Research Article

On Entire and Meromorphic Functions That Share One Small Function with Their Differential Polynomial

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We study the uniqueness of meromorphic functions that share one small function with more general differential polynomial $P[f]$. As corollaries, we obtain results which answer open questions posed by Yu (2003).

1. Introduction and Main Results

In this paper, a meromorphic functions mean meromorphic in the whole complex plane. We use the standard notations of Nevanlinna theory (see [1]). A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if $T(r,a) = S(r,f)$, that is, $T(r,a) = o(T(r,f))$ as $r \to \infty$ possibly outside a set of finite linear measure. If $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with same multiplicities (ignoring multiplicities), then we say that $f(z)$ and $g(z)$ share $a(z)$ CM(IM).

For any constant $a$, we denote by $N_{k}(r,1/(f - a))$ the counting function for zeros of $f(z) - a$ with multiplicity no more than $k$ and $N_{k}(r,1/(f - a))$ the corresponding for which multiplicity is not counted. Let $N_{k}(r,1/(f - a))$ be the counting function for zeros of $f(z) - a$ with multiplicity at least $k$ and $N_{k}(r,1/(f - a))$ the corresponding for which the multiplicity is not counted.

Let $f$ and $g$ be two nonconstant meromorphic functions sharing value 1IM. Let $z_0$ be common one point of $f$ and $g$ with multiplicity $p$ and $q$, respectively. We denote by $N_{k}(r,1/(f - 1))$ ($N_{k}(r,1/(g - 1))$) the counting function of those 1-points of $f$ where $p > q$; by $N_{k}^{(1)}(r,1/(f - 1))$ the counting function of those 1-points of $f$ where $p = q = 1$; by $N_{k}^{(2)}(r,1/(f - 1))$ the counting function of those 1-points of $f$ where $p = q \geq 2$. In the same way, we can define $N_{k}(r,1/(g - 1))$, $N_{k}^{(1)}(r,1/(g - 1))$ and $N_{k}^{(2)}(r,1/(g - 1))$ (see [2]).


**Conjecture 1.** Let $f$ be a nonconstant entire function such that the hyper-order $\sigma_{2}(f)$ of $f$ is not a positive integer and $\sigma_{2}(f) < \infty$. If $f$ and $f^{(k)}$ share a finite value $a$ CM, then $(f^{(k)} - a)/(f - a) = c$, where $c$ is a nonzero constant.

Many people extended this theorem and obtained many results. In 2003, Yu [4] proved the following theorem.

**Theorem B.** Let $k \geq 1$. Let $f$ be a nonconstant meromorphic function and $a(z)$ a meromorphic function such that $a(z) \neq 0, \infty$, $f$ and $a$ do not have any common pole and $T(r,a) = \sigma(T(r,f))$ as $r \to \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and

$$4\delta(0,f) + 2(8 + k)\Theta(\infty,f) > 19 + 2k,$$

then $f \equiv f^{(k)}$. 

In [3], under an additional hypothesis, Brück proved that the conjecture holds when $a = 1$.

**Theorem A.** Let $f$ be a nonconstant entire function. If $f$ and $f^{(k)}$ share the value 1 CM and $N(r,1/(f^{(k)})) = S(r,f)$, then $(f^{(k)} - 1)/(f - 1) = c$, for some constant $c \in \mathbb{C} \setminus \{0\}$.

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**Theorem B.** Let $k \geq 1$. Let $f$ be a nonconstant meromorphic function and $a(z)$ a meromorphic function such that $a(z) \neq 0, \infty$, $f$ and $a$ do not have any common pole and $T(r,a) = \sigma(T(r,f))$ as $r \to \infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and

$$4\delta(0,f) + 2(8 + k)\Theta(\infty,f) > 19 + 2k,$$

then $f \equiv f^{(k)}$. 

In this paper, we use the standard notations of Nevanlinna theory (see [1]). A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if $T(r,a) = S(r,f)$, that is, $T(r,a) = o(T(r,f))$ as $r \to \infty$ possibly outside a set of finite linear measure. If $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with same multiplicities (ignoring multiplicities), then we say that $f(z)$ and $g(z)$ share $a(z)$ CM(IM).
Theorem C. Let $k \geq 1$. Let $f$ be a nonconstant entire function and $a(z)$ be a meromorphic function such that $a(z) \neq 0, \infty$ and $T(r, a) = o(T(r, f))$ as $r \to \infty$. If $f - a$ and $f^{(k)} - a$ share the value $0$ CM and
\[
\delta(0, f) > \frac{3}{4},
\]
then $f \equiv f^{(k)}$.

In the same paper, the author posed the following questions.

Question 1. Can a CM shared value be replaced by an IM shared value in Theorem C?

Question 2. Is the condition $\delta(0, f) > 3/4$ sharp in Theorem C?

Question 3. Is the condition $4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k$ sharp in Theorem B?

In 2004, Liu and Gu [5] applied different method and obtained the following theorem which answers some questions.

Theorem D. Let $k \geq 1$. Let $f$ be a nonconstant meromorphic function and $a(z)$ a meromorphic function such that $a(z) \neq 0, \infty$ and $T(r, a) = S(r, f)$ as $r \to \infty$. If $f - a$ and $f^{(k)} - a$ share the value $0$ CM and $f^{(k)}$ and $a(z)$ do not have any common poles of same multiplicity and
\[
2\delta(0, f) + 4\Theta(\infty, f) > 5,
\]
then $f \equiv f^{(k)}$.

Theorem E. Let $k \geq 1$. Let $f$ be a nonconstant entire function and $a(z)$ a meromorphic function such that $a(z) \neq 0, \infty$ and $T(r, a) = S(r, f)$ as $r \to \infty$. If $f - a$ and $f^{(k)} - a$ share the value $0$ CM and
\[
\delta(0, f) > \frac{1}{2},
\]
then $f \equiv f^{(k)}$.

Recently, Zhang and Lü [6] considered the problem of meromorphic functions sharing one small function with its $k$th derivative and proved the following theorem.

Theorem F. Let $k(\geq 1), n(\geq 1)$ be integers and $f$ a nonconstant meromorphic function. Also let $a(z) \neq 0, \infty$ be a small meromorphic function with respect to $f$. If $f^n$ and $f^{(k)}$ share the value $a(z)$ IM and
\[
(2k + 6)\Theta(\infty, f) + 4\Theta(0, f) + 2\delta_{k+2}(0, f) > 2k + 12 - n,
\]
then $f \equiv f^{(k)}$.

or $f^n$ and $f^{(k)}$ share the value $a(z)$ CM and
\[
(k + 3)\Theta(\infty, f) + 2\Theta(0, f) + \delta_{k+2}(0, f) > k + 6 - n,
\]
then $f \equiv f^{(k)}$.

Regarding these results, a natural question is what can be said when a nonconstant meromorphic function $f$ shares one nonzero small meromorphic function $a(z)$ with $P[f]$, where $P[f]$ is a differential polynomial in $f$.

Definition 2. Any expression of the type
\[
P[f] = \sum_{j=0}^{n} a_j(z) f^{n_j}(f^{(ij)})^{n_{ij}} \cdots (f^{(mj)})^{n_{mj}}
\]
is called differential polynomial in $f$ of degree $\overline{d}(P)$, lower degree $\underline{d}(P)$, and weight $\Gamma_p$, where $n_0, n_1, \ldots, n_m$ are non-negative integers, $a_i = a_i(z)$ are meromorphic functions satisfying $T(r, a_i) = S(r, f)$ and
\[
\overline{d}(P) = \max \left\{ \sum_{j=0}^{m} n_{ij} : 1 \leq i \leq n \right\},
\]
\[
\underline{d}(P) = \min \left\{ \sum_{j=0}^{m} n_{ij} : 1 \leq i \leq n \right\},
\]
\[
\Gamma_p = \max \left\{ \sum_{j=0}^{m} (j + 1) n_{ij} : 1 \leq i \leq n \right\}.
\]

Further, if $\overline{d}(P) = \underline{d}(P) = n$ (say), then the differential polynomial $P[f]$ is called a homogeneous differential polynomial in $f$ of degree $n$.

Correspond to the above question, we obtain the following results, which extend and improve Theorems A–F and give answers to the questions posed by Yu [4] for more general differential polynomial.

Theorem 3. Let $f$ be a nonconstant meromorphic function and $a(z)$ be a small meromorphic function such that $a(z) \neq 0, \infty$. If $f$ and $P[f]$ be a nonconstant differential polynomial in $f$ as defined in (7). If $f$ and $P[f]$ share the value a IM and
\[
(2Q + 6)\Theta(\infty, f) + (2 + 3\overline{d}(P))\delta(0, f) > 2Q + 2\underline{d}(P) + \overline{d}(P) + 7,
\]
then $f \equiv P[f]$.

Remark 4. Taking $P[f] = f^{(k)}$, that is, $Q = k$, $\overline{d}(P) = \underline{d}(P) = 1$ in (9), we get $(2k + 6)\Theta(\infty, f) + 5\delta(0, f) > 2k + 10$, which improves (3) and extends the theorem to more general differential polynomial $P[f]$ as defined in (7).

Theorem 5. Let $f$ be a nonconstant meromorphic function and $a(z)$ be a small meromorphic function such that $a(z) \neq 0, \infty$. Let $P[f]$ a nonconstant differential polynomial in $f$ as defined in (7). If $f$ and $P[f]$ share the value a CM and
\[
3\Theta(\infty, f) + (\underline{d}(P) + 1)\delta(0, f) > 4,
\]
then $f \equiv P[f]$. 

Remark 6. Taking \( P[f] = f^{(k)} \), that is, \( Q = k, \widetilde{d}(P) = d(P) = 1 \) in (10), we get \( 3\Theta(\infty, f) + 2\delta(0, f) > 4 \), which improves (6) and extends the theorem to more general differential polynomial \( P[f] \) as defined in (7).

Remark 6 gives answer to Question 3 of [4].

Theorem 7. Let \( f \) be a nonconstant entire function and \( a(z) \) a small meromorphic function such that \( a(z) \not\equiv 0, \infty \). Let \( P[f] \) be a nonconstant differential polynomial in \( f \) as defined in (7) If \( f \) and \( P[f] \) share the value \( a \) IM and

\[
(3\delta(P) + 2) \delta(0, f) > 2\delta(P) + 2, \tag{11}
\]

then \( f \equiv P[f] \).

Remark 8 gives answer to Question 1 of Yu [4].

Theorem 9. Let \( f \) be a nonconstant entire function and \( a(z) \) be a small meromorphic function such that \( a(z) \not\equiv 0, \infty \). Let \( P[f] \) be a nonconstant differential polynomial in \( f \) as defined in (7) If \( f \) and \( P[f] \) share the value \( a \) CM and

\[
(\delta(P) + 1) \delta(0, f) > 1, \tag{13}
\]

then \( f \equiv P[f] \).

Remark 10. Taking \( P[f] = f^{(k)} \), that is, \( Q = k, \widetilde{d}(P) = d(P) = 1 \) in (13), we get \( \delta(0, f) > 1/2 \), which improves Theorem C and extends the theorem to more general differential polynomial \( P[f] \) as defined in (7).

Remark 10 gives answer to Question 2 of Yu [4].

Remark 11. By proving Remarks 6, 8, and 10 we have answered Questions 3, 1, and 2 (of [4]), respectively, for the case \( f^{(k)} \). Theorems 3–9 improve and generalize Theorems A–F for more general differential polynomial \( P[f] \).
Lemma 15. Let \( f \) be a transcendental meromorphic function. Let \( P[f] \) be defined as in (7). If \( P[f] \not\equiv 0 \), we have
\[
N\left(r, \frac{1}{P[f]}\right) \leq T(r, P[f]) + \left(\overline{d}(P) - \underline{d}(P)\right) m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f}\right) + S(r, f),
\]
where
\[
N(\overline{d}(P)) = \overline{N}(r, f) + Q\overline{N}(r, f) + S(r, f).
\]

Proof. By the first fundamental theorem, we have
\[
N\left(r, \frac{1}{P[f]}\right) = T(r, P[f]) - m\left(r, \frac{1}{P[f]}\right) + O(1).
\]

We have
\[
m\left(r, \frac{1}{\overline{d}(P)}\right) = m\left(r, \frac{P[f]}{\overline{d}(P)}\right) + m\left(r, \frac{1}{P[f]}\right)
\]
and
\[
m\left(r, \frac{1}{\underline{d}(P)}\right) = m\left(r, \frac{P[f]}{\underline{d}(P)}\right) \leq m\left(r, \frac{1}{P[f]}\right)
\]
or
\[
m\left(r, \frac{1}{P[f]}\right) \leq -m\left(r, \frac{1}{\overline{d}(P)}\right) + m\left(r, \frac{P[f]}{\overline{d}(P)}\right).
\]

By (21), (23) and Lemma 12, we obtain (19). Since
\[
T(r, P[f]) = m\left(r, \frac{P[f]}{\overline{d}(P)}\right) + m\left(r, \frac{P[f]}{\overline{d}(P)}\right) + \overline{d}(P) N\left(r, \frac{1}{f}\right) + Q\overline{N}(r, f) + S(r, f)
\]
we get
\[
T(r, P[f]) \leq \overline{d}(P) T(r, f) + Q\overline{N}(r, f) + S(r, f).
\]
Substituting (25) in (19), we obtain (20).

Lemma 16 (see [10]). Let \( f \) be a transcendental meromorphic function, \( P[f] \) a differential polynomial in \( f \) of degree \( \overline{d}(P) \) and weight \( \Gamma_{P} \). Then \( T(r, P) = O(T(r, f)) \), \( S(r, P) = S(r, f) \).

3. Proof of Theorems

Proof of Theorem 3. Let
\[
F = \frac{P[f]}{a}, \quad G = \frac{f}{a}.
\]
From the conditions of Theorem 3, we know that \( F \) and \( G \) share 1IM. From (26), we have
\[
T(r, F) = O(T(r, f)) + S(r, f), T(r, G) \leq T(r, f) + S(r, f),
\]
\[
\overline{N}(r, F) = \overline{N}(r, G) + S(r, f),
\]
\[
\overline{N}(r, F) = \overline{N}(r, f) + S(r, f), \overline{N}(r, G)
\]
\[
= \overline{N}(r) + S(r, f),
\]
\[
N_{E}^{1}\left(r, \frac{1}{F - 1}\right) + N_{E}^{2}\left(r, \frac{1}{G - 1}\right) + S(r, f)\]
\[
N_{L}\left(r, \frac{1}{F - 1}\right) \leq N_{E}^{1}\left(r, \frac{1}{F - 1}\right) + N_{E}^{2}\left(r, \frac{1}{G - 1}\right) + S(r, f)\]
\[
N_{L}\left(r, \frac{1}{F - 1}\right) = \overline{N}\left(r, \frac{1}{F - 1}\right) + S(r, f)\]
\[
\leq N_{E}^{1}\left(r, \frac{1}{F - 1}\right) + N_{E}^{2}\left(r, \frac{1}{G - 1}\right) + S(r, f)\]
\[
\leq N_{E}^{1}\left(r, \frac{1}{F - 1}\right) + N_{E}^{2}\left(r, \frac{1}{G - 1}\right) + S(r, f)\]
\[
N_{L}\left(r, \frac{1}{F - 1}\right) + N_{L}\left(r, \frac{1}{G - 1}\right) + S(r, f)\]
\[
\leq N_{E}^{1}\left(r, \frac{1}{F - 1}\right) + N_{E}^{2}\left(r, \frac{1}{G - 1}\right) + S(r, f)\]

Let \( H \) be defined by (17). Suppose that \( H \not\equiv 0 \). By Lemma 14, (18) holds.

From (17) and (28), we have
\[
N(r, H) \leq N_{E}^{1}\left(r, \frac{1}{G - 1}\right) + N_{E}^{2}\left(r, \frac{1}{G - 1}\right) + \overline{N}(r, G)
\]
\[
+ N_{L}\left(r, \frac{1}{F - 1}\right) + N_{L}\left(r, \frac{1}{G - 1}\right)
\]
where \( N_{E}(r, 1/F') \) denotes the counting function corresponding to the zeros of \( F' \) which are not the zeros of \( F \) and \( F - 1 \).

Similarly, \( N_{0}(r, 1/G') \) is defined.
From the second fundamental theorem, we have

\[ T(r, F) + T(r, G) \leq \mathcal{N}(r, F) + \mathcal{N}(r, G) \]

\[ + \mathcal{N}\left(r, \frac{1}{F-1}\right) + \mathcal{N}\left(r, \frac{1}{G-1}\right) \]

\[ - N_0(r, \frac{1}{F}) - N_0(r, \frac{1}{G}) + S(r, f). \]

(35)

Since \( F \) and \( G \) share 1IM, we get from (33):

\[ \mathcal{N}\left(r, \frac{1}{F-1}\right) + \mathcal{N}\left(r, \frac{1}{G-1}\right) \]

\[ = 2N^1_E\left(r, \frac{1}{F-1}\right) + 2N_L\left(r, \frac{1}{G-1}\right) \]

\[ + 2N_L\left(r, \frac{1}{G-1}\right) + 2N^{(2)}_E\left(r, \frac{1}{F-1}\right). \]

(36)

From this, (18), and (34), we have

\[ \mathcal{N}\left(r, \frac{1}{F-1}\right) + \mathcal{N}\left(r, \frac{1}{G-1}\right) \]

\[ \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \mathcal{N}(r, G) + N_0\left(r, \frac{1}{F'}\right) \]

\[ + 3N_L\left(r, \frac{1}{G-1}\right) + 3N_L\left(r, \frac{1}{F-1}\right) + \mathcal{N}^{(2)}_E\left(r, \frac{1}{F-1}\right) \]

\[ + 2N^{12}_E\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r, f). \]

(37)

It is clear that

\[ N_L\left(r, \frac{1}{F-1}\right) + 2N_L\left(r, \frac{1}{G-1}\right) \]

\[ + 2N^2_E\left(r, \frac{1}{G-1}\right) + N^{(1)}_E\left(r, \frac{1}{F-1}\right) \]

\[ \leq N\left(r, \frac{1}{G-1}\right) \]

\[ \leq T(r, G) + O(1). \]

(38)

Combining (37), and (38), we obtain

\[ \mathcal{N}\left(r, \frac{1}{F-1}\right) + \mathcal{N}\left(r, \frac{1}{G-1}\right) \]

\[ \leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \mathcal{N}(r, G) \]

\[ + 2N_L\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{G-1}\right) + T(r, G) \]

\[ + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r, f). \]

(39)

Substituting (39) in (35) and using (28), we obtain

\[ T(r, F) \leq 3\mathcal{N}(r, G) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) \]

\[ + 2N_L\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{G-1}\right) + S(r, f). \]

(40)

Using (26) and (19), we get

\[ \overline{d}(P)T(r, f) \]

\[ \leq 3\mathcal{N}(r, G) + \left(\overline{d}(P) - \overline{d}(P)\right)m\left(r, \frac{1}{f}\right) \]

\[ + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^\overline{d}(P)}\right) \]

\[ + 2N_L\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{G-1}\right) + S(r, f). \]

(41)

From (16), (20), and (26) we have

\[ 2N_L\left(r, \frac{1}{F-1}\right) + N_L\left(r, \frac{1}{G-1}\right) \]

\[ \leq 2N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) \]

\[ \leq 2\left[N\left(r, \frac{1}{F}\right) + \mathcal{N}(r, F)\right] + N\left(r, \frac{1}{f}\right) \]

\[ + \mathcal{N}(r, f) + S(r, f) \]

\[ \leq 2N\left(r, \frac{1}{P[f]}\right) + 3\mathcal{N}(r, f) \]

\[ + N\left(r, \frac{1}{f}\right) + S(r, f) \]

\[ \leq 2Q\mathcal{N}(r, f) + 2N\left(r, \frac{1}{\overline{d}(P)}\right) \]

\[ + 2\left(\overline{d}(P) - \overline{d}(P)\right)m\left(r, \frac{1}{f}\right) \]

\[ \leq 2Q\mathcal{N}(r, f) + 2N\left(r, \frac{1}{\overline{d}(P)}\right) \]

\[ \leq 2\left(\overline{d}(P) - \overline{d}(P)\right)m\left(r, \frac{1}{f}\right) \]

\[ + 3\mathcal{N}(r, f) + S(r, f) \]

\[ \leq 2Q\mathcal{N}(r, f) + 2\mathcal{N}\left(r, \frac{1}{P[f]}\right) \]

\[ + 2\left(\overline{d}(P) - \overline{d}(P)\right)m\left(r, \frac{1}{f}\right) \]

\[ \leq 2Q\mathcal{N}(r, f) + 2\mathcal{N}\left(r, \frac{1}{P[f]}\right) \]

\[ + 2\left(\overline{d}(P) - \overline{d}(P)\right)m\left(r, \frac{1}{f}\right) + S(r, f). \]

(42)
From (41) and (42), we get
\[
\bar{d}(P) T(r, f) \\
\leq (2Q + 6) N(r, f) + (2 + 3\bar{d}(P)) N\left(r, \frac{1}{f}\right) \\
+ 3 \left(\bar{d}(P) - \bar{d}(P)\right) m\left(r, \frac{1}{f}\right) + S(r, f) \\
\leq (2Q + 6) N(r, f) + (2 + 3\bar{d}(P)) N\left(r, \frac{1}{f}\right) \\
+ 3 \left(\bar{d}(P) - \bar{d}(P)\right) T\left(r, \frac{1}{f}\right) + S(r, f) \\
+ (3\bar{d}(P) - 2\bar{d}(P)) T(r, f) \\
\leq (2Q + 6) N(r, f) + (2 + 3\bar{d}(P)) N\left(r, \frac{1}{f}\right) \\
+ (2 + 3\bar{d}(P)) N\left(r, \frac{1}{f}\right) + S(r, f) \\
\leq \{2Q + 3\bar{d}(P) + 8\} - \left[(2Q + 6) \Theta(\infty, f) + (2 + 3\bar{d}(P)) \delta(0, f)\right] T(r, f) + S(R, f).
\]
(43)

Therefore, we have
\[
\{2Q + 6\} \Theta(\infty, f) + (2 + 3\bar{d}(P)) \delta(0, f) \\
- \left[(2Q + 3\bar{d}(P) + 8)\right] T(r, f) \leq S(r, f),
\]
(44)

which is a contradiction to our hypothesis (9). Thus \(H \equiv 0\). By integration, we get from (17) that
\[
\frac{1}{G - 1} = \frac{A}{F - 1} + B,
\]
(45)

where \((A \neq 0)\) and \(B\) are constants. Thus
\[
G = \frac{G}{B} + \frac{(A - B - 1)}{A}, \\
F = \frac{G}{B} + \frac{(A - B - 1)}{A}.
\]
(46)

We discuss the following three cases.

Case 1. Suppose that \(B \neq 0, -1\). From (46), we have
\[
N\left(r, \frac{1}{G - (B + 1)/B}\right) = N(r, F).
\]
(47)

From this and second fundamental theorem, we have
\[
T(r, f) \leq T(r, G) + S(r, f) \\
\leq \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right) \\
+ \overline{N}\left(r, \frac{1}{G - (B + 1)/B}\right) + S(r, f) \\
\leq \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r, F) + S(r, f) \\
\leq 2\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{G}\right) + S(r, f) \\
\leq (2Q + 6) N(r, f) + (2 + 3\bar{d}(P)) N\left(r, \frac{1}{f}\right) \\
+ (2 + 3\bar{d}(P)) N\left(r, \frac{1}{f}\right) + S(r, f) \\
\leq \{2Q + 3\bar{d}(P) + 8\} - \left[(2Q + 6) \Theta(\infty, f) + (2 + 3\bar{d}(P)) \delta(0, f)\right] T(r, f) + S(R, f).
\]
(48)

Therefore, we have
\[
\{2Q + 6\} \Theta(\infty, f) + (2 + 3\bar{d}(P)) \delta(0, f) \\
- \left[(2Q + 3\bar{d}(P) + 7)\right] T(r, f) \leq S(r, f),
\]
(49)

which is a contradiction to our hypothesis (9).

Case 2. Suppose that \(B = 0\). From (46), we get
\[
G = \frac{F + (A - 1)}{A}, \\
F = AG - (A - 1),
\]
(50)

we claim \(A = 1\).

If \(A \neq 1\) from (50), we obtain
\[
N\left(r, \frac{1}{G - (A - 1)/A}\right) = N\left(r, \frac{1}{F}\right).
\]
(51)
From this, second fundamental theorem, and (20), we have
\[ T(r, f) \leq T(r, G) + S(r, f) \]
\[ \leq N(r, G) + N(r, \frac{1}{G}) + S(r, f) \]
\[ \leq N(r, G) + N(r, \frac{1}{G}) + N(r, \frac{1}{f}) + S(r, f) \]
\[ \leq N(r, G) + N(r, \frac{1}{f}) + QN(r, f) \]
\[ + (\overline{d}(P) - \overline{d}(P)) m\left(r, \frac{1}{f}\right) \]
\[ + N\left(r, \frac{1}{f \overline{d}(P)}\right) + S(r, f) \]
\[ \leq (Q + 1) N(r, f) + (1 + \overline{d}(P)) N\left(r, \frac{1}{f}\right) \]
\[ + (\overline{d}(P) - \overline{d}(P)) T\left(r, \frac{1}{f}\right) + S(r, f) \]
\[ \leq (2Q + 6) N(r, f) + (2 + 3\overline{d}(P)) N\left(r, \frac{1}{f}\right) \]
\[ + S(r, f) \]
\[ \leq \left[(2Q + 6) \Theta(\infty, f) + (2 + 3\overline{d}(P)) \delta(0, f)\right] T(r, f) + S(R, f). \]
(52)

Hence, we have
\[ \left[(2Q + 6) \Theta(\infty, f) + (2 + 3\overline{d}(P)) \delta(0, f)\right] T(r, f) + S(r, f), \]
which is a contradiction to our hypothesis (9)

Thus, \( A = 1 \).

From (50) we have \( F \equiv G \).

Therefore, we have \( f \equiv P[f] \).

Case 3. Suppose that \( B = -1 \), from (46) we have
\[ G = \frac{A}{-F + A + 1}, \quad F = \frac{1 + A}{G}. \]
(54)

If \( A \neq -1 \), we obtain from (54) that
\[ N\left(r, \frac{1}{G - A/(A + 1)}\right) = N\left(r, \frac{1}{F}\right). \]
(55)

By the same argument as in Case 2, we obtain a contradiction.

Hence, \( A = -1 \).

From (54), we get
\[ FG \equiv 1, \]
that is,
\[ f \cdot P[f] \equiv a^2. \]
(57)

From (57), we have
\[ N(r, f) + N\left(r, \frac{1}{f}\right) = S(r, f). \]
(58)

Using (54), (57), Lemma 12, and first fundamental theorem, we get
\[ (\overline{d}(P) + 1) T(r, f) \]
\[ = T\left(r, \frac{1}{f \overline{d}(P) + 1}\right) \]
\[ = T\left(r, \frac{1}{f \overline{d}(P)}\right) \]
\[ = T\left(r, \frac{P[f]}{f \overline{d}(P)a^2}\right) + S(r, f) \]
\[ = m\left(r, \frac{P[f]}{f \overline{d}(P)}\right) + N\left(r, \frac{P[f]}{f \overline{d}(P)}\right) + S(r, f) \]
\[ \leq (\overline{d}(P) - \overline{d}(P)) m\left(r, \frac{1}{f}\right) + (\overline{d}(P) - \overline{d}(P)) N\left(r, \frac{1}{f}\right) \]
\[ + Q \left[N(r, f) + N\left(r, \frac{1}{f}\right)\right] + S(r, f) \]
\[ \leq (\overline{d}(P) - \overline{d}(P)) m\left(r, \frac{1}{f}\right) + S(r, f) \]
\[ \leq (\overline{d}(P) - \overline{d}(P)) T\left(r, \frac{1}{f}\right) + S(r, f). \]
(59)

From this, we have
\[ (\overline{d}(P) + 1) T(r, f) \leq S(r, f), \]
(60)

which is a contradiction. This completes the proof of Theorem 3.

Proof of Theorem 5. Let \( F \) and \( G \) be given by (26). From the assumption of Theorem 5, we know that \( F \) and \( G \) share 1 CM:
\[ N_L\left(r, \frac{1}{F - 1}\right) = N_L\left(r, \frac{1}{G - 1}\right) = 0. \]
(61)

Proceeding as in Theorem 3, we obtain (41).
Using (61) in (41), we get
\[
\overline{d}(P) T(r, f) 
\leq 3 N(r, G) + (\overline{d}(P) - d(P)) m\left(r, \frac{1}{f}\right) 
+ N\left(r, \frac{1}{\overline{d}(P)}\right) + S(r, f) 
\leq 3 N(r, f) + \left((\overline{d}(P) - d(P))\right) \left(T(r, f) - N\left(r, \frac{1}{f}\right)\right) 
+ (\overline{d}(P) + 1) N\left(r, \frac{1}{f}\right) + S(r, f),
\]
\[
\overline{d}(P) T(r, f) 
\leq 3 N(r, f) + (\overline{d}(P) + 1) N\left(r, \frac{1}{f}\right) + S(r, f) 
\leq \left(\overline{d}(P) + 4\right) 
- \left[3\Theta(\infty, f) + (\overline{d}(P) + 1) \delta(0, f)\right] T(r, f) + S(r, f),
\]
(62)

We have
\[
\left[3\Theta(\infty, f) + (\overline{d}(P) + 1) \delta(0, f) - 4\right] T(r, f) \leq S(r, f),
\]
(63)

which contradicts (10).

Thus, \( H \equiv 0 \). Proceeding as in Theorem 3, we prove Theorem 5.

*Proof of Theorem 7.* \( f \) is a nonconstant entire function. Taking \( N(r, f) = 0 \) in proof of Theorem 3, we obtain Theorem 7.

*Proof of Theorem 9.* \( f \) is a nonconstant entire function. Taking \( N(r, f) = 0 \) in proof of Theorem 5, we obtain Theorem 9.

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