \doteq x_3 \Rightarrow x_2 \doteq x_3)$.

The inference rules are Modus Ponens and generalization (G) "if ϕ is a theorem and x is not bound in ϕ , then $(\forall x)\phi$ is a theorem."

The standard FOL is considered as an extensional logic because two open sentences with the same tuple of variables $\phi(x_1, \dots, x_m)$ and $\psi(x_1, \dots, x_m)$ are equal if and only if they have the *same extension* in a given interpretation I_T , that is, if and only if $I_T^*(\phi(x_1, \dots, x_m)) = I_T^*(\psi(x_1, \dots, x_m))$, where I_T^* is the unique extension of I_T to all formulae, as follows.

- (1) For a (closed) sentence ϕ/g , we have that $I_T^*(\phi/g) = t$ if and only if g satisfies ϕ , as recursively defined above.
- (2) For an open-sentence ϕ with the tuple of free variables $(x_1, \dots, x_m)$, we have that $I_T^*(\phi(x_1, \dots, x_m)) =_{\text{def}} \{(g(x_1), \dots, g(x_m)) \mid g \in \mathcal{D}^{\mathcal{V}} \text{ and } I_T^*(\phi/g) = t\}$.

It is easy to verify that for a formula ϕ with the tuple of free variables $(x_1, \dots, x_m)$, $I_T^*(\phi(x_1, \dots, x_m)/g) = t$ if and only if $(g(x_1), \dots, g(x_m)) \in I_T^*(\phi(x_1, \dots, x_m))$.

This extensional *equality* of two virtual predicates can be generalized to the extensional *equivalence* when both predicates ϕ, ψ have the same set of free variables but their ordering in the *tuples* of free variables is not identical: such two virtual predicates are equivalent if the extension of the first is equal to the proper permutation of columns of the extension of the second virtual predicate. It is easy to verify that such an extensional equivalence corresponds to the logical equivalence denoted by $\phi \equiv \psi$.

This extensional equivalence between two relations $R_1, R_2 \in \mathfrak{R}$ with the same arity will be denoted by $R_1 \equiv R_2$, while the extensional identity will be denoted in the standard way by $R_1 = R_2$.

Let $\mathcal{A}_{\text{FOL}} = (\mathcal{L}, \doteq, \top, \wedge, \neg, \exists)$ be a free syntax algebra for "first-order logic with identity $\doteq$ ", with the set $\mathcal{L}$ of first-order logic formulae, with $\top$ denoting the tautology formula (the contradiction formula is denoted by $\neg\top$), with the set of variables in $\mathcal{V}$ and the domain of values in $\mathcal{D}$. It is well known that we are able to make the extensional algebraization of the FOL by using the *cylindric* algebras [25] that are the extension of Boolean algebras with a set of binary operators for the FOL identity relations and a set of unary algebraic operators ("projections") for each case of FOL quantification $(\exists x)$. In what follows, we will make an analog extensional algebraization over $\mathfrak{R}$ but by interpretation of the logic conjunction $\wedge$ by a set of *natural join* operators over relations introduced by Codd's relational algebra [9] and [26] as a kind of a predicate calculus whose interpretations are tied to the database.

Corollary 11 (extensional FOL semantics [15]). *Let us define the extensional relational algebra for the FOL by*

$$\mathcal{A}_{\mathfrak{R}} = (\mathfrak{R}, R_-, \{\langle \cdot \rangle\}, \{\mathfrak{M}_S\}_{S \in \mathcal{P}(\mathbb{N}^2)}, \sim, \{\pi_{-n}\}_{n \in \mathbb{N}}), \quad (10)$$

where $\{\langle \rangle\} \in \mathfrak{R}$ is the algebraic value correspondent to the logic truth and $R_ =$ is the binary relation for extensionally equal elements. We will use “=” for the extensional identity for relations in $\mathfrak{R}$.

Then, for any Tarski’s interpretation I_T its unique extension to all formulae $I_T^* : \mathcal{L} \rightarrow \mathfrak{R}$ is also the homomorphism $I_T^* : \mathcal{A}_{\text{FOL}} \rightarrow \mathcal{A}_{\mathfrak{R}}$ from the free syntax FOL algebra into this extensional relational algebra.

Proof. Directly from definition of the semantics of the operators in $\mathcal{A}_{\mathfrak{R}}$ defined in precedence, let us take the case of conjunction of logic formulae of the definition above where $\varphi(x_i, x_j, x_k, x_l, x_m, y_i, y_j)$ (its tuple of variables is obtained by the method defined in the FOL introduction) is the virtual predicate of the logic formula $\phi(x_i, x_j, x_k, x_l, x_m) \wedge \psi(x_i, y_i, x_j, y_j)$: $I_T^*(\phi \wedge \psi) = I_T^*(\phi) = \{(g(x_i), g(x_j), g(x_k), g(x_l), g(x_m), g(y_i), g(y_j)) \mid I_T^*(\phi/g) = t\} = \{(g(x_i), g(x_j), g(x_k), g(x_l), g(x_m), g(y_i), g(y_j)) \mid I_T^*(\phi/g \wedge \psi/g) = t\} = \{(g(x_i), g(x_j), g(x_k), g(x_l), g(x_m), g(y_i), g(y_j)) \mid I_T^*(\phi/g) = t \text{ and } I_T^*(\psi/g) = t\} = \{(g(x_i), g(x_j), g(x_k), g(x_l), g(x_m), g(y_i), g(y_j)) \mid (g(x_i), g(x_j), g(x_k), g(x_l), g(x_m)) \in I_T^*(\phi) \text{ and } (g(x_i), g(y_i), g(x_j), g(y_j)) \in I_T^*(\psi)\} = I_T^*(\phi) \bowtie_{\{(4,1),(2,3)\}} I_T^*(\psi)$.

Thus, it is enough to show that $I_T^*(\top) = \{\langle \rangle\}$ is also valid, and $I_T^*(\neg \top) = \emptyset$. The first property comes from the fact that $\top$ is a tautology, thus satisfied by every assignment g , that is, it is true, that is, $I_T^*(\top) = t$ (and t is equal to the empty tuple $\{\langle \rangle\}$). The second property comes from the fact that $I_T^*(\neg \top) = \sim(I_T^*(\top)) = \sim(\{\langle \rangle\}) = \mathcal{D}^0 \setminus \{\langle \rangle\} = \{\langle \rangle\} \setminus \{\langle \rangle\} = \emptyset$. That is, the tautology and the contradiction have the true and false logic value, respectively, in $\mathfrak{R}$.

We have also that $I_T^*(\doteq (x, y)) = I_T(\doteq) = R_ =$ for every interpretation I_T because $\doteq$ is the built-in binary predicate, that is, with the same extension in every Tarski’s interpretation.

Consequently, the mapping $I_T^* : (\mathcal{L}, \doteq, \top, \wedge, \neg, \exists) \rightarrow \mathcal{A}_{\mathfrak{R}}$ is a homomorphism that represents the extensional Tarskian semantics of the FOL. $\square$

Consequently, we obtain the following Intensional/extensional FOL semantics [15].

For any Tarski’s interpretation I_T of the FOL, the following diagram of homomorphisms commutes.

$$\begin{array}{ccc}
 & \mathcal{A}_{\text{int}} \text{ (concepts/meaning)} & \\
 \text{Intensional interpret. } I & \xrightarrow{\text{Frege/Russell semantics}} & h \text{ (extensionalization)} \\
 & \searrow & \swarrow \\
 \mathcal{A}_{\text{FOL}} \text{ (syntax)} & \xrightarrow{I_T^* \text{ (Tarski's interpretation)}} & \mathcal{A}_{\mathfrak{R}} \text{ (denotation)}
 \end{array} \quad (11)$$

where $h = \text{is}(w)$ and $w = I_T \in \mathcal{W}$ is the explicit possible world (extensional Tarski’s interpretation).

This homomorphic diagram formally expresses the fusion of Frege’s and Russell’s semantics [27–29] of meaning and denotation of the FOL language and renders mathematically correct the definition of what we call an “intuitive notion of

intensionality,” in terms of which a language is intensional if denotation is distinguished from sense: that is, if both denotation and sense are ascribed to its expressions. This notion is simply adopted from Frege’s contribution (without its infinite sense-hierarchy, avoided by Russell’s approach where there is only one meaning relation, one fundamental relation between words and things, here represented by one fixed intensional interpretation I), where the sense contains mode of presentation (here described algebraically as an algebra of concepts (intensions) $\mathcal{A}_{\text{int}}$), and where sense determines denotation for any given extensionalization function h (correspondent to a given Tarski’s interpretation I_T). More about the relationships between Frege’s and Russell’s theories of meaning may be found in the Chapter 7, “Extensionality and Meaning”, in [18].

As noted by Gottlob Frege and Rudolf Carnap (he uses terms Intension/extension in the place of Frege’s terms sense/denotation [30]), the two logic formulae with the same denotation (i.e., the same extension for a given Tarski’s interpretation I_T) need not have the same sense (intension), thus such codenotational expressions are not *substitutable* in general.

In fact there is exactly *one* sense (meaning) of a given logic formula in $\mathcal{L}$, defined by the uniquely fixed intensional interpretation I , and a *set* of possible denotations (extensions) each determined by a given Tarski’s interpretation of the FOL as follows from Definition 4:

$$\mathcal{L} \xrightarrow{I} \mathcal{D} \xRightarrow{h=\text{is}(I_T) \& I_T \in \mathcal{W} = \mathfrak{S}_T(I)} \mathfrak{R}. \quad (12)$$

Often “intension” has been used exclusively in connection with possible worlds semantics; however, here we use (as many others; as Bealer for example) “intension” in a more wide sense, that is, as an *algebraic expression* in the intensional algebra of meanings (concepts) $\mathcal{A}_{\text{int}}$, which represents the structural composition of more complex concepts (meanings) from the given set of atomic meanings. Consequently, not only the denotation (extension) is compositional, but also the meaning (intension) is compositional.

4. Conclusion

Semantics is a theory concerning the fundamental relations between words and things. In Tarskian semantics of the FOL, one defines what it takes for a sentence in a language to be truly relative to a model. This puts one in a position to define what it takes for a sentence in a language to be valid. Tarskian semantics often proves quite useful in logic. Despite this, Tarskian semantics neglects meaning, as if truth in language were autonomous. Because of that the Tarskian theory of truth becomes inessential to the semantics for more expressive logics, or more “natural” languages.

Both Montague’s and Bealer’s approaches were useful for this investigation of the intensional FOL with intensional abstraction operator, but the first is not adequate and explains why we adopted two-step intensional semantics (intensional interpretation with the set of extensionalization functions).

At the end of this work, we defined an extensional algebra for the FOL (different from standard cylindric algebras) and

the commutative homomorphic diagram that expresses the generalization of the Tarskian theory of truth for the FOL into the Frege/Russell's theory of meaning.

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