Research Article

On $q$-Difference Riccati Equations and Second-Order Linear $q$-Difference Equations

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Received 29 August 2012; Accepted 17 October 2012

1. Introduction and Main Results

In this paper, a meromorphic function means meromorphic in the whole complex plane $\mathbb{C}$, unless stated otherwise. We also assume that the reader is familiar with the standard symbols and fundamental results such as $m(r, f), N(r, f)$, and $T(r, f)$ of Nevanlinna theory, see, for example, [1, 2], for a given meromorphic function $f(z)$. A meromorphic function $a(z)$ is said to be a small function relative to $f(z)$ if $T(r, a) = S(r, f)$, where $S(r, f)$ is used to denote any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside of a set of finite logarithmic measure, furthermore, possibly outside of a set $E$ of logarithmic density $\log \text{dens}(E) = \lim_{r \to \infty} \int_{[1,r]}^E (dt)/\log r = 0$. For a small function $a(z)$ relative to $f(z)$, we define

$$
\delta(a, f) = \lim_{r \to \infty} \frac{m(r, 1/(f - a))}{T(r, f)} = 1 - \lim_{r \to \infty} \frac{N(r, 1/(f - a))}{T(r, f)}.
$$

(1)

Recently, Ishizaki [3] considered difference Riccati equation

$$
\Delta f(z) + \frac{f(z)^2 + A(z)}{f(z) - 1} = 0,
$$

and second-order linear difference equation

$$
\Delta^2 y(z) + A(z) y(z) = 0,
$$

where $A(z)$ is meromorphic function, and gave surveys of basic properties of (2) and (3), which are analogues in the differential cases.

Now, we are concerned with $q$-difference Riccati equation

$$
g(qz) = -\frac{a_1(z) g(z) + a_0(z)}{g(z)},
$$

and second-order linear $q$-difference equation

$$
f(qz) + a_1(z) f(qz) + a_0(z) f(z) = 0,
$$

where $q \in \mathbb{C} \setminus \{0\}, |q| 
eq 1, a_1(z)$ and $a_0(z) \neq 0$ are rational functions and will obtain some parallel results for $q$-difference case. For a meromorphic function $h(z)$, the $q$-difference operator $\Delta_q$ is defined by $\Delta_q h(z) = h(qz) - h(z)$.

This paper is organized as follows. In Section 2, we describe the transformation between $q$-difference Riccati equation (4) and second-order linear $q$-difference equation (5). In Section 3, we present some properties of $q$-difference Riccati equation (4), such as $q$-difference analogue on the property of a cross ratio for four distinct meromorphic
solutions of a differential Riccati equation, the meromorphic solutions concerning with \( q \)-Gamma function. In Section 4, we study the value distribution of transcendental meromorphic solutions of \( q \)-difference Riccati equation (4) and the form of meromorphic solutions of second-order linear \( q \)-difference equation (5). In Section 5, we discuss the properties on the \( q \)-Casorati determinant of meromorphic solutions of second-order linear \( q \)-difference equation (5).

2. Transformations between \( q \)-Difference Riccati Equations and Linear \( q \)-Difference Equations of Second-Order

It is well known that a differential Riccati equation

\[ w' (z) + w(z)^2 + A(z) = 0 \tag{6} \]

and second-order linear differential equation

\[ u'' (z) + A(z) u(z) = 0 \tag{7} \]

are closely related by the transformation

\[ w(z) = -u(z)/u(z) \tag{8} \]

where \( A(z) \) is a meromorphic function, see, for example, [4, pages 103–106].

Ishizaki [3] considered a difference analogue of (6) and (7) and obtained that difference Riccati equation (2) and second-order linear difference equation (3) are closely linked by the transformation

\[ w(z) = -u(z)/u(z) \]

where \( A(z) \) is a meromorphic function.

Here, we are concerned with a transformation between (4) and (5), see [5]. For a nontrivial meromorphic solution \( f(z) \) of (5), we take

\[ g(z) = \frac{f(qz)}{f(z)}. \tag{10} \]

Then \( g(z) \) satisfies \( q \)-difference Riccati equation (4). In fact, we deduce from (5) that

\[ \frac{f(q^2z)}{f(qz)} + a_1(z) + a_0(z) \frac{f(z)}{f(qz)} = 0, \tag{11} \]

which implies the desired form of (4).

Conversely, if (4) admits a nontrivial meromorphic solution \( g(z) \), then meromorphic function \( f(z) \) of \( q \)-difference equation (10) satisfies (5). In fact, we conclude from (4) and (10) that

\[ f(q^2z) = g(qz) f(qz) = \left( \frac{a_1(z) g(z) + a_0(z) g(z)}{g(z)} \right) f(qz) = -a_1(z) f(qz) - a_0(z) f(z), \tag{12} \]

which implies (5).

Example 1. Suppose that \( q \in \mathbb{C} \setminus \{0\} \) and \( |q| \neq 1 \). Let \( a_0(z) = (q^2 z^2 - (q^2 - 2q - 1) z + 1)/(1 - z^2) \) and \( a_1(z) = 2/(z - 1) \). Then \( g(z) = (qz + 1)/(z + 1) \) and \( f(z) = z + 1 \) satisfy \( q \)-difference Riccati equation (4) and second-order linear \( q \)-difference equation (5), respectively, which both satisfy the transformation (10).

3. Representations of Solutions of \( q \)-Difference Riccati Equations

The representations on meromorphic solutions of Riccati equations are interesting. Bank et al. [6, pages 371–373] obtained that differential Riccati equation (6) possesses a one parameter family of meromorphic solutions \( f(x) \_x \in \mathbb{C} \) if (6) has three distinct meromorphic solutions \( \alpha_1(z), \alpha_2(z), \) and \( \alpha_3(z) \). Ishizaki extended this property to difference Riccati equation (2) and obtained a difference analogue of this property, see [3, Proposition 2.1]. Now, we present this property for \( q \)-difference case below, which can also be seen as a \( q \)-difference analogue of the fact that a cross ratio for four distinct meromorphic solutions of a differential Riccati equation is a constant, see, for example, [4, pages 108-109]. Furthermore, we find that meromorphic solutions of \( q \)-difference Riccati equations (4) are concerned with \( q \)-Gamma function if \( q \in \mathbb{C} \) such that \( 0 < |q| < 1 \).

Theorem 2. Suppose that (4) possesses three distinct meromorphic solutions \( g_1(z), g_2(z), \) and \( g_3(z) \). Then any meromorphic solution \( g(z) \) of (4) can be represented by

\[ g(z) = \left( g_1(z) g_2(z) - g_2(z) g_3(z) - g_1(z) g_3(z) \phi(z) \right) \]

\[ + g_1(z) g_3(z) \left( \phi(z) \right) \]

\[ \times \left( g_1(z) - g_3(z) - g_2(z) \phi(z) + g_3(z) \phi(z) \right)^{-1}, \tag{13} \]

where \( \phi(z) \) is a meromorphic function satisfying \( \phi(qz) = \phi(z) \). Conversely, if for any meromorphic function \( \phi(z) \) satisfying \( \phi(qz) = \phi(z) \), we define a function \( g(z) \) by (13), then \( g(z) \) is a meromorphic solution of (4).

Proof of Theorem 2. Let \( h_j(z) \), \( j = 1, 2, 3, 4 \) be distinct meromorphic functions. We denote a cross ratio of \( h_j(z) \), \( j = 1, 2, 3, 4 \) by

\[ R(h_1, h_2, h_3; h_4; z) = \frac{h_1(z) - h_3(z)}{h_3(z) - h_4(z)} : \frac{h_2(z) - h_3(z)}{h_2(z) - h_4(z)}. \tag{14} \]

Suppose that \( g(z) \) is meromorphic solution of (4) and is also distinct from \( g_1(z), g_2(z), \) and \( g_3(z) \). We first show that \( g(z) \) is a meromorphic solution of \( q \)-difference Riccati equation (4)
if and only if \( R(qz) = R(z) \), where \( R(z) = R(g_1, g_2, g_3, g; z) \). In fact, we conclude from (4) that

\[
R(qz) = \frac{g_1(qz) - g_3(qz)}{g_1(qz) - g(qz)} = \frac{a_0(z)}{-a_1(z)} \frac{(g_1(z) - g_3(z))}{(g_1(z) - g(z))} = \frac{a_0(z)}{-a_1(z)} \frac{g_1(z) - g_3(z)}{g_1(z) - g(z)}
\]

Conversely, if \( R(qz) = R(z) \), then

\[
\frac{a_0(z)}{-a_1(z)} \frac{g_1(z) - g_3(z)}{g_1(z) - g(z)} = \frac{a_0(z)}{-a_1(z)} \frac{g_1(z) - g(z)}{g_1(z) - g(z)} = R(z).
\]

We conclude from (16) that \( g(qz) = -(a_1(z)g(z) + a_0(z))/g(z) \), which shows that \( g(z) \) satisfies (4).

Thus, for any meromorphic function \( \phi(z) \) satisfying \( \phi(qz) = \phi(z) \), we define \( g(z) \) by

\[
R(g_1, g_2, g_3, g; z) = \phi(z).
\]

Then \( g(z) \) is represented by (13), and also satisfies \( q \)-difference Riccati equation (4). The proof of Theorem 2 is completed.

Now, we recall some results of transcendental meromorphic solutions concerned with \( q \)-difference Riccati equation (4). Bergweiler et al. [7, 8] pointed out that all transcendental meromorphic solutions of (5) satisfy \( T(r, f) = O((\log r)^2) \) if \( q \in \mathbb{C} \) and \( 0 < |q| < 1 \). Since (10) is a transformation between (4) and (5), we obtain that all transcendental meromorphic solutions of (4) are of order zero if \( q \in \mathbb{C} \) and \( 0 < |q| < 1 \). On the other hand, if \( g(z) \) is a transcendental meromorphic solution of

\[
g(qz) = R(z, g(z)),
\]

where \( q \in \mathbb{C} \), \( |q| > 1 \) and the coefficients of \( R(z, g(z)) \) are small functions relative to \( g(z) \), Gundersen et al. [9] showed that the order of growth of (18) is equal to \( \log \deg(R)/\log |q| \), where \( \deg(R) \) is the degree of irreducible rational function \( R(z, g(z)) \) in \( g(z) \). Thus, from the above two cases, we obtain that all transcendental meromorphic solutions of (4) are of order zero for all \( q \in \mathbb{C} \setminus \{0\} \) and \( |q| \neq 1 \).

We also illustrate some of the results on \( q \)-difference equations, which are explicitly solvable in terms of known zero-order meromorphic functions (see [5]). Let \( q \in \mathbb{C} \) be such that \( 0 < |q| < 1 \). Then \( q \)-Gamma function \( \Gamma_q(x) \) is defined by

\[
\Gamma_q(x) := \frac{(q; q)_\infty}{(q^q; q)_\infty} (1 - q)^{1-q},
\]

where \((a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k)\). It is a meromorphic function with poles at \( x = -n + 2\pi i k/\log q \), where \( k \) and \( n \) are nonnegative integers, see [10]. By defining

\[
\gamma_q(x) := (1 - q)^{-x} \Gamma_q(x), \quad x = q^x,
\]

and \( \gamma_q(0) := (q; q)_\infty \), we see that \( \gamma_q(x) \) is a meromorphic function of zero-order with no zeros, having its poles at \( \{q^k\}_k \).

Therefore, the first-order linear \( q \)-difference equation

\[
h(qz) = (1 - z)h(z)
\]

is solved by the function \( \gamma_q(z) \). Moreover, for general first-order linear \( q \)-difference equation,

\[
h(qz) = a(z)h(z),
\]

where \( a(z) \) is a rational function. If \( a(z) \equiv a \) is a constant, (22) is solvable in terms of rational functions if and only if \( \log_q a \) is an integer. If \( a(z) \) is nonconstant, let \( a_i, i = 1, 2, \ldots, n \) and \( \beta_j, j = 1, 2, \ldots, m \) be the zeros and poles of \( a(z) \), respectively, repeated according to their multiplicities. Then \( a(z) \) can be written in the form

\[
a(z) = \frac{c(1 - z/\alpha_1)(1 - z/\alpha_2) \cdots (1 - z/\alpha_n)}{(1 - z/\beta_1)(1 - z/\beta_2) \cdots (1 - z/\beta_m)},
\]

where \( c \neq 0 \) is a complex number depending on \( a(z) \). So, (22) is solved by

\[
h(qz) = z^{\log_q c} \gamma_q(z/\alpha_1) \gamma_q(z/\alpha_2) \cdots \gamma_q(z/\alpha_n) \gamma_q(z/\beta_1) \gamma_q(z/\beta_2) \cdots \gamma_q(z/\beta_m),
\]

which is meromorphic if and only if \( \log_q c \) is an integer.

Now, let \( c_1(z) \) and \( c_2(z) \) be two distinct rational solutions of the differential Riccati equation (6). If there exists a rational solution \( c_3(z) \) distinct from \( c_1(z), j = 1, 2 \), then all meromorphic solutions of (6) are rational solutions. If there exists a transcendental meromorphic solution \( w(z) \), then there is no rational solution other than \( c_1(z), j = 1, 2 \), see, for example, [6, pages 393-394]. For difference Riccati equation (2), Ishizaki obtained a difference analogue, see [3, Proposition 2.2]. In the following, we give a \( q \)-difference case for \( q \)-difference Riccati equation (4).

**Theorem 3.** Let \( q \in \mathbb{C} \) be such that \( 0 < |q| < 1 \). Suppose that \( q \)-difference Riccati equation (4) possesses two distinct rational solutions \( g_1(z) \) and \( g_2(z) \). Then there exists a meromorphic solution \( g_3(z) \) distinct from \( g_1(z) \) and \( g_2(z) \) so that any meromorphic solution \( g(z) \) of (4) is represented in the form (13).
Proof of Theorem 3. Since \( g_1(z) \) and \( g_2(z) \) are two distinct rational solutions of (4), we define a translation

\[
g(z) = \frac{g_1(z) h(z) + g_2(z)}{h(z) + 1}.
\]  
(25)

Then \( \sigma(h) = \sigma(g) = 0 \). Substituting (25) into (4), we conclude that

\[
h(qz) = \frac{g_1(z)}{g_2(z)} h(z),
\]  
(26)

which is type of (22). So, \( h(z) \) is a meromorphic solution of (25) as in the form (24). Therefore, we conclude from (25) that \( g_3(z) \) is a meromorphic solution of (4), which is distinct from \( g_1(z) \) and \( g_2(z) \). So, we now deduce from Theorem 2 that any meromorphic solution of (4) is represented in the form (13). The proof of Theorem 3 is completed.

Example 4. Let \( q = (1/2) \), \( a_1(z) = (5z + 4)/2(z + 2) \) and \( a_0(z) = (z - 4)/(z + 2) \) in (4) and (5). Then functions

\[
g_1(z) = -2, \quad g_2(z) = -\frac{z - 2}{2(z + 1)}
\]  
(27)

satisfy \( q \)-difference Riccati equation (4), and (26) turns into

\[
h\left(-\frac{1}{2}\right) = \frac{4(z + 1)}{2(z - 2)} h(z) = -\frac{2}{1 - z/(-1)} h(z).
\]  
(28)

We note that

\[
y_q'(z) = (1 - q)^{-1} y_q(z) = \frac{(q; q)_\infty}{(q^2; q^2)_\infty} = \frac{(q; q)_\infty}{(z; q)_\infty}.
\]  
(29)

Thus, we conclude from (24) and (29) that

\[
h(z) = z^{\log_{1/2}(-2)} (\frac{y_{-1/2}(z)}{y_{-1/2}(z/2)})
\]  
\[
= z^{-1} (\frac{z/2; -1/2}_\infty}{-z/2; -1/2}_\infty)
\]  
\[
= z^{-1} \prod_{k=0}^\infty \left( 1 - \frac{(z/2)(-1/2)^k}{1 + (z(-1/2)^k)} \right)
\]  
\[
= z^{-1} \prod_{k=0}^\infty \left( 1 + \frac{z(-1/2)^k}{1 - z(-1/2)^k} \right)
\]  
\[
= z^{-1} \left[ 1 + \left(1 + \frac{1}{2}z \right) \left(1 + \frac{1}{2}z \right) \cdots \left(1 + \frac{1}{2}z \right) \right]
\]  
\[
\times \left[ 1 + \left(1 + \frac{1}{2}z \right) \left(1 + \frac{1}{2}z \right) \cdots \left(1 + \frac{1}{2}z \right) \right]^{-1} = \frac{1}{z(z + 1)},
\]

\[
g_3(z) = \frac{g_1(z) h(z) + g_2(z)}{h(z) + 1} = -\frac{(z - 2)^2}{2(z^2 + z + 1)}
\]  
(30)

is a meromorphic solution of (4), which is distinct from \( g_1(z) \) and \( g_2(z) \). Moreover, we also conclude from (10), (27), and (5) that

\[
f_1\left(\frac{1}{z^2}\right) = -2f_1(z), \quad f_2\left(\frac{1}{z^2}\right) = \frac{1 - z/2}{1 - z/(-1)} f_2(z),
\]  
(31)

which are corresponding to \( g_1(z) \) and \( g_2(z) \), respectively, and are also the types of (22). Thus, we deduce from (24) that

\[
f_1(z) = \frac{z}{z}, \quad f_2(z) = \frac{y_{-1/2}(z/2)}{y_{-1/2}(-z)} = z + 1
\]  
(32)

satisfy second-order linear \( q \)-difference equation (5).

4. Value Distribution of Solutions of \( q \)-Difference Riccati Equations and Form of Solutions of Second-Order Linear \( q \)-Difference Equations

We first consider the value distribution of transcendental meromorphic solution of \( q \)-difference Riccati equation (4).

Theorem 5. Let \( a_1(z) \) and \( a_0(z) \) be nonconstant rational functions. If \( g(z) \) is a zero-order transcendental meromorphic solution of \( q \)-difference Riccati equation

\[
g(z) = -\frac{a_1(z) g(z) + a_0(z)}{g(z)}
\]  
(33)

with \( q \in \mathbb{C} \setminus \{0\} \) and \( |q| \neq 1 \), then

(i) if

\[
|q| > 1, \quad N(r, g) + N\left(r, \frac{1}{g}\right) = S(r, g),
\]  
(34)

then \( g(z) \) has at most one Borel exceptional value;

(ii) if \( |q| \neq 1 \), then Nevanlinna deficiencies \( \delta(0, g) = \delta(\infty, g) = 0; \)

(iii) if \( |q| \neq 1 \) and \( qa(z) + a_0(z) \neq 0 \), then \( g(z) \) has infinitely many fixed points.

In particular, we obtain the following theorem.

Theorem 6. If \( a_1(z) = a_1 \) and \( a_0(z) = a_0 (\neq 0) \) are constants, and if \( q \in \mathbb{C} \setminus \{0\} \) and \( |q| \neq 1 \), then \( q \)-difference Riccati equation (4) has only rational solutions. Furthermore, if \( a_1(z) \equiv 0 \) and \( a_0(z) = a_0 \) is nonzero constant, then (4) has only a nonzero constant solution \( g(z) = \hat{d} \), which satisfies \( d^2 + a_0 = 0 \).

We need some preliminaries to prove Theorems 5 and 6.

The theorem of Tumura and Clunie is an important result in Nevanlinna theory, see [11, 12]. Weissenborn extended it and obtained the following lemma.
Lemma 7 (see [13, Theorem]). Let \( h(z) \) be a meromorphic function and let \( \phi \) be given by
\[
\phi(z) = c_n(z) h(z)^n + c_{n-1}(z) h(z)^{n-1} + \cdots + c_0(z),
\]
\[
T(r, c_j) = S(r, h), \quad j = 0, 1, \ldots, n-1, n.
\] Then either
\[
\phi = \left( h + \frac{c_{n-1}(z)}{nc_n(z)} \right)^n,
\]
or
\[
T(r, h) \leq \mathcal{N}(r, 1) + \mathcal{N}(r, h) + S(r, h).
\] (35)

Lemma 8. Suppose that \( h(z) \) is a nonconstant meromorphic function satisfying
\[
\mathcal{N}(r, h) + \mathcal{N}(r, \frac{1}{\phi}) = S(r, h).
\] (38)

Let
\[
\phi(z) = c_n(z) h(z)^n + c_{n-1}(z) h(z)^{n-1} + \cdots + c_0(z)
\]
be a polynomial in \( h(z) \) with \( n \in \mathbb{N} \), and coefficients satisfying
\[
T(r, c_j) = S(r, h), \quad j = 0, 1, \ldots, n-1, n, \quad c_n(z) c_0(z) \neq 0.
\] (39)

Then
\[
\mathcal{N}(r, \frac{1}{\phi}) = nT(r, h) + S(r, h).
\] (41)

or
\[
\mathcal{N}(r, \frac{1}{\phi}) \geq T(r, h) + S(r, h).
\] (42)

Thus, \( \phi(z) \neq 0 \).

Proof of Lemma 8. By differentiating both sides of (39), we conclude that
\[
\phi'(z) = \sum_{j=1}^{n} \left( c_j(z) + j c_j(z) \frac{h'(z)}{h(z)} \right) h(z)^j.
\] (43)

Thus, we deduce from (39) and (43) that
\[
\left( c_n'(z) + nc_n(z) \frac{h'(z)}{h(z)} \right) \phi(z) - c_n \phi'(z)
= \sum_{j=1}^{n-1} \left[ c_j(z) \left( c_n'(z) + nc_n(z) \frac{h'(z)}{h(z)} \right) - c_n(z) \left( c'_j(z) + j c_j(z) \frac{h'(z)}{h(z)} \right) \right] h(z)^j
+ c_0(z) \left( c_n'(z) + nc_n(z) \frac{h'(z)}{h(z)} \right).
\] (44)

Therefore, \( (c_n'(z) + nc_n(z)(h'(z)/h(z)))\phi(z) - c_n \phi'(z) \) is a polynomial in \( h(z) \) with degree no greater than \( n-1 \) and the term of degree zero is \( c_0(z)(c_n(z) + nc_n(z)(h'(z)/h(z))) \neq 0 \). Then
\[
\frac{c_n'(z) + nc_n(z) h'(z)/h(z)}{h(z)} \neq 0.
\] (45)

Otherwise, if \( c_n'(z) + nc_n(z)(h'(z)/h(z)) = 0 \), then \( c_n(z) h(z)^n \) is a nonzero constant, a contradiction. We also note that \( c_n(z)(c_n(z) + nc_n(z)(h'(z)/h(z))) \) is a small function relative to \( h(z) \) by (38) and the lemma of logarithmic derivative. Set
\[
\mu_1(z) = c_n'(z) + nc_n(z) \frac{h'(z)}{h(z)}, \quad \nu_1(z) = c_n.
\] (46)

Then \( \mu_1(z) \) and \( \nu_1(z) \) are small functions relative to \( h(z) \) and
\[
\phi_1(z) = \mu_1(z) \phi(z) - \nu_1(z) \phi_1(z)
\]
is a polynomial in \( h(z) \) with degree no greater than \( n-1 \) and the term of degree zero is small function relative to \( h(z) \).

If the degree of \( \phi_1(z) \) is greater than zero, then by repeating the above process, we can get two small functions \( \mu_2(z) \) and \( \nu_2(z) \) such that
\[
\phi_2(z) = \mu_2(z) \phi_1(z) - \nu_2(z) \phi_1(z)
\]
is a polynomial in \( h(z) \) with a degree less than the degree of \( \phi_1(z) \) and the term of degree zero is a small function relative to \( h(z) \).

We note that such process will be terminated at most \( n \) times. Thus, we can proceed this process to obtain small functions \( \mu_j(z) \) and \( \nu_j(z) \), where \( j = 1, 2, \ldots, s, s+1 \) and \( s \leq n \), such that
\[
\phi_j(z) = \mu_j(z) \phi_{j-1}(z) - \nu_j(z) \phi_{j-1}(z)
\]
are polynomial in \( h(z) \) with \( \deg \phi_j(z) > \deg \phi_{j-1}(z) \) (\( j = 1, 2, \ldots, s \)), where \( \phi_0(z) \equiv \phi(z) \) and
\[
\phi_{s+1}(z) = \mu_{s+1}(z) \phi_s(z) - \nu_{s+1}(z) \phi_s(z)
\]
is a small function relative to \( h(z) \). Thus, we deduce that the small function \( \phi_{s+1}(z) \) can be expressed as a linear differential polynomial in \( \phi(z) \) with coefficients being small functions relative to \( h(z) \). So,
\[
m \left( r, \frac{1}{\phi} \right) = S(r, h).
\] (51)

On the other hand, we deduce from Lemma 7 that either
\[
\phi \equiv \left( h + \frac{c_{n-1}(z)}{nc_n(z)} \right)^n
\]
or
\[
T(r, h) \leq \mathcal{N}(r, \frac{1}{\phi}) + \mathcal{N}(r, h) + S(r, h).
\] (53)

Thus, we deduce from Valiron-Mohon’ko Lemma, (51), and (52) that (41) holds and obtain from (38) and (53) that (42) holds. Therefore, \( \phi(z) \neq 0 \). The proof of Lemma 8 is completed. \( \square \)
Lemma 9 (see [9, Theorem 5.2]). Let \( h(z) \) be a transcendental meromorphic solution of
\[
h(z) = R(z, h(z)) = \sum_{i=0}^{p} a_i(z) h(z)^i \tag{54}\]
with meromorphic coefficients \( a_i(z), b_i(z) \) relative to \( h(z) \) and \( q \in \mathbb{C} \) such that \( |q| > 1 \). If \( N(r, h) + N(r, 1/h) = S(r, h) \), then (54) is either of the form
\[
f(z) = a_p(z) f(z)^p \quad \text{or} \quad f(z) = \frac{a_0(z)}{f(z)^n}. \tag{55}\]

Lemma 10 (see [5, Theorem 2.2]). Let \( f(z) \) be a nonconstant zero-order meromorphic solution of
\[
P(z, f) = 0, \tag{56}\]
where \( P(z, f) \) is a \( q \)-difference polynomials in \( f(z) \). If \( P(z, \alpha) \text{ is a polynomial} \), then
\[
m \left( r, \frac{1}{f - \alpha} \right) = o(T(r, f)) \tag{57}\]
on a set of logarithmic density 1.

Lemma 11 (see [2, Theorem 2.2.5 and Corollary 2.2.7]). Let \( f(z) \) be a meromorphic function. Then for all irreducible rational functions in \( f(z), \)
\[
R(z, f(z)) = \frac{\sum_{i=0}^{p} a_i(z) f(z)^i}{\sum_{j=0}^{q} b_j(z) f(z)^j}, \tag{58}\]
with meromorphic coefficients \( a_i(z), b_j(z) \), the characteristic function of \( R(z, f(z)) \) satisfies
\[
T \left( R(z, f(z)) \right) = dT(r, f) + O(\Psi(r)), \tag{59}\]
where \( d = \max{\{p, q\}} \) and \( \Psi(r) = \max_{i,j} \{T(r, a_i), T(r, b_j)\} \).

In the particular case when
\[
T(r, a_i) = S(r, f), \quad i = 0, 1, \ldots, p, \tag{60}\]
\[
T(r, b_j) = S(r, f), \quad j = 0, 1, \ldots, q,
\]
we have
\[
T(r, R(z, f(z))) = dT(r, f) + S(r, f). \tag{61}\]

We also use the observation [7, page 2] that, for any meromorphic function \( f(z) \) and any constant \( q \in \mathbb{C} \setminus \{0\}, \)
\[
T(r, f(z)) = T(|q| r, f) + O(1). \tag{62}\]

Proof of Theorem 5. Suppose that \( g(z) \) is a zero-order transcendental meromorphic solution of \( q \)-difference Riccati equation (4).

(i) Suppose that \( g(z) \) has two finite Borel exceptional values \( a \) and \( b \neq 0, a \). For the case where one of \( a \) and \( b \) is infinite, we can use a similar method to prove. Set
\[
h(z) = \frac{g(z) - a}{g(z) - b}. \tag{63}\]
Since \( \overline{N}(r, g) + \overline{N}(r, 1/g) = S(r, g) \), we deduce from (63) that
\[
\overline{N}(r, h) + \overline{N} \left( r, \frac{1}{h} \right) = S(r, h). \tag{64}\]
We also conclude from (63) that
\[
g(z) = \frac{a - bh(z)}{1 - h(z)}. \tag{65}\]
Now, substituting (65) into (4), we conclude that
\[
h(z) = \frac{(ba_1(z) + a_0(z) + ab)}{(ba_1(z) + a_0(z) + b^2)} h(z) - \frac{(aa_1(z) + a_0(z) + a^2)}{(ba_1(z) + a_0(z) + b^2)} h(z) - \frac{(aa_1(z) + a_0(z) + a^2)}{(ba_1(z) + a_0(z) + a^2)}. \tag{66}\]

By the assumptions of Theorem 5, we get
\[
(ba_1(z) + a_0(z) + ab) \neq 0, \quad (ba_1(z) + a_0(z) + b^2) \neq 0. \tag{67}\]
Thus, we deduce from Lemma 9, (64), and (66) that
\[
h(qz) = c(z) h(z)^k, \quad k \in \mathbb{Z} \setminus \{0\}, \tag{68}\]
where \( T(r, c(z)) = S(r, h) \). If \( k > 1 \), we conclude from (66) and (68) that
\[
c(z) (ba_1(z) + a_0(z) + b^2) h(z)^{k+1} - c(z) (aa_1(z) + a_0(z) + ab) h(z)^k - (ba_1(z) + a_0(z) + ab) h(z) + (aa_1(z) + a_0(z) + a^2) = 0. \tag{69}\]
Thus, we deduce from Lemma 8 and (64) that (69) is a contradiction. If \( k \leq -1 \), we use the same method as above to get another contradiction. Therefore, \( g(z) \) at most one Borel exceptional value.

(ii) We first prove \( \delta(0, g) = 0 \). We obtain from (4) that
\[
P_1(z, g) = g(z) g(qz) + a_1(z) g(z) + a_0(z) = 0. \tag{70}\]
Since \( P_1(z, 0) = a_0(z) \neq 0 \), we deduce from Lemma 10 and (70) that
\[
m \left( r, \frac{1}{g} \right) = S(r, g) \tag{71}\]
on a set \( E \) of logarithmic density 1. Therefore,
\[
0 \leq \delta(0, g) = \lim_{r \to \infty} \frac{m \left( r, \frac{1}{g} \right)}{T(r, g)} \leq \lim_{r \to \infty} \frac{m \left( r, \frac{1}{h} \right)}{T(r, h)} = 0. \tag{72}\]
Thus, \( \delta(0, g) = 0 \).

We second prove \( \delta(\infty, g) = 0 \). Set \( y(z) = 1/g(z) \). Then
\[
T(r, y) = T(r, g) + O(1), \quad S(r, y) = S(r, g). \tag{73}\]
Now, substituting \( g(z) = 1/y(z) \) into (4), we conclude that
\[
P_2(z, y) = y(qz) (a_0(z) y(z) + a_1(z)) + 1 = 0. \tag{74}
\]
Since \( P_2(z, 0) = 1 \neq 0 \), we obtain from Lemma 10 and (74) that
\[
m \left( r, \frac{1}{y} \right) = S(r, y) \tag{75}
\]
on a set \( E \) of logarithmic density 1. Therefore,
\[
N \left( r, \frac{1}{y} \right) = T(r, y) + S(r, y) \tag{76}
\]
on a set \( E \) of logarithmic density 1. Thus, we conclude from
\( y(z) = 1/g(z) \) and (76) that
\[
N \left( r, \frac{1}{y} \right) = T(r, y) + o(T(r, y)) \tag{77}
\]
and
\[
N \left( r, \frac{1}{y} \right) = T(r, y) + S(r, y) \tag{76}
\]
on a set \( E \) of logarithmic density 1, and so,
\[
0 \leq \delta(\infty, g) = 1 - \lim_{r \to \infty} \frac{N(r, g)}{T(r, g)} \tag{78}
\]
and
\[
0 \leq \delta(\infty, g) = 1 - \lim_{r \to \infty} \frac{N(r, g)}{T(r, g)} = 0.
\]
Thus, \( \delta(\infty, g) = 0 \).

(iii) Set \( y(z) = g(z) - z \). Then
\[
T(r, y) = T(r, g) + S(r, g), \quad S(r, y) = S(r, g) \tag{79}
\]
Substituting \( g(z) = y(z) + z \) into (4), we conclude that
\[
P_3(z, y) = y(qz) + z y(qz) + q z y(z) \tag{80}
\]
and
\[
P_3(z, 0) = q z^2 + z a_1(z) + a_0(z) \neq 0.
\]
Since \( P_3(z, 0) = q z^2 + z a_1(z) + a_0(z) \neq 0 \), we deduce from Lemma 10 and (80) that
\[
m \left( r, \frac{1}{g - z} \right) = S(r, y) \tag{81}
\]
on a set \( E \) of logarithmic density 1. Therefore
\[
N \left( r, \frac{1}{g - z} \right) = N \left( r, \frac{1}{y} \right) = T(r, y) + o(T(r, y)) \tag{82}
\]
on a set \( E \) of logarithmic density 1. This shows that \( g(z) \) has infinitely many fixed points if \( q z^2 + z a_1(z) + a_0(z) \neq 0 \).

Proof of Theorem 6. Suppose first that \( 0 < |q| < 1 \) and (4) with nonzero constant coefficients \( a_1(z) \) and \( a_0(z) \) admits a meromorphic solution \( g(z) \). We assert that \( g(z) \) is rational. In fact, we conclude from Lemma 11, (4), and (62) that
\[
T(r, f) \leq T(|q|, r, f) + A, \quad r \geq R_0, \tag{83}
\]
where \( A > \Psi(r) = \max\{T(r, a_0), T(r, a_1)\} \geq 0 \), \( R_0 > 0 \) is fixed number.

Thus, for any \( r \geq R_0 \), there exists an \( n \in \mathbb{N} \) such that
\[
\frac{R_0}{|q|^n} \leq r < \frac{R_0}{|q|^n}. \tag{84}
\]
By an inductive argument, we deduce from (84) that
\[
T(r, f) \leq T(|q|^n, r, f) + An \tag{85}
\]
and
\[
0 \leq T(R_0, f) + A \left( \frac{\log r}{\log (1/|q|)} - \frac{\log R_0}{\log (1/|q|)} + 1 \right) \tag{85}
\]
Suppose now that \( |q| > 1 \) and (4) with nonzero constant coefficients \( a_1(z) \) and \( a_0(z) \) admits a meromorphic solution \( g(z) \). Replacing \( z \) by \( z/|q| \) in (4), we proceed in a similar method as above to get (85) again. Therefore, \( g(z) \) is rational solution of (4).

Now, we affirm that \( g(z) \) must be nonzero constant if \( a_1(z) \equiv 0 \) and \( a_0(z) = a_0(\neq 0) \) is a constant. Otherwise, if \( g(z) \) is nonconstant rational and has a pole \( z_0 \neq 0 \), we conclude from (4) that \( g(z) \) has infinitely many poles of the forms \( q^{2(n-1)}z_0 \) and infinitely many zeros of the forms \( q^{2(n-1)+1}z_0 \) for all \( n \in \mathbb{N} \). Conversely, If \( g(z) \) is nonconstant rational and has a zero \( z_0 \neq 0 \), we conclude from (4) that \( g(z) \) has infinitely many zeros of the forms \( q^{2(n-1)}z_0 \) and infinitely many poles of the forms \( q^{2(n-1)+1}z_0 \) for all \( n \in \mathbb{N} \). These are both impossible since \( g(z) \) is rational. Thus, the only possible pole (resp. zero) of \( g(z) \) is at 0. So \( g(z) \) may have the form \( g(z) = dz^k(k \in \mathbb{Z}) \), where \( d \) is a nonzero constant. If \( k \neq 0 \), we get a contradiction from (4). Therefore, \( k = 0 \) and (4) has only a nonzero constant solution \( g(z) = d \), which satisfies \( d^2 + a_0 = 0 \). The proof of Theorem 6 is completed.

We now consider the form of meromorphic solutions of (5), which is according to Theorem 6. In fact, more details about meromorphic solutions of (5) have been studied in [7, 14]. Here, we only consider the case that all coefficients are constants.

Theorem 12. If \( a_1(z) \equiv 0 \) and \( a_0(z) = a_0 \) is constant, and if \( q \in \mathbb{C} \setminus \{0\} \) and \( |q| \neq 1 \), then every meromorphic solution \( f(z) \) of second-order linear \( q \)-difference equation (5) has the form \( f(z) = \beta z^k \), where \( \beta \in \mathbb{C} \setminus \{0\} \) and \( k \in \mathbb{Z} \) satisfying \( q^{2k} + a_0 = 0 \).

We first list a lemma needed below.

Lemma 13 (see [14, Theorem 2.1]). Suppose that \( q \in \mathbb{C} \setminus \{0\} \) and \( |q| \neq 1 \). Let \( a_0, a_1, \ldots, a_n \) be complex constants and let \( Q(z) \) be of the reduced form \( Q(z) = p_1(z)/z^l \), where \( p_1(z) \) is a polynomial of degree \( d \) and \( l \in \mathbb{N} \cup \{0\} \). Then all meromorphic solutions \( f(z) \) of
\[
\sum_{j=0}^{n} a_j(z) f(q^j z) = Q(z) \tag{86}
\]
are of the reduced form \( f(z) = p_2(z)/z^p \), where \( p_2(z) \) is a polynomial and \( p \geq 1 \).

**Proof of Theorem 12.** We deduce from Lemma 13 that all meromorphic solutions \( f(z) \) of (5) are of the form:

\[
d = g(z) = \frac{f(qz)}{f(z)} = \frac{1}{ql} \cdot \frac{p_2(qz)}{p_2(z)},
\]

where \( d \) is defined as Theorem 6. From (87), we obtain that there exists \( \beta \in \mathbb{C} \setminus \{0\} \) and \( m \in \mathbb{N} \) such that \( p_2(z) = \beta z^m \), and so \( f(z) = p_2(z)/z^p = \beta z^m/z^p = \beta z^{m-p} \), where \( k = m - p \in \mathbb{Z} \). Now, substituting \( f(z) = \beta z^{m-p} \) into (5), we conclude that \( k \) satisfies \( q^k + a_0 = 0 \). The proof of Theorem 12 is completed.

**Example 14.** Let \( q \in \mathbb{C} \setminus \{0\}, \|q\| \neq 1, a_1(z) \equiv 0 \) and \( a_0(z) = -1/q^2 \). Then second-order \( q \)-difference equation (5) is solved by \( f(z) = 1/z \). Obviously, \( f(z) = 1/z \) and \( k = -1 \) satisfy the conclusions described by Theorem 12.

## 5. Linear \( q \)-Difference Equations of Second-Order

Let \( y_1(z) \) and \( y_2(z) \) be meromorphic solutions of (5). We define the \( q \)-Casoratian determinant of meromorphic functions \( y_1(z) \) and \( y_2(z) \) by

\[
\cC_q(y_1, y_2; qz) = \begin{vmatrix} y_1(z) & y_2(z) \\ y_1(qz) & y_2(qz) \end{vmatrix}.
\]

Then the \( q \)-Casoratian determinant \( \cC_q(z) \) vanishes identically on \( \mathbb{C} \) if and only if the functions \( y_1(z) \) and \( y_2(z) \) are linearly dependent over the field of functions \( \phi(qz) = \phi(z) \). On the other hand, \( g_1(z) \) and \( g_2(z) \) are linearly independent if and only if \( \cC_q(g_1, g_2; z) \neq 0 \). From this definition, we have some properties on the \( q \)-Casoratian determinant \( \cC_q(z) \) as follows.

**Theorem 15.** If \( y_1(z) \) and \( y_2(z) \) are nontrivial meromorphic solutions of (5), then \( q \)-Casoratian determinant \( \cC_q(y_1, y_2; z) \) satisfies a first-order \( q \)-difference equation

\[
\Delta_q \cC_q(z) = (a_0 - 1) \cC_q(z).
\]

Conversely, we assume that \( y_1(z)(\neq 0) \) and \( y_2(z) \) satisfy (89). If \( y_1(z) \) is a meromorphic solution of (5), then \( y_2(z) \) is also a meromorphic solution of (5).

**Proof of Theorem 15.** Suppose first that \( y_1(z) \) and \( y_2(z) \) are nontrivial meromorphic solutions of (5), we conclude that

\[
\cC_q(qz) = \cC_q(y_1, y_2; qz) = \begin{vmatrix} y_1(qz) & y_2(qz) \\ y_1(q^2z) & y_2(q^2z) \end{vmatrix} = \begin{vmatrix} y_1(qz) & y_2(qz) \\ -a_1 y_1(qz) - a_0(z) y_1(z) & -a_1 y_2(qz) - a_0(z) y_2(z) \end{vmatrix} = \begin{vmatrix} y_1(qz) & y_2(qz) \\ -a_1 y_1(qz) - a_0(z) y_1(z) & -a_0(z) y_2(z) \end{vmatrix} = a_0(z) \cC_q(z).
\]

Therefore,

\[
\Delta_q \cC_q(z) = \cC_q(qz) - \cC_q(z) = (a_0(z) - 1) \cC_q(z).
\]

Second, if \( y_1(z)(\neq 0) \) and \( y_2(z) \) satisfy (89), then we have

\[
\begin{vmatrix} y_1(qz) & y_2(qz) \\ y_1(q^2z) & y_2(q^2z) \end{vmatrix} = a_0(z) \begin{vmatrix} y_1(qz) & y_2(qz) \\ y_1(q^2z) & y_2(q^2z) + c(z) y_1(qz) \end{vmatrix}.
\]

We note that, for any meromorphic function \( c(z) \neq 0 \),

\[
\begin{vmatrix} y_1(qz) & y_2(qz) \\ y_1(q^2z) & y_2(q^2z) + c(z) y_1(qz) \end{vmatrix} = \begin{vmatrix} y_1(q^2z) + a_1(z) y_1(qz) & y_2(q^2z) + a_1(z) y_2(qz) \\ a_0(z) y_1(qz) & y_2(q^2z) \end{vmatrix}.
\]

In particular, we take \( c(z) = a_1(z) \). Thus,

\[
\begin{vmatrix} y_1(qz) & y_2(qz) \\ y_1(q^2z) + a_1(z) y_1(qz) & y_2(q^2z) + a_1(z) y_2(qz) \end{vmatrix} = a_0(z) \begin{vmatrix} y_1(qz) & y_2(qz) \\ y_1(q^2z) & y_2(q^2z) \end{vmatrix}.
\]

So, we have

\[
\begin{vmatrix} y_1(qz) & y_2(qz) \\ y_1(q^2z) + a_1(z) y_1(qz) & y_2(q^2z) + a_1(z) y_2(qz) \end{vmatrix} = \begin{vmatrix} y_1(qz) & y_2(qz) \\ a_0(z) y_1(qz) & a_0(z) y_2(qz) \end{vmatrix} = 0.
\]
From this, we conclude that
\[
y_1(qz) \left[y_2(q^2z) + a_1(z) y_2(qz) + a_0(z) y_2(z) \right] \\
= y_2(qz) \left[y_1(q^2z) + a_1(z) y_1(qz) + a_0(z) y_1(z) \right].
\] (96)

Since \( y_1(z) \neq 0 \) is a meromorphic solution of (5), we have
\[
y_1(q^2z) + a_1(z) y_1(qz) + a_0(z) y_1(z) = 0,
\] (97)
and so,
\[
y_2(q^2z) + a_1(z) y_2(qz) + a_0(z) y_2(z) = 0.
\] (98)

This shows that \( y_2(z) \) is a meromorphic solution of (5). The proof of Theorem 15 is completed.

**Theorem 16.** (i) Let \( y_1(z) \) and \( y_2(z) \) be linear independent meromorphic solutions of (5), and let \( \mathcal{C}_q(z) \) be the \( q \)-Casoratian determinant of \( y_1(z) \) and \( y_2(z) \). Then \( y_2(z) \) is represented as \( y_2(z) = g(z) y_1(z) \), where \( g(z) \) satisfies
\[
\Delta_q g(z) = \frac{\mathcal{C}_q(z)}{y_1(z) y_1(qz)}.
\] (99)

(ii) Let \( y_1(z) \) be a nontrivial meromorphic solution of (5), and let \( \mathcal{C}_q(z) \) be a meromorphic solution of (89). If \( g(z) \) satisfies (99), then \( y_2(z) = g(z) y_1(z) \) is a meromorphic solution of (5).

**Proof of Theorem 16.** (i) From the definition of \( \mathcal{C}_q(z) \), we obtain
\[
\mathcal{C}_q(z) = y_1(z) y_2(qz) - y_2(z) y_1(qz).
\] (100)
This shows that \( y_2(z) \) satisfies first-order \( q \)-difference equation of type
\[
y_2(qz) = y_2(z) \frac{y_1(qz)}{y_1(z)} + \mathcal{C}_q(z) \frac{y_1(qz)}{y_1(z)}. \] (101)

By substituting \( y_2(z) = g(z) y_1(z) \) into (101), we conclude that
\[
g(qz) y_1(qz) = g(z) y_1(z) \cdot \frac{y_1(qz)}{y_1(z)} + \mathcal{C}_q(z) \frac{y_1(qz)}{y_1(z)}, \] (102)
and so we obtain the desired form (99).

(ii) Obviously, we conclude from (99) and (89) that
\[
g(qz) = g(z) + \frac{\mathcal{C}_q(z)}{y_1(z) y_1(qz)},
\]
\[
g(q^2z) = g(qz) + \frac{\mathcal{C}_q(qz)}{y_1(qz) y_1(q^2z)} \] (103)
\[
= g(qz) + \frac{a_0(z) \mathcal{C}_q(z)}{y_1(qz) y_1(q^2z)}.
\]

Since \( y_2(z) = g(z) y_1(z), \mathcal{C}_q(z) = y_1(z) y_2(qz) - y_2(z) y_1(qz), \) and
\[
y_1(q^2z) + a_1(z) y_1(qz) + a_0(z) y_1(z) = 0,
\] (104)
we conclude from (103), and (104) that
\[
y_2(q^2z) = g(q^2z) y_1(q^2z)
\]
\[
= \left( g(z) + \frac{\mathcal{C}_q(z)}{y_1(z) y_1(qz)} + \frac{a_0(z) \mathcal{C}_q(z)}{y_1(qz) y_1(q^2z)} \right) y_1(q^2z)
\]
\[
= g(z) y_1(q^2z) + \frac{a_1(z) y_1(qz)}{y_1(z) y_1(qz)} \cdot \mathcal{C}_q(z)
\]
\[
= g(z) y_1(q^2z) + \frac{a_1(z) y_1(qz)}{y_1(z) y_1(qz)} \cdot \mathcal{C}_q(z)
\]
\[
= \frac{y_2(z)}{y_1(z)} \cdot (-a_1(z) y_1(qz) - a_0(z) y_1(z)) - \frac{a_1(z)}{y_1(z)}
\]
\[
\cdot (y_1(z) y_2(qz) - y_2(z) y_1(qz))
\]
\[
= -a_1(z) y_2(qz) - a_0(z) y_2(z).
\] (105)

This yields that \( y_2(z) = g(z) y_1(z) \) is a meromorphic solution of (5). The proof of Theorem 16 is completed.

**Acknowledgments**

The author would like to thank the referees for their helpful remarks and suggestions to improve this paper. The author also wishes to express his thanks to Professors Risto Korhonen, Ilpo Laine, Benharrat Belaïdi, and Kuldeep Singh Charak for their valuable advice during the preparation of this paper. This research was supported by National Natural Science Foundation of China (no. 11171119 and no. 11171121).

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