Hindawi Publishing Corporation
Journal of Calculus of Variations
Volume 2013, Article ID 936915, 11 pages
http://dx.doi.org/10.1155/2013/936915

Research Article

Existence and Uniqueness of a Solution in the Space of BV Functions to the Equation of a Vibrating Membrane with a "Viscosity" Term

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Received 30 April 2013; Accepted 11 July 2013

Academic Editor: Adam Kowalewski

A nonlinear equation of motion of vibrating membrane with a "viscosity" term is investigated. Usually, the term $-\Delta u_t$ is added, and it is well known that this equation is well posed in the space of $W^{1,2}$ functions. In this paper, the viscosity term is changed to $-(\operatorname{div}(\nabla u/\sqrt{1+|\nabla u|^2}))_t$, and it is proved that if initial data is slightly smooth (but belonging to $W^{2,2}$ is sufficient), then a weak solution exists uniquely in the space of BV functions.

1. Introduction

Let Ω be a bounded domain in $\mathbf{R}^n$ with the Lipschitz continuous boundary $\partial\Omega$. In [1] and in the author's previous works [2–4], the following:

$$u_{tt} - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \quad x \in \Omega \tag{1}$$

is investigated, which is in these works referred to as the equation of motion of vibrating membrane. Up to now, neither existence nor uniqueness of a solution to (1) is obtained. In [1–3], we only have that *a sequence of approximate solutions to* (1) *converges to a function u in an appropriate function space*, and that *if u satisfies the energy conservation law, it is a weak solution to* (1). In [1], approximate solutions are constructed by the Ritz-Galerkin method and in [2, 3] by Rothe's method. In [2], the boundary condition is not essentially discussed, and the observation is added in [3]. In these works, the limit should satisfy the energy conservation law, and existence theorem of a global weak solution has not been established yet. Instead, in [4], linear approximation for (1) is established. On the other hand, the equation with the strong viscosity term $-\Delta u_t$ is investigated by several authors. For example, in [5], it is investigated in the context of control theory, and it is asserted that if $u(0) \in W_0^{1,2}(\Omega)$ and $u_t(0) \in L^2(\Omega)$, there exists a unique solution $u \in L^2((0,T); W_0^{1,2}(\Omega)) \cap W^{1,2}((0,T); L^2(\Omega))$ for each $T > 0$. Namely, the equation with strong viscosity term is well posed in $W^{1,2}(\Omega)$, and since $W^{1,2}$ is a smaller class than the space of BV functions, this suggests that the influence of the term $-\Delta u_t$ is too strong.

In this paper, replacing the strong viscosity term $-\Delta u_t$ with $-(\operatorname{div}(\nabla u/\sqrt{1+|\nabla u|^2}))_t$, we investigate it in the space of BV functions. Namely, our problem of this paper is as follows:

$$\begin{aligned} u_{tt} - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \\ - \left(\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\right)_t = 0, \quad x \in \Omega, \end{aligned} \tag{2}$$

with initial and boundary conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = v_0(x), \quad x \in \Omega, \tag{3}$$

$$u(t,x) = 0, \quad x \in \partial\Omega. \tag{4}$$

We should note that the term "viscosity" probably means implying regularity. However, in this paper, we only investigate existence and uniqueness of (2)–(4), regularity is not investigated. This is the reason that in the title there is a quotation mark.

A function u is said to be a *function of bounded variation* or a BV *function* in Ω if the distributional derivative Du is an $\mathbf{R}^n$ valued finite Radon measure in Ω. The vector space of all functions of bounded variation in Ω is denoted by $\mathrm{BV}(\Omega)$. It is a Banach space equipped with the norm $\|u\|_{\mathrm{BV}} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$ (see, e.g., [6–8]). We should note that, for $u \in \mathrm{BV}(\Omega)$, the operator $\mathrm{div}(\nabla u/\sqrt{1+|\nabla u|^2})$ is multivalued. It is usually defined by the use of the subdifferential of the area functional. Namely, for a function $u \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$, we regard $-\mathrm{div}(\nabla u/\sqrt{1+|\nabla u|^2})$ as

$$\partial J(u) = \Big\{ f \in \left(\mathrm{BV}(\Omega) \cap L^2(\Omega)\right)'; J(u+\phi) - J(u) \geq (f,\phi) \text{ for each } \phi \in \mathrm{BV}(\Omega) \cap L^2(\Omega) \Big\}, \tag{5}$$

where

$$J(u) = \sqrt{1+|Du|^2}\left(\overline{\Omega}\right). \tag{6}$$

Here, readers should note that $J(u)$ is not $\sqrt{1+|Du|^2}(\Omega)$. We are imposing (4), and in the analysis in the space of BV functions, the most appropriate weak formulation of (4) is to replace $\sqrt{1+|Du|^2}(\Omega)$ with $\sqrt{1+|Du|^2}(\overline{\Omega})$ (cf. [3], see, also [4, Appendix C]).

Now, we present our definition of a weak solution to (2)–(4).

Definition 1. A function u is a *weak solution* to (2)–(4) in $[0,T) \times \Omega$ if u satisfies that

(i) $u \in L^\infty((0,T); \mathrm{BV}(\Omega) \cap L^2(\Omega))$, $u_t \in L^2((0,T) \times \Omega)$,

(ii) $\text{s-lim}_{t \searrow 0} u(t,x) = u_0(x)$ in $L^2(\Omega)$,

(iii) there exist $f_0 \in \partial J(u_0)$ and $f_1 \in L^\infty((0,T); (\mathrm{BV}(\Omega) \cap L^2(\Omega))')$ such that for $\mathscr{L}^1$-a.e. t ($\mathscr{L}^1$ denotes the one-dimensional Lebesgue measure), $f_1 \in \partial J(u)$ and, for any $\phi \in W^{1,\infty}((0,T); \mathrm{BV}(\Omega) \cap L^2(\Omega)) \cap C_0^0([0,T); L^2(\Omega))$,

$$\int_0^T \int_\Omega u_t(t)\phi_t(t)\,dx\,dt - \int_0^T (f_1, \phi(t))\,dt + \int_0^T (f_1, \phi_t(t))\,dt + (f_0, \phi(0)) + \int_\Omega v_0 \phi(0)\,dx = 0. \tag{7}$$

If a function $u \in L^\infty((0,\infty); \mathrm{BV}(\Omega) \cap L^2(\Omega))$ is a weak solution to (2)–(4) in $[0,T) \times \Omega$ for each $T > 0$, then we say that u is a weak solution to (2)–(4) in $[0,\infty) \times \Omega$.

Our main theorem is as follows.

Theorem 2. *Suppose that $u_0 \in \mathrm{BV}(\Omega) \cap \mathrm{L}^2(\Omega)$ and $v_0 \in L^2(\Omega)$. We further suppose that $J(u_0) \cap L^2(\Omega) \neq \emptyset$. Then, there exists a unique weak solution to* (2)–(4) *in* $[0,\infty) \times \Omega$.

Remark 3. If $J(u_0) \cap L^2(\Omega) \neq \emptyset$, then the element is unique. Indeed, for each $\phi \in C_0^\infty(\Omega)$, $J(u_0 + \varepsilon\phi)$ is differentiable at $\varepsilon = 0$ and $(d/d\varepsilon)J(u_0+\varepsilon\phi)|_{\varepsilon=0} = (f_0, \phi)$ for each $f_0 \in J(u_0) \cap L^2(\Omega)$. Since $f_0 \in L^2(\Omega)$ and ϕ is arbitrary, f_0 is uniquely determined.

2. Reduction of the Problem

In order to solve (2)–(4), we give a formal observation. Let us put

$$f = -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right), \qquad g = u_{tt}, \tag{8}$$

then (2) becomes $f_t = -f - g$, which can be regarded as an ordinary differential equation to f. By the variation-of-constants formula, we obtain that $f(t) = f_0 e^{-t} - e^{-t}\int_0^t g(s)e^s ds$. Noting that $g = u_{tt}$, we have

$$f(t) = f_0 e^{-t} - u_t + v_0 e^{-t} + u - u_0 e^{-t} - e^{-t}\int_0^t u(s)e^s ds, \tag{9}$$

where $f_0 = f(0) = -\mathrm{div}(\nabla u_0/\sqrt{1+|\nabla u_0|^2})$. Hence, formally, (2) is reduced to

$$u_t - \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f_0 e^{-t} + v_0 e^{-t} + u - u_0 e^{-t} - e^{-t}\int_0^t u(s)e^s ds. \tag{10}$$

Definition of a weak solution to this equation is as follows.

Definition 4. Let $f_0 \in \partial J(u_0)$, $u_0 \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$, and $v_0 \in L^2(\Omega)$. A function u is a *weak solution* to (10) with (3) and (4) in $[0,T) \times \Omega$ if u satisfies that

(i) $u \in L^\infty((0,\infty); \mathrm{BV}(\Omega) \cap L^2(\Omega))$, $u_t \in L^2((0,T) \times \Omega)$,

(ii) $\text{s-lim}_{t \searrow 0} u(t,x) = u_0(x)$ in $L^2(\Omega)$,

(iii) for any $\phi \in L^2(\Omega) \cap \mathrm{BV}(\Omega)$ and for $\mathscr{L}^1$-a.e. t,

$$J(u+\phi) - J(u) \geq -\int_\Omega u_t \phi(t,x)\,dx + \int_\Omega u\phi(t,x)\,dx + e^{-t}(f_0, \phi(t)) + e^{-t}\int_\Omega \left(v_0(x) - u_0(x) - \int_0^t u(s,x)e^s ds\right)\phi(t,x)\,dx. \tag{11}$$

Similar to the case of (2), we say that u is a weak solution to (10) with (3) and (4) in $[0,\infty)\times\Omega$ if a function $u \in L^\infty((0,\infty);\mathrm{BV}(\Omega)\cap L^2(\Omega))$ is a weak solution to (10) with (3) and (4) in $[0,T)\times\Omega$ for each $T>0$.

The previous observation is just formal. In the following proposition, we show it rigorously.

Proposition 5. *Definitions 1 and 4 are equivalent.*

Proof. It is sufficient to show that, for each $T>0$, a function u is a weak solution to (2)–(4) in $[0,T)\times\Omega$ if and only if it is a weak solution to (10) with (3) and (4) in $[0,T)\times\Omega$.

Suppose that u is a weak solution to (10) with (3) and (4) in $[0,T)\times\Omega$. Conditions (i) and (ii) of Definition 1 are the same as those of Definition 4. Thus, we only have to show (iii) of Definition 1. Let

$$f_1 = -u_t + u + e^{-t} f_0 + e^{-t}\left(v_0 - u_0 - \int_0^t u e^s ds\right). \tag{12}$$

Then, by (iii) of Definition 4, we have that $f_1 \in \partial J(u)$ for $\mathscr{L}^1$-a.e. t. Thus, by a direct calculation, we have that u satisfies (iii) of Definition 1.

Next, we suppose that u is a weak solution to (2)–(4) in $[0,T)\times\Omega$. For each $\phi \in L^\infty((0,T);L^2(\Omega)\cap\mathrm{BV}(\Omega))$ and each $\rho \in C_0^0([0,T))$, we put

$$\psi(t) = e^t \int_t^T e^{-s}\rho(s)\phi(s)\,ds. \tag{13}$$

Then $\psi \in W^{1,\infty}((0,T);\mathrm{BV}(\Omega)\cap L^2(\Omega))\cap C_0^0([0,T);L^2(\Omega))$, and since $\psi_t(t) = \psi(t) - \rho(t)\phi(t)$, we have the following by (iii) of Definition 1:

$$\begin{aligned}
&\int_0^T\int_\Omega u_t(t)\psi(t)\,dx\,dt \\
&\quad - \int_0^T \rho(t)\int_\Omega u_t(t)\phi(t)\,dx\,dt \\
&\quad - \int_0^T \rho(t)\left(f_1,\phi(t)\right)dt + \left(f_0,\psi(0)\right) \\
&\quad + \int_\Omega v_0\psi(0)\,dx = 0.
\end{aligned} \tag{14}$$

By integration by parts, we have

$$\begin{aligned}
&\int_0^T\int_\Omega u_t(t)\psi(t)\,dx\,dt \\
&\quad = \left[\int_\Omega u(t)\psi(t)\,dx\right]_0^T - \int_0^T\int_\Omega u(t)\psi_t(t)\,dx\,dt \\
&\quad = -\int_\Omega u_0\psi(0)\,dx - \int_0^T\int_\Omega u(t)\psi(t)\,dx\,dt \\
&\qquad + \int_0^T \rho(t)\int_\Omega u(t)\phi(t)\,dx\,dt.
\end{aligned} \tag{15}$$

Furthermore, we have the following by Fubini's theorem:

$$\begin{aligned}
&\int_0^T\int_\Omega u(t)\psi(t)\,dx\,dt \\
&\quad = \int_0^T \rho(s)\int_\Omega\int_0^s u(t)e^t dt e^{-s}\phi(s)\,dx\,ds.
\end{aligned} \tag{16}$$

Finally, noting that $\psi(0) = \int_0^T e^{-s}\phi(s)ds$, we have the following by (14), (15), and (16):

$$\begin{aligned}
&\int_0^T \rho(t)\left(f_1,\phi(t)\right)dt \\
&\quad = -\int_0^T \rho(t)\int_\Omega u_t(t)\phi(t)\,dx\,dt \\
&\qquad + \int_0^T \rho(t)\int_\Omega u(t)\phi(t)\,dx\,dt \\
&\qquad + \int_0^T e^{-t}\rho(t)\left(f_0,\phi(t)\right)dt \\
&\qquad + \int_0^T e^{-t}\rho(t)\int_\Omega v_0\phi(t)\,dx\,dt \\
&\qquad - \int_0^T e^{-t}\rho(t)\int_\Omega u_0\phi(t)\,dx\,dt \\
&\qquad + \int_0^T \rho(t)\int_\Omega\int_0^t u(s)e^s ds e^{-t}\phi(t)\,dx\,dt.
\end{aligned} \tag{17}$$

Since ϕ and ρ are arbitrary, we have, for $\mathscr{L}^1$-a.e. t,

$$f_1 = -u_t + u + e^{-t} f_0 + e^{-t}\left(v_0 - u_0 - \int_0^t u e^s ds\right), \tag{18}$$

which means that u satisfies Definition 4, (iii). □

Now, Theorem 2 is reduced to the following.

Theorem 6. *Suppose that $u_0 \in \mathrm{BV}(\Omega)\cap \mathrm{L}^2(\Omega)$ and $v_0 \in L^2(\Omega)$. We further suppose that $J(u_0)\cap L^2(\Omega)\neq\emptyset$ and let $f_0 \in J(u_0)\cap L^2(\Omega)$. Then, there exists a unique weak solution to* (10), (3), *and* (4) *in* $[0,\infty)\times\Omega$.

Our strategy of proving Theorem 6 is the contracting mapping theorem. For this purpose, given that $\tilde{u} \in L^2((0,T)\times\Omega)$, we solve

$$\begin{aligned}
u_t - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) &= f_0 e^{-t} + v_0 e^{-t} \\
&\quad + \tilde{u} - u_0 e^{-t} - e^{-t}\int_0^t \tilde{u}(s)e^s ds
\end{aligned} \tag{19}$$

and show that the map $\tilde{u} \mapsto u$ is a contraction. A weak solution to (19) with (3) and (4) is defined as follows.

Definition 7. Let $f_0 \in \partial J(u_0)$. A function u is a *weak solution* to (19) with (3) and (4) in $[0,T)\times\Omega$ if u satisfies that

(i) $u \in L^\infty((0,T);\mathrm{BV}(\Omega)\cap L^2(\Omega))$, $u_t \in L^2((0,T)\times\Omega)$,

(ii) $\text{s-lim}_{t\searrow 0} u(t,x) = u_0(x)$ in $L^2(\Omega)$,

(iii) for any $\phi \in L^2(\Omega)\cap \mathrm{BV}(\Omega)$ and for $\mathscr{L}^1$-a.e. t,

$$\begin{aligned} J(u+\phi) - J(u) &\geq -\int_\Omega u_t \phi(t,x)\,dx \\ &\quad + \int_\Omega \tilde{u}\phi(t,x)\,dx - e^{-t}(f_0,\phi(t)) \\ &\quad + e^{-t}\int_\Omega \Big(v_0(x) - u_0(x) \\ &\qquad - \int_0^t \tilde{u}(s,x)e^s ds\Big)\phi(t,x)\,dx. \end{aligned} \tag{20}$$

The proof of Theorem 6 consists of two parts. The first part is solving (19), and the second part is to show that the map $\tilde{u} \mapsto u$ is a contraction.

3. Existence and Uniqueness of a Solution to (19)

Let u_0 and v_0 be as in Theorem 6. In this section, we show that there exists a unique solution to (19) with (3) and (4) in $[0,T)\times\Omega$ for each $T>0$.

Uniqueness is easy. Suppose that u and v are solutions to (19) with (3) and (4) in $[0,T)\times\Omega$, and inserting $\phi = v-u$ to (iii) of Definition 7, integrating it from 0 to T, obtaining another inequality by replacing u and v, and adding these two inequalities, we have

$$\begin{aligned} 0 &\geq \int_0^T\int_\Omega (v_t - u_t)(v-u)\,dx\,dt \\ &= \frac{1}{2}\Big(\int_\Omega |v(T)-u(T)|^2 dx \\ &\qquad - \int_\Omega |v(0)-u(0)|^2 dx\Big). \end{aligned} \tag{21}$$

Since $u(0) = v(0) = u_0$, we have the uniqueness of a solution to (19).

It is sufficient to show the existence in $[0,T)\times\Omega$ for $\mathscr{L}^1$-a.e. T. Approximate solutions are constructed by Rothe's time semidiscretization method. In Rothe's method, we should solve elliptic equations with respect to space variables. Here, we solve them by a direct variational method (namely, this is the method of discrete Morse semiflow, cf. [9] and references cited therein).

Suppose that $u_0 \in \mathrm{BV}(\Omega)\cap L^2(\Omega)$ with $J(u_0)\cap L^2(\Omega)\neq\emptyset$ and $v_0 \in L^2(\Omega)$, and let $f_0 \in J(u_0)\cap L^2(\Omega)$. For a positive number h, we construct a sequence $\{u_l\}_{l=0}^\infty$ in the following way. For $l=0$, we let u_0 be as in (3), and for $l\geq 1$, it is defined as a minimizer of the following functional:

$$\mathscr{F}_l(v) = \frac{1}{2}\int_\Omega \frac{|v-u_{l-1}|^2}{h}dx + J(v) + \frac{1}{h}\int_{(l-1)h}^{lh}(G,v)\,dt, \tag{22}$$

in the class $L^2(\Omega)\cap\mathrm{BV}(\Omega)$, where

$$G(t,x) = -e^{-t}f_0 - v_0e^{-t} - \tilde{u} + u_0e^{-t} + e^{-t}\int_0^t \tilde{u}(s,x)e^s ds. \tag{23}$$

Since

$$\begin{aligned} &\frac{1}{h}\int_{(l-1)h}^{lh}(G,v)\,dt \\ &\quad \geq -\Big(\frac{1}{2h}\int_{(l-1)h}^{lh}\int_\Omega |G(t,x)|^2 dx\,dt \\ &\qquad + \frac{1}{2}\int_\Omega |v|^2 dx\Big), \end{aligned} \tag{24}$$

F_l is bounded from below, and hence, the existence of a minimizer of $\mathscr{F}_l$ follows.

Lemma 8 (energy inequality). *consider the following:*

$$\begin{aligned} &\sum_{j=1}^{l}\Big[\frac{1}{2}\int_\Omega \frac{|u_j-u_{j-1}|^2}{h}dx \\ &\quad + \int_{(j-1)h}^{jh}\Big(G, \frac{u_j-u_{j-1}}{h}\Big)dt\Big] + J(u_l) \\ &\quad \leq J(u_0). \end{aligned} \tag{25}$$

Proof. Since u_l is a minimizer of $\mathscr{F}_l$, we have

$$\mathscr{F}_l(u_l) \leq \mathscr{F}_l(u_{l-1}) = J(u_{l-1}) + \frac{1}{h}\int_{(l-1)h}^{lh}(G,u_{l-1})\,dt. \tag{26}$$

Hence, for each l,

$$\begin{aligned} &\frac{1}{2}\int_\Omega \frac{|u_l-u_{l-1}|^2}{h}dx + J(u_l) \\ &\quad + \int_{(l-1)h}^{lh}\Big(G, \frac{u_l-u_{l-1}}{h}\Big)dt \\ &\quad \leq J(u_{l-1}). \end{aligned} \tag{27}$$

Thus, by induction on l, we have the conclusion. □

Next, we define approximate solutions $u^h(t,x)$ and $\overline{u}^h(t,x)$ for $(t,x)\in(-h,\infty)\times\Omega$ as follows: for $(l-1)h < t \leq lh$,

$$\begin{aligned} u^h(t,x) &= \frac{t-(l-1)h}{h}u_l(x) + \frac{lh-t}{h}u_{l-1}(x), \\ \overline{u}^h(t,x) &= u_l(x). \end{aligned} \tag{28}$$

Then Lemma 8 shows for each $T > 0$

$$\begin{aligned}\frac{1}{2}\int_0^T\int_\Omega \left|u_t^h(t,x)\right|^2 dx\,dt &+ \int_0^T \left(G, u_t^h\right) dt + J\left(\overline{u}^h(T,\cdot)\right)\\ &\le J\left(u_0\right).\end{aligned} \tag{29}$$

Now, we estimate the second term of the left hand side of (29). Then,

$$\begin{aligned}&\int_0^T\int_\Omega \left|e^{-t}\int_0^t \widetilde{u}(s,x)\,e^s ds\right|^2 dx\,dt\\ &\le \int_0^T\int_\Omega e^{-2t}\int_0^t e^{2s}ds\int_0^t |\widetilde{u}(s,x)|^2 ds\,dx\,dt\\ &\le \int_0^T\int_\Omega e^{-2t}2^{-1}\left(e^{2t}-1\right)\int_0^t |\widetilde{u}(s,x)|^2 ds\,dx\,dt\\ &= \int_0^T\int_s^T 2^{-1}\left(1-e^{-2t}\right)dt\int_\Omega |\widetilde{u}(s,x)|^2 ds\,dx\\ &= \int_0^T 2^{-1}\left[t+2^{-1}e^{-2t}\right]_s^T\int_\Omega |\widetilde{u}(s,x)|^2 ds\,dx\\ &= 2^{-1}\int_0^T\left(T+2^{-1}e^{-2T}-s-2^{-1}e^{-2s}\right)\int_\Omega |\widetilde{u}(s,x)|^2 ds\,dx\\ &\le C(T)\int_0^T\int_\Omega |\widetilde{u}(s,x)|^2 ds\,dx,\end{aligned} \tag{30}$$

where $C(T) = 2^{-1}(T+2^{-1}e^{-2T})$, and thus, it is easy to see that

$$\begin{aligned}\left|\int_0^T\left(G,u_t^h\right)dt\right| \le \Big[&\|f_0\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Omega)} + \|u_0\|_{L^2(\Omega)}\\ &+ (1+C(T))\,\|\widetilde{u}\|_{L^2((0,T)\times\Omega)}\Big]\left\|u_t^h\right\|_{L^2((0,T)\times\Omega)}.\end{aligned} \tag{31}$$

By (29), we have, for each $\varepsilon > 0$ and for each $T > 0$,

$$\begin{aligned}\frac{1-\varepsilon}{2}\int_0^T\int_\Omega \left|u_t^h(t,x)\right|^2 dx\,dt &+ J\left(\overline{u}^h(T,\cdot)\right)\\ &\le C_0\left(u_0,v_0,f_0,\widetilde{u},\varepsilon,T\right),\end{aligned} \tag{32}$$

where

$$\begin{aligned}&C_0\left(u_0,v_0,f_0,\widetilde{u},\varepsilon,T\right)\\ &= \frac{1}{2\varepsilon}\Big[\|f_0\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Omega)} + \|u_0\|_{L^2(\Omega)}\\ &\quad + (1+C(T))\,\|\widetilde{u}\|_{L^2((0,T)\times\Omega)}\Big]^2 + J\left(u_0\right).\end{aligned} \tag{33}$$

Proposition 9. *It holds that*

(1) $\{\|u_t^h\|_{L^2((0,\infty)\times\Omega)}\}$ *is uniformly bounded with respect to* h;

(2) *for any* $T > 0$, $\{\|\overline{u}^h\|_{L^\infty((0,T);L^2(\Omega)\cap \mathrm{BV}(\Omega))}\}$ *is uniformly bounded with respect to* h;

(3) *for any* $T > 0$, $\{\|u^h\|_{L^\infty((0,T);L^2(\Omega)\cap \mathrm{BV}(\Omega))}\}$ *is uniformly bounded with respect to* h;

Then there exist a sequence $\{h_j\}$ *with* $h_j \to 0$ *as* $j \to \infty$ *and a function* u *such that*

(4) $u_t^{h_j}$ *converges to* u_t *as* $j \to \infty$ *weakly in* $L^2((0,\infty)\times\Omega)$;

(5) *for any* $T > 0$, u^{h_j} *converges to* u *as* $j \to \infty$ *weakly star in* $L^\infty((0,T);L^2(\Omega))$;

(6) *for any* $T > 0$, u^{h_j} *converges to* u *as* $j \to \infty$ *strongly in* $L^\infty((0,T);L^p(\Omega))$ *for each* $1 \le p < 1^* = n/(n-1)$;

(7) *for any* $T > 0$, $\overline{u}^{h_j}$ *converges to* u *as* $j \to \infty$ *strongly in* $L^\infty((0,T);L^p(\Omega))$ *for each* $1 \le p < 1^*$;

(8) $u \in L^\infty((0,\infty);\mathrm{BV}(\Omega))$;

(9) *for* $\mathscr{L}^1$*-a.e.* $t \in (0,\infty)$, $D\overline{u}^{h_j}(t,\cdot)$ *converges to* $Du(t,\cdot)$ *as* $j \to \infty$ *in the sense of distributions;*

(10) $s\text{-}\lim_{t\searrow 0} u(t) = u_0$ *in* $L^2(\Omega)$.

Proof. Assertion (1) immediately follows from (32). Since we have

$$u^h(t,x) - u^h\left(t',x\right) = \int_{t'}^t u_t^h(s,x)\,ds, \tag{34}$$

for each $t, t' \ge 0$, Assertion (1) implies that, for each $T > 0$, $\{\|u^h\|_{L^\infty((0,T);L^2(\Omega))}\}$ is uniformly bounded with respect to h. Given that $t > 0$, we let l be an integer such that $(l-1)h < t \le lh$. Then,

$$\begin{aligned}&\left\|\overline{u}^h(t) - u^h(t)\right\|_{L^2(\Omega)}\\ &= \left\|u_l - \frac{t-(l-1)h}{h}u_l - \frac{lh-t}{h}u_{l-1}\right\|_{L^2(\Omega)}\\ &= \left\|\frac{lh-t}{h}\left(u_l-u_{l-1}\right)\right\|_{L^2(\Omega)}\\ &\le \|u_l-u_{l-1}\|_{L^2(\Omega)}.\end{aligned} \tag{35}$$

By (32),

$$\begin{aligned}&\int_0^{Lh}\int_\Omega \left|u_t^h(t,x)\right|^2 dx\,dt\\ &= \sum_{l=1}^L\int_\Omega \frac{\left|u_l-u_{l-1}\right|^2}{h}dx\\ &\le \frac{2}{1-\varepsilon}C_0\left(u_0,v_0,f_0,\widetilde{u},\varepsilon,T\right).\end{aligned} \tag{36}$$

Thus, we have

$$\|u_l-u_{l-1}\|_{L^2(\Omega)}^2 \le C\left(u_0,v_0,f_0,\widetilde{u},\varepsilon,T\right)h, \tag{37}$$

for each l, where $C(u_0, v_0, \tilde{u}, \varepsilon, T) = (2/(1-\varepsilon))C_0(u_0, v_0, f_0, \tilde{u}, \varepsilon, T)$. Hence,

$$\sup_{t>0}\left\|\overline{u}^h(t) - u^h(t)\right\|_{L^2(\Omega)} \le \sqrt{C(u_0, v_0, f_0, \tilde{u}, \varepsilon, T)\, h}. \quad (38)$$

Now, we have that $\{\|\overline{u}^h\|_{L^\infty((0,T);L^2(\Omega))}\}$ is uniformly bounded with respect to h since

$$\begin{aligned}\left\|\overline{u}^h\right\|_{L^\infty((0,T);L^2(\Omega))} &\le \left\|\overline{u}^h - u^h\right\|_{L^\infty((0,T);L^2(\Omega))} \\ &\quad + \left\|u^h\right\|_{L^\infty((0,T);L^2(\Omega))}.\end{aligned} \quad (39)$$

Since C_0 is increasing with respect to T, Assertion (2) follows from (32). Since J is convex, we have

$$\begin{aligned}J\left(u^h(t,\cdot)\right) &\le \frac{t-(\ell-1)h}{h} J\left(\overline{u}^h(t,\cdot)\right) \\ &\quad + \frac{\ell h - t}{h} J\left(\overline{u}^h(t-h,\cdot)\right),\end{aligned} \quad (40)$$

and Assertion (3) also holds.

Assertion (4) is a direct consequence of Assertion (1). Assertion (5) follows from Assertion (3). Furthermore, (34) and Assertion (1) imply that the function $t \mapsto u^h(t,\cdot) \in L^2(\Omega)$ is equicontinuous with respect to h. By Sobolev's theorem $\mathrm{BV}(\Omega) \subset L^p(\Omega)$ compactly for each $1 \le p < 1^*$. This means that, for any $T > 0$, $\{u^h(t,\cdot)\}$ is contained in a sequentially compact subset of $L^p(\Omega)$ which is independent of h and $t \in [0,T]$. Thus, by the Ascoli-Arzela theorem, we obtain Assertion (6).

Now, we have, for $1 \le p < 1^*$,

$$\begin{aligned}\left\|\overline{u}^h - u\right\|_{L^\infty((0,T);L^p(\Omega))} &\le \left\|\overline{u}^h - u^h\right\|_{L^\infty((0,T);L^p(\Omega))} \\ &\quad + \left\|u^h - u\right\|_{L^\infty((0,T);L^p(\Omega))},\end{aligned} \quad (41)$$

the right hand side of which converges to 0 as $h \to 0$ by (38) and Assertion (6). Now, we have Assertion (7). Assertions (2) and (7) imply Assertions (8) and (9).

Letting $t' = 0$ in (34), we have

$$\left\|u^h(t) - u_0\right\|_{L^2(\Omega)} \le \left\|u_t^h\right\|_{L^2((0,\infty)\times\Omega)} \sqrt{t}. \quad (42)$$

Thus, by Assertion (1) the left hand side is uniformly bounded with respect to h and, hence, passing to a subsequence if necessary, $\{u^h(t) - u_0\}_{h>0}$ converges weakly in $L^2(\Omega)$, and by Assertion (6), the weak limit is $u(t) - u_0$. Then, by the lower semicontinuity of L^2 norm, we have

$$\begin{aligned}\left\|u(t) - u_0\right\|_{L^2(\Omega)} &\le \liminf_{h\to 0}\left\|u^h(t) - u_0\right\|_{L^2(\Omega)} \\ &\le \text{Constant}\sqrt{t},\end{aligned} \quad (43)$$

which implies Assertion (10). □

Now, our purpose is to show that u is a weak solution to (19). Proposition 9 implies that u satisfies (i) and (ii) of Definition 7.

Since u_l is a minimizer of $\mathscr{F}_l(v)$, we have

$$\partial\mathscr{F}_l(u_l) = \frac{u_l - u_{l-1}}{h} + \partial J(u_l) + \frac{1}{h}\int_{(l-1)h}^{lh} G\,dt \ni 0. \quad (44)$$

Let us write, for $(l-1)h \le t < lh$, $G^h(t,x) = (1/h)\int_{(l-1)h}^{lh} G(s,x)\,ds$. Then, for each j and for $\mathscr{L}^1$-a.e. $t \in (0,\infty)$,

$$u_t^{h_j}(t) + \partial J\left(\overline{u}^{h_j}(t)\right) + G^{h_j}(t) \ni 0, \quad (45)$$

namely, for each $v \in L^2(\Omega)$,

$$\begin{aligned}&J(v) - J\left(\overline{u}^{h_j}(t)\right) \\ &\quad \ge -\int_\Omega u_t^{h_j}(t,x)\left(v(x) - \overline{u}^{h_j}(t,x)\right)dx \\ &\qquad - \int_\Omega G^{h_j}(s,x)\left(v(x) - \overline{u}^{h_j}(t,x)\right)dx.\end{aligned} \quad (46)$$

By Proposition 9 (7) we have that, for $\mathscr{L}^1$-a.e. $t \in (0,\infty)$,

$$\overline{u}^{h_j}(t) \longrightarrow u(t) \quad (47)$$

strongly in $L^1(\Omega)$ as $j \to \infty$. Let T be a number t such that (46) and (47) hold. We insert an arbitrary function $v \in L^2((0,T)\times\Omega)$ in (46). Integrating it from 0 to T, we have the following by Proposition 9 (4), (7), Fatou's lemma, and the lower semicontinuity:

$$\begin{aligned}&\int_0^T (J(v(t)) - J(u(t)))\,dt \\ &\quad \ge -\int_0^T\int_\Omega u_t(t,x)\,v(t,x)\,dt \\ &\qquad + \liminf_{j\to\infty}\int_0^T\int_\Omega u_t^h(t,x)\,\overline{u}^{h_j}(t,x)\,dx\,dt \\ &\qquad - \limsup_{j\to\infty}\int_0^T\int_\Omega G^{h_j}(s,x)\left(v(t,x)\right. \\ &\qquad \left. - \overline{u}^{h_j}(t,x)\right)dx\,dt.\end{aligned} \quad (48)$$

For a while, we write $h_j = h$ for simplicity. First, we note the following identity:

$$\begin{aligned}&\int_\Omega \frac{u_l(x) - u_{l-1}(x)}{h} u_l(x)\,dx \\ &\quad = \frac{1}{2h}\int_\Omega |u_l(x)|^2 dx + \int_\Omega \frac{|u_l(x) - u_{l-1}(x)|^2}{2h} dx \\ &\qquad - \frac{1}{2h}\int_\Omega |u_{l-1}(x)|^2 dx.\end{aligned} \quad (49)$$

Let L be the integer such that $(L-1)h < T \le Lh$. By (49), we have

$$\begin{aligned}
&\int_0^T \int_\Omega u_t^h(t,x)\,\overline{u}^h(t,x)\,dx\,dt \\
&\quad = -\int_T^{Lh} \int_\Omega u_t^h(t,x)\,\overline{u}^h(t,x)\,dx\,dt \\
&\qquad + \int_0^{Lh} \int_\Omega u_t^h(t,x)\,\overline{u}^h(t,x)\,dx\,dt \\
&\quad = -(Lh-T)\int_\Omega \frac{u_L(x)-u_{L-1}(x)}{h} u_L(x)\,dx \\
&\qquad + h\sum_{l=1}^{L}\int_\Omega \frac{u_l(x)-u_{l-1}(x)}{h} u_l(x)\,dx \\
&\quad = -\frac{Lh-T}{h}\int_\Omega \left(u_L(x)-u_{L-1}(x)\right) u_L(x)\,dx \\
&\qquad + \frac{1}{2}\int_\Omega \left|u_L(x)\right|^2 dx \\
&\qquad + h\sum_{l=1}^{L}\int_\Omega \frac{\left|u_l(x)-u_{l-1}(x)\right|^2}{2h} dx - \frac{1}{2}\int_\Omega \left|u_0(x)\right|^2 dx \\
&\quad =: -I + II + III - IV.
\end{aligned} \tag{50}$$

By (42) and Proposition 9 (1), $\|u^h(t)\|_{L^2(\Omega)}$ is uniformly bounded with respect to $t \in [0,T]$ and h. On the other hand, since $(L-1)h < T \le Lh$,

$$\begin{aligned}
u^h(T) &= \frac{T-(L-1)h}{h} u_L + \frac{Lh-T}{h} u_{L-1} \\
&= u_L - \frac{Lh-T}{h}\left(u_L - u_{L-1}\right).
\end{aligned} \tag{51}$$

Hence, we have by (37)

$$\begin{aligned}
\|u_L\|_{L^2(\Omega)} &\le \|u^h(T)\|_{L^2(\Omega)} \\
&\quad + \frac{Lh-T}{h}\|u_L - u_{L-1}\|_{L^2(\Omega)} \\
&\le C''\left(u_0, v_0, \tilde{u}, \varepsilon\right),
\end{aligned} \tag{52}$$

and thus by (37) again

$$\begin{aligned}
|I| &\le \|u_L - u_{L-1}\|_{L^2(\Omega)} \|u_L\|_{L^2(\Omega)} \\
&\le \sqrt{C'\left(u_0, v_0, \tilde{u}, \varepsilon\right)} \\
&\quad \times \sqrt{C''\left(u_0, v_0, \tilde{u}, \varepsilon\right)}\sqrt{h} \longrightarrow 0,
\end{aligned} \tag{53}$$

as $h \to 0$. By Proposition 9 (1),

$$\begin{aligned}
|III| &= h \cdot h \sum_{l=1}^{L} \int_\Omega \frac{\left|u_l(x)-u_{l-1}(x)\right|^2}{2h^2} dx \\
&= \frac{h}{2}\int_0^{Lh}\int_\Omega \left|u_t^h\right|^2 dx\,dt \\
&\le \frac{h}{2}\int_0^{\infty}\int_\Omega \left|u_t^h\right|^2 dx \longrightarrow 0,
\end{aligned} \tag{54}$$

as $h \to 0$. Since $\overline{u}^h(T)$, $u(T) \in L^2(\Omega)$, (47) implies that $\overline{u}^{h_j}(T) \rightharpoonup u(T)$ in $L^2(\Omega)$. In particular, we have $\liminf_{j\to\infty}\|\overline{u}^{h_j}(T)\|^2_{L^2(\Omega)} \ge \|u(T)\|^2_{L^2(\Omega)}$. Then, since $u_L(x) = \overline{u}^h(T,x)$, we have

$$\begin{aligned}
&\liminf_{j\to\infty}(II - IV) \\
&\quad = \liminf_{j\to\infty}\left(\left\|\overline{u}^{h_j}(T)\right\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)}\right) \\
&\quad \ge \|u(T)\|^2_{L^2(\Omega)} - \|u_0\|^2_{L^2(\Omega)} \\
&\quad = \int_0^T \frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} dt \\
&\quad = \int_0^T \int_\Omega u_t(t,x)\,u(t,x)\,dx\,dt.
\end{aligned} \tag{55}$$

Summing up, we have

$$\begin{aligned}
&\liminf_{j\to\infty}\int_0^T\int_\Omega u_t^{h_j}(t,x)\,\overline{u}^{h_j}(t,x)\,dx\,dt \\
&\quad \ge \int_0^T\int_\Omega u_t(t,x)\,u(t,x)\,dx\,dt.
\end{aligned} \tag{56}$$

It is not difficult to show that $G^h \to G$ strongly in $L^2((0,T)\times\Omega)$. Hence, by (48), we finally have

$$\begin{aligned}
&\int_0^T \left(J(v) - J(u)\right) dt \\
&\quad \ge -\int_0^T\int_\Omega u_t(v-u)\,dx\,dt - \int_0^T\int_\Omega G(v-u)\,dx\,dt,
\end{aligned} \tag{57}$$

for each $v \in L^2((0,T)\times\Omega)$. By the convexity of J, for each function $\psi \in L^2(0,T)$ with $0 \le \psi \le 1$, we have $\psi J(v) - \psi J(u) \ge J(u + \psi(v-u)) - J(u)$. Thus, (57) implies that

$$\begin{aligned}
&\int_0^T \psi(t)\left[J(v) - J(u)\right] dt \\
&\quad \ge \int_0^T \left[J\left(u + \psi(v-u)\right) - J(u)\right] dt \\
&\quad \ge -\int_0^T \psi(t)\int_\Omega u_t(v-u)\,dx\,dt \\
&\qquad - \int_0^T \psi(t)\int_\Omega G(v-u)\,dx\,dt.
\end{aligned} \tag{58}$$

It is easy to extend this inequality to all nonnegative functions $\psi \in L^2(0,T)$. Hence, (iii) of Definition 7 holds for $\mathscr{L}^1$-a.e. $t \in (0,\infty)$.

4. Proof That $\tilde{u} \mapsto u$ Is a Contraction

Let $\tilde{u}, \tilde{v}$ be functions in $L^2((0,T)\times\Omega)$. In this section, we write

$$\begin{aligned} G(\tilde{u}) &= e^{-t} f_0 - v_0 e^{-t} - \tilde{u} \\ &\quad + u_0 e^{-t} + e^{-t} \int_0^t \tilde{u}(s,x) e^s ds, \\ G(\tilde{v}) &= e^{-t} f_0 - v_0 e^{-t} - \tilde{v} \\ &\quad + u_0 e^{-t} + e^{-t} \int_0^t \tilde{v}(s,x) e^s ds. \end{aligned} \tag{59}$$

Then,

$$G(\tilde{u}) - G(\tilde{v}) = -(\tilde{u} - \tilde{v}) + e^{-t} \int_0^t (\tilde{u}(s,x) - \tilde{v}(s,x)) e^s ds. \tag{60}$$

Let u, v be a solution to (19) with (3) and (4) for $\tilde{u}$, $\tilde{v}$, respectively. By (iii) of Definition 7,

$$\begin{aligned} J(v) - J(u) &\geq -\int_\Omega u_t (v-u)\, dx + \int_\Omega G(\tilde{u})(v-u)\, dx, \\ J(u) - J(v) &\geq -\int_\Omega v_t (u-v)\, dx + \int_\Omega G(\tilde{v})(u-v)\, dx. \end{aligned} \tag{61}$$

Summing these, we have

$$\begin{aligned} 0 &\geq \int_\Omega (u_t - v_t)(u-v)\, dx \\ &\quad - \int_\Omega (G(\tilde{u}) - G(\tilde{v}))(u-v)\, dx. \end{aligned} \tag{62}$$

Integrating from 0 to t, we have the following by (60) and by the fact that $u(0) = v(0) = u_0$:

$$\begin{aligned} 0 &\geq \frac{1}{2} \int_\Omega |u(t) - v(t)|^2 dx \\ &\quad + \int_0^t \int_\Omega (\tilde{u} - \tilde{v})(u-v)\, dx\, d\tau \\ &\quad - \int_0^t e^{-\tau} \int_\Omega \int_0^\tau (\tilde{u}(s,x) - \tilde{v}(s,x)) \\ &\qquad \times (u(\tau,x) - v(\tau,x)) e^s ds\, dx\, d\tau. \end{aligned} \tag{63}$$

We further integrate this from 0 to T and write it $I + II + III$. Then,

$$\begin{aligned} II &= \int_0^T \int_0^t \int_\Omega (\tilde{u}(\tau,x) - \tilde{v}(\tau,x)) \\ &\qquad \times (u(\tau,x) - v(\tau,x))\, dx\, d\tau\, dt \\ &= \int_\Omega dx \int_0^T d\tau \int_\tau^T (\tilde{u}(\tau,x) - \tilde{v}(\tau,x)) \\ &\qquad \times (u(\tau,x) - v(\tau,x))\, dt \\ &= \int_\Omega dx \int_0^T (T-\tau)(\tilde{u}(\tau,x) - \tilde{v}(\tau,x)) \\ &\qquad \times (u(\tau,x) - v(\tau,x))\, d\tau \\ &\leq \frac{T}{2} \int_\Omega \int_0^T |\tilde{u}(\tau,x) - \tilde{v}(\tau,x)|^2 d\tau\, dx \\ &\quad + \frac{T}{2} \int_\Omega \int_0^T |u(\tau,x) - v(\tau,x)|^2 d\tau\, dx, \end{aligned} \tag{64}$$

$$\begin{aligned} III &= \int_0^T \int_0^t e^{-\tau} \int_\Omega \int_0^\tau (\tilde{u}(s,x) - \tilde{v}(s,x)) \\ &\qquad \times (u(\tau,x) - v(\tau,x)) e^s ds\, dx\, d\tau\, dt \\ &= \int_\Omega dx \int_0^T dt \int_0^t e^{-\tau} \left[\int_0^\tau (\tilde{u}(s,x) - \tilde{v}(s,x)) e^s ds \right] \\ &\qquad \times (u(\tau,x) - v(\tau,x))\, d\tau. \end{aligned} \tag{65}$$

Here,

$$\begin{aligned} &\int_0^\tau (\tilde{u}(s,x) - \tilde{v}(s,x)) e^s ds\, (u(\tau,x) - v(\tau,x)) \\ &\quad \leq \frac{1}{2} \left(\int_0^\tau (\tilde{u}(s,x) - \tilde{v}(s,x)) e^s ds \right)^2 \\ &\qquad + \frac{1}{2} |u(\tau,x) - v(\tau,x)|^2 \\ &\quad \leq \frac{1}{2} \int_0^\tau |\tilde{u}(s,x) - \tilde{v}(s,x)|^2 ds \int_0^\tau e^{2s} ds \\ &\qquad + \frac{1}{2} |u(\tau,x) - v(\tau,x)|^2 \\ &\quad \leq \frac{1}{4} \left(e^{2\tau} - 1 \right) \int_0^\tau |\tilde{u}(s,x) - \tilde{v}(s,x)|^2 ds \\ &\qquad + \frac{1}{2} |u(\tau,x) - v(\tau,x)|^2. \end{aligned} \tag{66}$$

Thus,

$$\begin{aligned} III &\le \int_\Omega dx \int_0^T dt \int_0^t e^{-\tau} \frac{1}{4}\left(e^{2\tau}-1\right) \\ &\qquad \times \int_0^\tau |\widetilde{u}(s,x)-\widetilde{v}(s,x)|^2 ds\, d\tau \\ &\quad + \int_\Omega dx \int_0^T dt \int_0^t e^{-\tau} \frac{1}{2} |u(\tau,x)-v(\tau,x)|^2 d\tau \\ &=: III_1 + III_2. \end{aligned} \tag{67}$$

These two terms are estimated as follows:

$$\begin{aligned} III_1 &= \int_\Omega dx \int_0^T dt \int_0^t e^{-\tau} \frac{1}{4}\left(e^{2\tau}-1\right) \\ &\quad \times \int_0^\tau |\widetilde{u}(s,x)-\widetilde{v}(s,x)|^2 ds\, d\tau \\ &= \int_\Omega dx \int_0^T dt \int_0^t \left[\int_s^t \frac{1}{4}\left(e^{\tau}-e^{-\tau}\right) d\tau\right] \\ &\qquad \times |\widetilde{u}(s,x)-\widetilde{v}(s,x)|^2 ds \\ &= \frac{1}{2} \int_\Omega dx \int_0^T dt \int_0^t \left[\int_s^t \sinh \tau d\tau\right] \\ &\qquad \times |\widetilde{u}(s,x)-\widetilde{v}(s,x)|^2 ds \\ &= \frac{1}{2} \int_\Omega dx \int_0^T dt \int_0^t (\cosh t - \cosh s) \\ &\qquad \times |\widetilde{u}(s,x)-\widetilde{v}(s,x)|^2 ds \\ &\le \frac{1}{2} \int_\Omega dx \int_0^T \cosh t\, dt \\ &\qquad \times \int_0^T |\widetilde{u}(s,x)-\widetilde{v}(s,x)|^2 ds \\ &= \frac{\sinh T}{2} \int_\Omega \int_0^T |\widetilde{u}(s,x)-\widetilde{v}(s,x)|^2 ds\, dx, \end{aligned} \tag{68}$$

$$\begin{aligned} III_2 &= \int_\Omega dx \int_0^T dt \int_0^t e^{-\tau} \frac{1}{2} |u(\tau,x)-v(\tau,x)|^2 d\tau \\ &= \int_\Omega dx \int_0^T d\tau \int_\tau^T e^{-\tau} \frac{1}{2} |u(\tau,x)-v(\tau,x)|^2 dt \\ &= \int_\Omega dx \int_0^T d\tau\, (T-\tau)\, e^{-\tau} \frac{1}{2} |u(\tau,x)-v(\tau,x)|^2 \\ &\le \frac{T}{2} \int_\Omega \int_0^T |u(\tau,x)-v(\tau,x)|^2 d\tau\, dx. \end{aligned} \tag{69}$$

Summing up, we have

$$\begin{aligned} 0 &\ge \frac{1}{2} \|u-v\|_{L^2((0,T)\times\Omega)} \\ &\quad - \frac{T}{2}\left(\|u-v\|_{L^2((0,T)\times\Omega)} + \|\widetilde{u}-\widetilde{v}\|_{L^2((0,T)\times\Omega)}\right) \\ &\quad - \left(\frac{\sinh T}{2} \|\widetilde{u}-\widetilde{v}\|_{L^2((0,T)\times\Omega)} + \frac{T}{2} \|u-v\|_{L^2((0,T)\times\Omega)}\right). \end{aligned} \tag{70}$$

Hence, when $0 < T < 1/2$, putting

$$K(T) := \frac{T+\sinh T}{1-2T}\ (>0), \tag{71}$$

we have

$$\|u-v\|_{L^2((0,T)\times\Omega)} \le K(T)\, \|\widetilde{u}-\widetilde{v}\|_{L^2((0,T)\times\Omega)}. \tag{72}$$

As $T \to 0$, $K(T)$ converges to 0. Thus, if T is sufficiently small, $0 < K(T) < 1$. This means that the map from $\widetilde{u}$ to u is a contraction in $L^2((0,T)\times\Omega)$. Hence, there is a fixed point $\overline{u}$ and it is a solution to (10) with (3) and (4) in $[0,T)\times\Omega$.

End of the proof of Theorem 6.

Uniqueness of a Local Solution. Let $\overline{u}$, $\overline{v}$ be solutions to (10) with (3) and (4). Then, in the same calculus as before, we obtain

$$(1-K(T))\, \|\overline{u}-\overline{v}\|_{L^2((0,T)\times\Omega)} \le 0. \tag{73}$$

This implies the uniqueness.

Existence of a Time Global Solution. Suppose that $\overline{u}$ is a solution to (10) with (3) and (4) in $[0,s]$. First, we remark that

$$\begin{aligned} \partial J(\overline{u}(s,\cdot)) &\ni -\overline{u}_t(s) + \overline{u}(s) \\ &\quad + e^{-t} f_0 + e^{-s}\left(v_0 - u_0 - \int_0^s \overline{u}(\sigma)\, e^{\sigma} d\sigma\right), \end{aligned} \tag{74}$$

and the right hand side belongs to $L^2(\Omega)$. Hence, we are able to solve (10) with (3) and (4) from $t = s$. By the change of variable $t = \widetilde{t} + s$, (10) becomes

$$\begin{aligned} u_{\widetilde{t}} - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) &= U_s e^{-\widetilde{t}} + u_t(s)\, e^{-\widetilde{t}} + u(\widetilde{t}+s) \\ &\quad - u(s)\, e^{-\widetilde{t}} - e^{-\widetilde{t}} \int_0^{\widetilde{t}} u(\sigma+s)\, e^{\sigma} d\sigma, \end{aligned} \tag{75}$$

where

$$\begin{aligned} U_s &= -u_t(s) + u(s) + e^{-s} f_0 \\ &\quad + e^{-s}\left(v_0(x) - u_0(x) - \int_0^s u(\sigma,x)\, e^{\sigma} d\sigma\right). \end{aligned} \tag{76}$$

Hence, solving (10) with (4) and initial condition $u(0) = \overline{u}(s)$, $u_t(0) = \overline{u}_t(s)$, we obtain a function u in $[0,T]\times\Omega$ which solves (10). For $t \in [s, s+T]$, put $\overline{u}(t,x) = u(t-s,x)$. Then, $\overline{u}$ is a solution to (10) with (3) and (4) in $[0, s+T]\times\Omega$. Repeating this process, we obtain a time global solution. Uniqueness of the local solution implies uniqueness of the global solution.

5. Uniform Estimates

Let T be the small number presented in the previous section, and let u be a solution to (10) with (3) and (4) in $[0,T)\times\Omega$. As we see in the previous section, it is obtained as a fixed point of the map $\tau : \widetilde{u} \mapsto u$ in $L^2((0,T)\times\Omega)$, where u is a solution to (19) with (3) and (4). The fixed point is obtained as in the following way. Let $u^{(0)}$ be an arbitrary element of $L^2((0,T)\times\Omega)$ and put $\tau^n(u^{(0)}) = u^{(n)}$. There is a constant σ which is determined by T such that $0 < \sigma < 1$ and $\|\tau(u) - \tau(v)\|_{L^2((0,T)\times\Omega)} \le \sigma\|u - v\|_{L^2((0,T)\times\Omega)}$. Thus, we have

$$\left\|u^{(k)} - u^{(k-1)}\right\|_{L^2((0,T)\times\Omega)} \le \sigma^{k-1}\left\|u^{(1)} - u^{(0)}\right\|_{L^2((0,T)\times\Omega)} \tag{77}$$

and hence,

$$\begin{aligned}&\left\|u^{(n)} - u^{(m)}\right\|_{L^2((0,T)\times\Omega)}\\ &\quad\le \sum_{k=m+1}^{n} \sigma^{k-1}\left\|u^{(1)} - u^{(0)}\right\|_{L^2((0,T)\times\Omega)},\end{aligned} \tag{78}$$

namely, $\{u^{(n)}\}$ is a Cauchy sequence in $L^2((0,T)\times\Omega)$ and it converges to a function u. Since $u^{(n)} = \tau(u^{(n-1)})$, we have that $u = \tau(u)$ by letting $n \to \infty$. Thus, u is the fixed point of τ.

Letting $m = 0$ in (78), we have

$$\begin{aligned}&\left\|u^{(n)}\right\|_{L^2((0,T)\times\Omega)}\\ &\quad\le \sum_{k=1}^{n}\sigma^{k-1}\left\|u^{(1)} - u^{(0)}\right\|_{L^2((0,T)\times\Omega)}\\ &\qquad + \left\|u^{(0)}\right\|_{L^2((0,T)\times\Omega)}.\end{aligned} \tag{79}$$

By the lower semicontinuity, an energy inequality for a solution to (19) with (3) and (4) is obtained by letting $h \to 0$ in (32):

$$\begin{aligned}&\frac{1-\varepsilon}{2}\int_0^T\int_\Omega \left|u_t(t,x)\right|^2 dx\,dt + J(u(T,\cdot))\\ &\quad\le C_0\left(u_0, v_0, f_0, \widetilde{u}, \varepsilon, T\right).\end{aligned} \tag{80}$$

Since C_0 is increasing with respect to T, we have $\text{ess.}\sup_{0<t\le T} J(u(t,\cdot)) \le C_0(T)$. Hence,

$$\begin{aligned}&\frac{1-\varepsilon}{2}\int_0^T\int_\Omega \left|u_t(t,x)\right|^2 dx\,dt\\ &\quad + \text{ess.}\sup_{0<t\le T} J(u(t,\cdot))\\ &\quad\le 2C_0\left(u_0, v_0, f_0, \widetilde{u}, \varepsilon, T\right).\end{aligned} \tag{81}$$

Recall that C_0 is presented as in (33). By the proof of Proposition 9 (5), we have

$$\begin{aligned}&\left\|u^{(1)}\right\|_{L^2((0,T)\times\Omega)}\\ &\quad\le \left\|u_0\right\|_{L^2((0,T)\times\Omega)}\\ &\qquad + \frac{\sqrt{2}T}{1-\varepsilon}C_0\left(u_0, v_0, f_0, u^{(0)}, \varepsilon, T\right)\\ &\quad =: C_1\left(u_0, v_0, f_0, u^{(0)}, \varepsilon, T\right).\end{aligned} \tag{82}$$

Letting $n \to \infty$ in (79), we finally have

$$\begin{aligned}&\|u\|_{L^2((0,T)\times\Omega)}\\ &\quad\le \frac{1}{1-\sigma}\left(\left\|u^{(1)}\right\|_{L^2((0,T)\times\Omega)}\right.\\ &\qquad\left. + \left\|u^{(0)}\right\|_{L^2((0,T)\times\Omega)}\right) + \left\|u^{(0)}\right\|_{L^2((0,T)\times\Omega)}\\ &\quad\le \frac{1}{1-\sigma}\left(C_1\left(u_0, v_0, f_0, u^{(0)}, \varepsilon, T\right)\right.\\ &\qquad\left. + \left\|u^{(0)}\right\|_{L^2(\Omega)}\right) + \left\|u^{(0)}\right\|_{L^2((0,T)\times\Omega)}\\ &\quad =: C_2\left(u_0, v_0, f_0, u^{(0)}, \varepsilon, T\right).\end{aligned} \tag{83}$$

Now, a solution u to (10) is a solution to (19) for $\widetilde{u} = u$, the fixed point of τ. Hence, by (81), we have

$$\begin{aligned}&\frac{1-\varepsilon}{2}\int_0^T\int_\Omega \left|u_t(t,x)\right|^2 dx\,dt\\ &\quad + \text{ess.}\sup_{0<t\le T} J(u(t,\cdot))\\ &\quad\le 2C_0\left(u_0, v_0, f_0, u, \varepsilon, T\right).\end{aligned} \tag{84}$$

By (83) and (84), there exists a constant C_3 such that

$$\begin{aligned}&\int_0^T\int_\Omega \left|u_t(t,x)\right|^2 dx\,dt\\ &\quad + \text{ess.}\sup_{0<t\le T} J(u(t,\cdot))\\ &\quad\le C_3\left(u_0, v_0, f_0, u^{(0)}, \varepsilon, T\right).\end{aligned} \tag{85}$$

Now, by (83) and (85), we obtain uniform estimates for $\|u\|_{L^2((0,T)\times\Omega)}$, $\|u\|_{L^\infty((0,T);\mathrm{BV}(\Omega)\cap L^2(\Omega))}$, $\|u_t\|_{L^2((0,T)\times\Omega)}$ for the small T.

Solving (10) from $t = s$ $(0 < s < T)$, we have the following by (83):

$$\|u\|_{L^2((s,s+T)\times\Omega)} \le C_2\left(u(s), u_t(s), U_s, u^{(0)}, \varepsilon, T\right), \tag{86}$$

where U_s is as in (76). By (81) and Chebyshev's inequality, for sufficiently large R, $\mathscr{L}^1(\{\|u_t(t)\|_{L^2(\Omega)} \geq R\}) \leq (1/R^2)(2C_0/(1-\varepsilon))$. Let δ be an arbitrary small positive number and put $R = \sqrt{(4C_0/\delta(1-\varepsilon))}$. Then, there exists an $s \in [T-\delta, T) = [T-(1/R^2)(4C_0/(1-\varepsilon), T))$ such that $\|u_t(s)\|_{L^2(\Omega)} < R$. Hereby, we have by (83) and (85) that, for such an s, there exists a constant C_4 such that

$$\begin{aligned} &\|u\|_{L^2((s,s+T)\times\Omega)} \\ &\quad + \int_s^{s+T}\int_\Omega \left|u_t(t,x)\right|^2 dx\,dt \\ &\quad + \operatorname*{ess.\,sup}_{s<t\leq s+T} J(u(t,\cdot)) \\ &\leq C_4\left(u_0, v_0, f_0, u^{(0)}, \varepsilon, T, \delta\right). \end{aligned} \tag{87}$$

Repeating this process, we have uniform estimates for $\|u\|_{L^2((0,\widetilde{T})\times\Omega)}$, $\|u\|_{L^\infty((0,\widetilde{T});\mathrm{BV}(\Omega)\cap L^2(\Omega))}$, $\|u_t\|_{L^2((0,\widetilde{T})\times\Omega)}$ for each $\widetilde{T} > 0$. However, their upper bounds depend on $\|f_0\|_{L^2(\Omega)}$.

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