Research Article

On the Classification of Almost Kenmotsu Manifolds of Dimension 3

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This paper deals with the classification of a 3-dimensional almost Kenmotsu manifold satisfying certain geometric conditions. Moreover, by applying our main classification theorem, we obtain some sufficient conditions for an almost Kenmotsu manifold of dimension 3 to be an Einstein-Weyl manifold.

1. Introduction

Contact metric manifolds known as a special class of almost contact metric manifolds are objects of increasing interest both from geometers and physicists [1] recently. We refer the reader to the recent monograph [2] for a wide and detailed overview of the results in this field. From (14) (see Section 3) we know that a normal almost contact metric manifold (which includes Sasakian and Kenmotsu manifolds as its special cases) of dimension 3 satisfies $Q\phi = \phi Q$. But the above property need not be true in an almost contact metric manifold. Blair et al. [3] obtained a classification of 3-dimensional contact metric manifold with $Q\phi = \phi Q$. However, in higher dimensions the classification of contact metric manifold with $Q\phi = \phi Q$ is still open. It is worthy to point out that Ghosh [4] recently proved that a contact metric manifold admitting the Einstein-Weyl structures $W^+ = (g, +\omega)$ and $W^- = (g, -\omega)$.

On the other hand, in 1972, Kenmotsu [5] introduced a class of almost contact metric manifolds which are known as Kenmotsu manifolds nowadays. Recently, almost Kenmotsu manifolds satisfying $\eta$-parallelism and locally symmetries are studied by Dileo and Pastore [6] and [7], respectively. We notice that Dileo and Pastore [8] complete the classification of 3-dimensional almost Kenmotsu manifold with the assumption that $\xi$ belongs to the $(k, \mu)\^0$-nullity distribution.

However, to the best of our knowledge the study of 3-dimensional almost Kenmotsu manifolds is still lacking so far. The object of this paper is to classify the 3-dimensional almost Kenmotsu manifolds satisfying $Q\phi = \phi Q$ and other geometric conditions, providing some results which show the differences between almost Kenmotsu manifolds and the contact metric manifolds of dimension 3 [3, 9]. Moreover, by applying our main classification theorem, we obtain some sufficient conditions for an almost Kenmotsu manifold of dimension 3 to be an Einstein-Weyl manifold.

This paper is organized as the following way. In Section 2, we provide some basic formulas and properties of almost Kenmotsu manifolds. Section 3 is devoted to present our main theorems and their proofs. Finally, in Section 4, we prove that if an almost Kenmotsu manifold of dimension 3 is $\eta$-Einstein with certain condition then it admits both Einstein-Weyl structures $W^+ = (g, +\omega)$ and $W^- = (g, -\omega)$.

2. Almost Kenmotsu Manifolds

First of all, we give some basic notions of almost Kenmotsu manifolds which follow from [5, 7]. An almost contact structure on a $(2n + 1)$-dimensional smooth manifold $M^{2n+1}$ is a triplet $(\phi, \xi, \eta)$, where $\phi$ is a $(1, 1)$-tensor, $\xi$ a global vector field, and $\eta$ a 1-form, such that

$$\phi^2 = -1\text{id} + \eta \otimes \xi, \quad \eta(\xi) = 1,$$  \hspace{1cm} (1)
which implies that $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and rank$(\phi) = 2n$. It follows from $[2, 10]$ that a Riemannian metric $g$ on $M^{2n+1}$ is said to be compatible with the almost contact structure $(\phi, \xi, \eta)$ if
\[ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y). \] (2)

An almost contact structure endowed with a compatible Riemannian metric is said to be an almost contact metric structure. The fundamental 2-form $\Phi$ is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields $X$ and $Y$ on $M^{2n+1}$. An almost Kenmotsu manifold is defined as an almost contact metric manifold together with $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. It is well known that the normality of almost contact structure is expressed by the vanishing of the tensor $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of $\phi$. A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold.

Now let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold. We denote by $l = R(\cdot, \xi)\xi$ and $h = (1/2)L\phi \phi$ on $M^{2n+1}$, where $R$ is the curvature tensor and $L\phi$ is the Lie differentiation, respectively. Thus, the two $(1,1)$-tensors $l$ and $h$ are symmetric and satisfy
\[ h\xi = 0, \quad l\xi = 0, \quad \text{tr} h = 0, \quad \text{tr}(h\phi) = 0, \quad h\phi + \phi h = 0. \] (3)

We also have the following formulas following from $[5, 7, 8]$:
\[ \nabla_X \xi = -\phi^2 X - \phi h X \quad (\Rightarrow \nabla_X \xi = 0), \] (4)
\[ \phi\phi - l = 2(h^2 - \phi^2), \] (5)
\[ \text{tr}(l) = S(\xi, \xi) = g(Q\xi, \xi) = -2n - \text{tr} h^2, \] (6)
\[ R(X, Y)\xi = \eta(X)(Y - \phi h Y) - \eta(Y)(X - \phi h X) + (\nabla_Y \phi h) X - (\nabla_X \phi h) Y, \] (7)
\[ \nabla_Y h = -\phi - 2h - \phi h^2 - \phi l, \] (8)

for any $X, Y \in \Gamma(TM^{2n+1})$, where $S$, $Q$, $\nabla$, and $\Gamma(TM)$ denote the Ricci curvature tensor, Ricci operator, the Levi-Civita connection of $g$, and the Lie algebra of vector fields in $M^{2n+1}$, respectively. From the above formulas we also have $\nabla_X \phi = 0$.

An almost contact manifold is said to be $\eta$-Einstein if
\[ Q = \alpha \phi d + \beta \eta \otimes \xi, \] (9)
where $\alpha$ and $\beta$ are both smooth functions on $M^{2n+1}$. It is easy to see that an $\eta$-Einstein almost Kenmotsu manifold satisfies $Q\phi = \phi Q$ because of $\phi\xi = 0$. We also recall that the $k$-nullity distribution $[11]$ is defined by
\[ N_k = \{ Z \in T_p M : R(X, Y) Z = k(g(Y, Z) X - g(X, Z) Y) \}, \] (10)

where $k$ is a real number. When $k$ in (10) is a smooth functions, then the nullity distributions are called the generalized nullity distributions $[12]$. Also, the sectional curvature $K(\xi, X)$ of a plane section spanned by $\xi$ and a vector $X$ orthogonal to $\xi$ is called a sectional curvature and the sectional curvature $K(\phi X, \phi X)$ of a plane section spanned by vectors $X$ and $\phi X$ with $X$ orthogonal to $\xi$ is called a $\phi$-sectional curvature $[3]$.

A Riemannian manifold $M$ of dimension $m$ is conformally flat if and only if the Weyl tensor $C$ defined by
\[ C(X, Y) Z = R(X, Y) Z - \frac{1}{m-2} \times \{ S(Y, Z) X - S(X, Z) Y + g(Y, Z) X - g(X, Z) Y \} \] (11)
vanishes for $m > 3$ and
\[ (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = \frac{1}{4} \{ X(r) g(Y, Z) - Y(r) g(X, Z) \}, \] (12)
for $m = 3$, where $S$, $Q$, and $r$ denote the Ricci tensor, Ricci operator, and the scalar curvature, respectively.

3. A Classification Theorem

Let $(M^3, \phi, \xi, \eta, g)$ be a normal almost contact metric manifold of dimension 3, then we have $Q\phi = \phi Q$. In fact, it follows from $[13, 14]$ that $M^3$ satisfies
\[ S(X, Y) = \left( \frac{r}{2} + \alpha^2 - \beta^2 \right) g(\phi X, \phi Y) - 2\left( \alpha^2 - \beta^2 \right) \eta(X) \eta(Y), \quad \forall X, Y \in \Gamma(TM), \] (13)
where $\alpha = \text{div}(\xi)$ and $\beta = \text{tr}(\phi V \xi)$ are both constants and $r$ denotes the scalar curvature of $M^3$. Then from the above equation it is easy to get
\[ Q = -\left( \frac{r}{2} + \alpha^2 - \beta^2 \right) \phi^2 - 2\left( \alpha^2 - \beta^2 \right) \eta \otimes \xi. \] (14)

Noticing that $\phi \xi = 0$, then from (14) we know that $Q\phi = \phi Q$. If $\beta = 0$ (resp., $\alpha = 0$) then $M^3$ is just the 3-dimensional Sasakian (resp., Kenmotsu) manifold $[13]$. However, the condition $Q\phi = \phi Q$ need not be true in a contact metric manifold as well as an almost Kenmotsu manifold. In a contact metric manifold $M^{2n+1}$ if the characteristic vector $\xi$ is a Killing vector field, then the manifold is said to be a K-contact manifold. A Sasakian manifold is always a K-contact one, but the converse need not hold only if $M$ is of dimension 3. We present the following result to characterize Kenmotsu manifold of dimension 3 which is analogous to the contact metric manifold of dimension 3.

Lemma 1. Let $(M^3, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold of dimension 3, then the following conditions are equivalent.
(1) \( M^3 \) is a Kenmotsu manifold.
(2) \( h = 0 \).
(3) \( \nabla \xi = -\phi^2 \).

Proof. It follows from [7, 8] that an almost Kenmotsu is a Kenmotsu manifold if and only if

\[
(\nabla_X\phi) Y = -g(X,\phi Y) - \eta(Y) \phi X, \quad \forall X, Y \in \Gamma(TM), \tag{15}
\]

Noticing that \( \nabla_X\phi = 0 \) and \( (\nabla_X\phi)Y + (\nabla_Y\phi)X = -\eta(Y)\phi X - 2g(X,\phi Y) - \eta(Y)h(X) \) (see [7]) for any \( X, Y \in \Gamma(TM) \), thus \( M^3 \) is a Kenmotsu manifold if and only if every nonvanishing vector field \( X \) in contact distribution \( \mathcal{D} \) (defined by \( \mathcal{D} = \ker(\eta) = \text{Im}(\phi) \)) satisfies the following equations:

\[
(\nabla_X\phi) X = 0, \tag{16}
\]
\[
(\nabla_X\phi) \phi X = g(X,X)\xi, \tag{17}
\]
\[
(\nabla_X\phi) \xi = -\phi X, \tag{18}
\]
\[
(\nabla_X\phi) \phi \xi = -2\phi^2 X - \phi h X. \tag{19}
\]

Comparing the above three equations with (16), we complete the proof of equivalence between (1) and (2). The equivalence between (2) and (3) follows from the fact that \( \nabla_X\xi = -\phi^2 X - \phi h X \) for any \( X \in \Gamma(TM) \).

Theorem 2. Let \((M^3,\phi,\xi,\eta,g)\) be an almost Kenmotsu manifold of dimension 3 for which the characteristic vector field \( \xi \) is an eigenvector field of the Ricci operator. If \( M^3 \) is conformally flat, then it satisfies

\[
h^lQ = Qh^l, \tag{19}
\]
\[
X(\text{tr}(l)) = \frac{1}{4}X(r), \tag{20}
\]

for any \( X \in \Gamma(TM) \), where \( r \) denotes the scalar curvature of \( M^3 \).

Proof. Let \( \xi \) be an eigenvector field of the Ricci operator corresponding to the eigenvalue \( \alpha \), that is, \( Q\xi = \alpha\xi \). Substituting the above equation into (6) implies that \( \alpha = \text{tr}(l) \).

Then we obtain

\[
Q\xi = \text{tr}(l)\xi. \tag{21}
\]

Differentiating (21) along an arbitrary vector field \( X \) and using (4) gives

\[
(\nabla_XQ)\xi = X(\text{tr}(l))\xi + Q(\phi^2 X + \phi h X) - \text{tr}(l)(\phi^2 X + \phi h X). \tag{22}
\]

Since \( M^3 \) is conformally flat, then substituting \( Z = \xi \) into (12) and using \( g((\nabla_XQ)Y,Z) = g((\nabla_X\phi)Z,Y) \) gives

\[
g((\nabla_XQ)Y - (\nabla_YQ)X,\xi) = \frac{1}{4}\{X(\eta(Y) - Y(\eta(X))\}, \tag{23}
\]

for any \( X, Y \in \Gamma(TM) \). Using the symmetry \( g((\nabla_XQ)Y,Z) = g((\nabla_X\phi)Z,Y) \), then from (23) we have

\[
g((\nabla_XQ)\xi,\xi) - g((\nabla_YQ)\xi,\xi) = \frac{1}{4}\{X(\eta(Y) - Y(\eta(X))\}, \tag{24}
\]

for any \( X, Y \in \Gamma(TM) \). Substituting (22) into (24) gives

\[
X(\text{tr}(l))\eta(Y) - Y(\text{tr}(l))\eta(X) + g(X,\phi h Y) - g(X,\phi h Y) = \frac{1}{4}\{X(\eta(Y) - Y(\eta(X))\}, \tag{25}
\]

for any \( X, Y \in \Gamma(TM) \). Replacing \( \phi X \) and \( \phi Y \) by \( X \) and \( Y \), respectively, in (25) gives

\[
g(\phi Y,\phi h Y) = g(\phi X,\phi h Y), \tag{26}
\]

for any \( X, Y \in \Gamma(TM) \). Thus, it follows from (26) that \( h^lQ = Qh^l \) and hence by substituting \( Y = \xi \) in (25) we obtain

\[
X(\text{tr}(l)) - \frac{1}{4}X(r) = \left[\xi(\text{tr}(l)) - \frac{1}{4}\xi(r)\right]\eta(X), \tag{27}
\]

for any \( X \in \Gamma(TM) \). Applying the exterior derivation \( d \) on both sides of (27) and using the well-known Poincaré lemma \( d^2 = 0 \), and then replacing \( X, Y \), respectively, by \( \phi X, \phi Y \) in the resulting equation, we obtain

\[
\xi(\text{tr}(l)) - \frac{1}{4}\xi(r) = 0. \tag{28}
\]

Thus, substituting (28) into (27) gives (20).

Theorem 3. Let \((M^3,\phi,\xi,\eta,g)\) be a 3-dimensional almost Kenmotsu manifold satisfying \( Q\phi = \phi Q \). Then one of the following cases occurs.

Case 1. \( \xi(\text{tr}(l)) = 0 \) and hence \( M^3 \) is a Kenmotsu manifold.

Case 2. \( \xi(\text{tr}(l)) \neq 0 \) and hence the eigenvalues of \( h \) are locally given by \([0,ce^{-2t},-ce^{-2t}]\) with coordinate \( t \) on \( R \) and \( c \) a nonzero number.

Proof. For an almost Kenmotsu manifold the operator \( l \) never vanishes. In fact, if \( l = 0 \), it follows from (6) that \( \text{tr}(h^2) = -2 \),
there is a contradiction. Now we suppose that $M^3$ is an almost Kenmotsu manifold of dimension 3 with $Q\phi = \phi Q$. Noticing that $\phi \xi = 0$ then we have $\phi Q\xi = 0$. Denoting by $P$ the projective component of $Q\xi$ on contact distribution $\mathcal{D}$, then from (6) we have $Q\xi = \text{tr}(l)\xi + P$. By using the hypothesis $Q\phi = \phi Q$ on the above equation we get $P = 0$, then we have

$$Q\xi = \text{tr}(l)\xi.$$  

(29)

It is well known that the curvature tensor of a 3-dimensional Riemannian manifold $(M^3, g)$ is given by

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X - g(QX,Z)Y + \frac{r}{2} [g(Y,Z)X - g(X,Z)Y],$$

(30)

for any $X,Y,Z \in \Gamma(TM)$, where $r$ denotes the scalar curvature of $M^3$. Noticing (29) and replacing $Y,Z$ by $\xi$ in (30) yields

$$Q \xi = lX - \left( \frac{\text{tr}(l) - r}{2} \right) X - \left( \frac{r}{2} - \text{tr}(l) \right) \eta(X)\xi, \quad \forall X \in \Gamma(TM).$$

(31)

Using $\phi \xi = 0$ and $\phi Q = Q\phi$, it follows from (31) that $\phi lX = \phi Q + (\text{tr}(l) - r/2)\phi X = l\phi X$ for any $X \in \Gamma(TM)$, that is,

$$\phi l = l\phi.$$  

(32)

On the other hand, substituting (32) into (5) gives that

$$l = \phi^2 - h^2.$$  

(33)

From (8) and (33) we obtain that

$$\nabla_\xi h = -2h, \quad \nabla_\xi l = 4h^2.$$  

(34)

On the other hand, since $\phi$ is antisymmetric on $\mathcal{D}$ then from (32) we have $g(\phi X, lX) = 0$ for any $X \in \mathcal{D}$. Thus, noticing that $g(\xi, lX) = 0$, we obtain $lX - (\text{tr}(l)/2)X$ for any $X \in \mathcal{D}$, that is, $lX = -\text{tr}(l)/2\phi X$ for any $X \in \Gamma(TM)$. Substituting the above equation into (31) gives

$$Q \xi = \frac{r - \text{tr}(l)}{2} X + \frac{3 \text{tr}(l) - r}{2} \eta(X)\xi, \quad \forall X \in \Gamma(TM).$$  

(35)

Differentiating (35) along $Y \in \Gamma(TM)$ we obtain

$$\nabla_Y Q \xi = Y\left( \frac{r - \text{tr}(l)}{2} \right) X + \frac{3 \text{tr}(l) - r}{2} \eta(X) \nabla_Y \xi + \frac{1}{2} \left[ Y(3 \text{tr}(l) - r) \eta(X) + (3 \text{tr}(l) - r) g(X,\nabla_Y \xi) \right] \xi,$$

(36)

$$\forall X,Y \in \Gamma(TM).$$

Also, it is well known that

$$\frac{1}{2} \nabla_Y \text{grad}(r) = (\nabla_X Q)X + (\nabla_X \xi) \phi X + (\nabla_X Q)\xi,$$

(37)

for any unit vector filed $X$ in contact distribution $\mathcal{D}$, where $\text{grad}(r)$ denotes the gradient of scalar curvature of $M^3$. Now letting $Y = X$ in (36) be unit vector fields in $\mathcal{D}$ gives

$$\left( \nabla_X Q \right) X = X\left( \frac{r - \text{tr}(l)}{2} \right) X + \frac{3 \text{tr}(l) - r}{2} g(X,X + h\phi X) \xi.$$  

(38)

Similarly, we have

$$\left( \nabla_X Q \right) \phi X = \phi X\left( \frac{r - \text{tr}(l)}{2} \right) \phi X + \frac{3 \text{tr}(l) - r}{2} g(X,X - h\phi X) \xi.$$  

(39)

Substituting (38) and (39) and $(\nabla_X Q)\xi = \xi(\text{tr}(l))\xi$ into (37) implies

$$\frac{1}{2} \text{grad}(r) = X\left( \frac{r - \text{tr}(l)}{2} \right) X + \phi X\left( \frac{r - \text{tr}(l)}{2} \right) \phi X + \{ (3 \text{tr}(l) - r) + \xi(\text{tr}(l)) \} \xi.$$  

(40)

By using (40) and taking an inner product with $X$ and $\phi X$, respectively, we obtain $X(\text{tr}(l)) = 0$ and $\phi X(\text{tr}(l)) = 0$ for any $X \in \mathcal{D}$, that is, $\text{tr}(l)$ is a constant on $\mathcal{D}$.

Case 1. If $\xi(\text{tr}(l)) = 0$, then we find that $\text{tr}(l)$ is a constant on $M$ and hence from (6) we see that $\text{tr}(h^2)$ is also a constant on $M$. Let $X \in \mathcal{D}$ be a unit eigenvector filed of $h$ with eigenvalue $\lambda$, that is, $hX = \lambda X$ (and hence $h\phi X = -\lambda\phi X$), then $\lambda$ is a constant since $\text{tr}(h^2)$ is a constant. Using the first term of (34) gives $g((\nabla_X h)X,X) = g(\xi(\lambda)X,X) = -2g(hX,X) = -2\lambda$; thus, noting that $\lambda$ is a constant then we obtain $\lambda = 0$, that is, $h = 0$ and hence from Lemma 1 we see that $M^3$ is a Kenmotsu manifold.

Case 2. If $\xi(\text{tr}(l)) \neq 0$, it follows from Case 1 that $\xi(\lambda) = -2\lambda \neq 0$, where $\pm \lambda$ denotes the eigenvalues of $h$ on $\mathcal{D}$. Locally, we can write $\xi = \partial/\partial t$ and hence $\lambda = ce^{-2t}$ with coordinate $t$ on $R$ and $c$ a nonzero number following the fact that $\lambda$ is a constant on contact distribution $\mathcal{D}$, which completes the proof.

**Corollary 4.** Let $(M^3, \phi, \xi, g)$ be an almost Kenmotsu manifold of dimension 3. Then the following assertions are equivalent:

(a) $M^3$ is an $\eta$-Einstein Kenmotsu manifold.

(b) $Q\phi = \phi Q$.

(c) $\xi$ belongs to the generalized $k$-nullity distribution.

Moreover, if one of the above conditions holds, then the $\xi$-sectional curvature of $M^3$ is $-1$ and the $\phi$-sectional curvature is $2 + r/2$, where $r$ denotes the scalar curvature of $M^3$.

**Proof.** Suppose that $M^3$ is an $\eta$-Einstein almost Kenmotsu manifold of dimension 3; from (9) it is easy to see (a) $\Rightarrow$ (b).
If \( Q\phi = \phi Q \), then replacing \( Z \) by \( \xi \) in (30) and using (35) we see that \( \xi \) belongs to the generalized \( k \)-nullity distribution, which means that \((b) \Rightarrow (c)\). Now letting \( \xi \) belongs to the generalized \( k \)-nullity distribution, then by a straightforward calculation we know that \( \xi \) is an eigenvector field of Ricci operator. Replacing \( Y = Z \) by \( \xi \) in (30) implies that \((b) \Rightarrow (a)\).

Finally, if one of the above statements holds, then from the above statements we have \( Q\phi = \phi Q \). We choose a unit nonvanishing vector field \( X \) in contact distribution \( \mathcal{D} \). Replacing \( Y = Z = \phi X \) on the resulting equation and taking into account \( Q\phi = \phi Q \) we obtain

\[
R(X,\phi X,\phi X,\phi X) = g(R(X,\phi X)\phi X,\phi X) = 2S(X,X) - \frac{r}{2},
\]

(41)

On the other hand, by the definition of Ricci curvature tensor we have

\[
S(X,X) = R(X,\xi,X,\xi) + R(X,\phi X,X,\phi X) = g(l(X),X) + R(X,\phi X,X,\phi X).
\]

(42)

Using \( IX = (trl)/2 \) for any \( X \in \mathcal{D} \) (see Theorem 3), then it follows from (41) into (42) that \( S(X,X) = 1 + r/2 \). Then from (41) we know that the \( \phi \)-sectional curvature of \( M^3 \) is \( 2 + r/2 \), which completes the proof.

Kenmotsu [5] proved that if a Kenmotsu manifold \( M^{2m+1} \) is a space of constant \( \phi \)-holomorphic sectional curvature \( H \) then \( M^{2m+1} \) is a space of constant curvature and \( H = -1 \). Thus, the following result follows from Corollary 4 and Theorem 3.

**Corollary 5.** Let \((M^3,\phi,\xi,\eta,g)\) be an \( \eta \)-Einstein almost Kenmotsu manifold of dimension 3 with \( \xi(trl)) = 0 \). If the scalar curvature of \( M^3 \) is a constant, then \( M^3 \) is locally isometric to a hyperbolic space \( H^1(-1) \) with constant scalar curvature \(-6\).

### 4. Einstein-Weyl Structures

Recall that a Weyl structure \([4,15]\) on a Riemannian manifold \((M,g)\) of dimension \( \geq 3 \) is defined by the pair \( W^\tau = (g,\omega) \) satisfying

\[
D\omega = \omega \otimes g,
\]

(43)

where \( D \) is a unique torsion-free connection which preserves the conformal class \([g] = \{\lambda g : \lambda \in C^\infty(M)\}\) on \( M \) and \( \omega \) is an 1-form on \( M \). It follows from (43) that

\[
D_XY = \nabla_XY - \frac{1}{2} \omega(Y)X + \frac{1}{2} g(X,Y)E,
\]

(44)

where \( \nabla \) and \( E \) denote the Levi-Civita connection of \( g \) and the dual vector field of \( \omega \) with respect to \( g \), respectively. The Weyl structure is said to be Einstein-Weyl if the symmetrized Ricci tensor associated with the Weyl connection is proportional to the Riemannian metric \( g \), that is,

\[
S^D(X,Y) + S^D(Y,X) = \Lambda g(X,Y), \quad \forall X, Y \in \Gamma(TM),
\]

(45)

where \( S^D \) denotes the Ricci tensor associated with \( D \) and \( \Lambda \) is a smooth function on \( M \). Notice that Narita [15] proved that an \( \eta \)-Einstein almost contact metric manifold satisfying \( \nabla_X\xi = -\phi X \) admits an Einstein-Weyl structure \( W^\tau = (g,\omega) \). However, (4) implies that an almost Kenmotsu manifold never satisfies Narita’s condition even if \( h = 0 \). Since then, we present the following sufficient conditions to characterize Einstein-Weyl structure on an almost Kenmotsu manifold of dimension 3.

**Theorem 6.** Let \((M^3,\phi,\xi,\eta,g)\) be an \( \eta \)-Einstein almost Kenmotsu manifold of dimension 3 with \( \xi(trl)) = 0 \), that is, \( S(X,Y) = \alpha g(X,Y) + \beta \eta(X)\eta(Y) \). If \( \beta \) is a constant \( \leq 1/4 \), then \( M^3 \) admits an Einstein-Weyl structure \( W^\tau = (g,\omega) \).

**Proof.** We define 1-form \( \omega \) by \( \omega = \lambda \eta \), where \( \lambda \) is a \( \omega \)-nonvanishing function on \( M^3 \). Then the dual vector field of \( \omega \) with respect to \( g \) is \( \lambda \xi \). We also define a connection \( D \) on \( M^3 \) by

\[
D_XY = \nabla_XY + \frac{1}{2} \gamma \eta(Y)X + \frac{1}{2} \eta(Y)X + \frac{1}{4} \gamma g(Y,Y)\xi,
\]

(46)

then from (46) it is easy to verify that \( D \) is torsion free and \( DG = \lambda \eta \otimes g \), that is, \((g,\omega)\) is a Weyl structure on \( M^3 \). It follows from [15, 16] that

\[
S^D(X,Y) = S(X,Y) + \frac{1}{2} \nabla_X\omega(Y) + \frac{1}{2} \nabla_Y\omega(X) + \frac{1}{4} \gamma g(Y,Y)\xi,
\]

(47)

for any \( X, Y \in \Gamma(TM^3) \).

By using (4) then a simple computation gives that

\[
(\nabla_X\omega)Y + (\nabla_Y\omega)X = X(\lambda) \eta(Y) + Y(\lambda) \eta(X) + 2f[g(X,Y) - \eta(X)\eta(Y) + g(h\phi X,Y)],
\]

(48)

for any \( X, Y \in \Gamma(TM^3) \) and

\[
\text{div}(\lambda \xi) = g(\nabla_{\xi}(\lambda \xi),\xi) + g(\nabla_{\xi}(\lambda \xi),\phi) + g(\nabla_{\xi}(\lambda \xi),\phi e), \quad \xi(\lambda) + 2\lambda,
\]

(49)
where \( e \in \mathcal{D} \) is a unit vector field on \( M^3 \). Thus, substituting (48) and (49) into (47) yields that

\[
S^D(X, Y) + S^D(Y, X) = 2S(X, Y) + \frac{1}{2}((\nabla_X \omega) Y + (\nabla_Y \omega) X) + \frac{1}{2} \omega(X) \omega(Y) + \left( \frac{\text{div}(\lambda \xi)}{2} - \frac{1}{2} \lambda^2 \right) g(X, Y)
\]

\[= \left( 2\alpha + 3\lambda + \xi(\lambda) - \frac{1}{2} \lambda^2 \right) g(X, Y) + \left( 2\beta - \lambda + \frac{1}{2} \lambda^2 \right) \eta(X) \eta(Y) + \left( 2\lambda \eta(\lambda) \eta(Y) + Y(\lambda) \eta(X) \right) + \lambda g(\eta(X), Y) \quad \forall X, Y \in \Gamma(TM^3).
\]  

(50)

On the other hand, noticing Theorem 3 and Corollary 5, we know that \( h = 0 \) and \( M^3 \) is a Kenmotsu manifold. We set \( \lambda = 1 + \sqrt{1 - 4\beta} \) or \( \lambda = 1 - \sqrt{1 - 4\beta} \), then it follows from (50) that

\[
S^D(X, Y) + S^D(Y, X) = 2(\alpha + \lambda + \beta) g(X, Y), \quad \forall X, Y \in \Gamma(M^3),
\]  

(51)

which completes the proof. \( \square \)

**Remark 7.** From Corollary 5 we know that an \( \eta \)-Einstein almost Kenmotsu manifold of dimension 3 with \( \xi(\text{tr}(l)) = 0 \) is a Kenmotsu manifold and \( h = 0 \); however, this property need not be true in higher dimensions more than 3. Thus, our result cannot be generalized to the case of higher dimensions.

A Weyl structure \( W^\perp = (g, -\omega) \) on a Riemannian manifold \( M \) of dimension \( (2n + 1) \) is defined by \( Dg = -\omega \otimes g \) for a unique torsion-free connection \( D \) and a 1-form \( \omega \). The Weyl structure \( W^\perp = (g, -\omega) \) is said to be an Einstein-Weyl structure if (45) holds for a smooth function \( \lambda \) on \( M \). For an Einstein-Weyl structure \( W^\perp = (g, -\omega) \) it follows from [4] that

\[
D_X Y = \nabla_X Y + \frac{1}{2} \eta(\nabla_X \omega) Y + \frac{1}{2} \eta(Y) X - \frac{1}{2} \eta g(X, Y) \xi,
\]

\[
S^D(X, Y) = S(X, Y) - n (\nabla_X \omega) Y + \frac{1}{2} \left( \frac{\text{div}(\lambda \xi)}{2} - \frac{1}{2} \lambda^2 \right) g(X, Y) + \left( \frac{1}{2} \omega(X) \omega(Y) + \frac{1}{4} \omega(X) \omega(Y) \right),
\]  

(52)

for any \( X, Y \in \Gamma(TM^{2n+1}) \), where \( E \) denotes the dual vector field of \( \omega \) with respect to \( g \). Thus, a straightforward calculation which is similar to the proof of Theorem 6 gives the following.

**Theorem 8.** Let \((M^3, \phi, \xi, \eta, g)\) be an \( \eta \)-Einstein almost Kenmotsu manifold of dimension 3 with \( \xi(\text{tr}(l)) = 0 \), that is, \( S(X, Y) = \alpha g(X, Y) + \beta \eta(X) \eta(Y) \). If \( \beta \) is a constant \( \leq 1/4 \), then \( M^3 \) admits both Einstein-Weyl structures \( W^\perp = (g, \pm \omega) \).

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**References**


