Sharp Large Deviation for the Energy of $\alpha$-Brownian Bridge

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We study the sharp large deviation for the energy of $\alpha$-Brownian bridge. The full expansion of the tail probability for energy is obtained by the change of measure.

1. Introduction

We consider the following $\alpha$-Brownian bridge:

$$dX_t = -\frac{\alpha}{T-t}X_t dt + dW_t, \quad X_0 = 0,$$

where $W$ is a standard Brownian motion, $t \in [0, T)$, $T \in (0, \infty)$, and the constant $\alpha > 1/2$. Let $P_\alpha$ denote the probability distribution of the solution $\{X_t, t \in [0, T]\}$ of (1). The $\alpha$-Brownian bridge is first used to study the arbitrage profit associated with a given future contract in the absence of transaction costs by Brennan and Schwartz [1].

$\alpha$-Brownian bridge is a time inhomogeneous diffusion process which has been studied by Barczy and Pap [2, 3], Jiang and Zhao [4], and Zhao and Liu [5]. They studied the central limit theorem and the large deviations for parameter estimators and hypothesis testing problem of $\alpha$-Brownian bridge. While the large deviation is not so helpful in some statistics problems since it only gives a logarithmic equivalent for the deviation probability, Bahadur and Ranga Rao [6] overcame this difficulty by the sharp large deviation principle for the empirical mean. Recently, the sharp large deviation principle is widely used in the study of Gaussian quadratic forms, Ornstein-Uhlenbeck model, and fractional Ornstein-Uhlenbeck (cf. Bercu and Rouault [7], Bercu et al. [8], and Bercu et al. [9, 10]).

In this paper we consider the sharp large deviation principle (SLDP) of energy $S_t$, where

$$S_t = \int_0^t \frac{X_s^2}{(s-T)^2} ds.$$  \hspace{1cm} (2)

Our main results are the following.

Theorem 1. Let $\{X_t, t \in [0, T]\}$ be the process given by the stochastic differential equation (1). Then $\{S_t/\lambda_t, t \in [0, T]\}$ satisfies the large deviation principle with speed $\lambda_t$ and good rate function $I(\cdot)$ defined by the following:

$$I(x) = \begin{cases} 
\frac{1}{8x}((2\alpha_0 - 1)x - 1)^2, & \text{if } x > 0; \\
+\infty, & \text{if } x \leq 0,
\end{cases}$$  \hspace{1cm} (3)

where $\lambda_t = \log(T/(T-t)).$

Theorem 2. $\{S_t/\lambda_t, t \in [0, T]\}$ satisfies SLDP; that is, for any $c > 1/(2\alpha - 1)$, there exists a sequence $b_{c,k}$ such that, for any $p > 0$, when $t$ approaches $T$ enough,

$$P(S_t \geq c\lambda_t) = \exp\left[-I(c)\lambda_t + H(a_t)\right] \sqrt{2\pi a_t}\beta_t \times \left(1 + \sum_{k=1}^L \frac{b_{c,k}}{\lambda_t} + O\left(\frac{1}{\lambda_t^{p+1}}\right)\right),$$  \hspace{1cm} (4)
where
\[\sigma_c^2 = 4c^2, \quad \beta_c = \sigma_c \sqrt{\lambda_t},\]
\[a_c = \frac{(1 - 2\alpha)c^2 - 1}{8c^2},\]
\[H(a_c) = -\frac{1}{2} \log \left(1 - \frac{1 - 2\alpha}{2}\right)\].

The coefficients \(b_{c,k}\) may be explicitly computed as functions of the derivatives of \(L\) and \(H\) (defined in Lemma 3) at point \(a_c\).

For example, \(b_{c,1}\) is given by
\[b_{c,1} = \frac{1}{\sigma_c} \left( \frac{h_2 - h_1^2}{2} + \frac{l_4}{8\sigma_c^2} + \frac{l_2h_1}{2\sigma_c^2} \right) - \frac{5l_2^2}{24\sigma_c^4} \frac{h_1}{a_c} - \frac{l_3}{2a_c} \left( \frac{1}{a_c^2} - \frac{1}{a_c^2} \right),\]
with \(l_k = L^{(k)}(a_c)\), and \(h_k = H^{(k)}(a_c)\).

2. Large Deviation for Energy

Given \(\alpha > 1/2\), we first consider the following logarithmic moment generating function of \(S_t\); that is,
\[L_t(u) := \log E_\alpha \exp \left( u \int_0^t \frac{X_s^2}{(s - T)^2} ds \right), \quad \forall \lambda \in \mathbb{R}.\]

And let
\[\mathcal{D}_L := \{ u \in \mathbb{R}, L_t(u) < +\infty \}\]
be the effective domain of \(L_t\). By the same method as in Zhao and Liu [5], we have the following lemma.

**Lemma 3.** Let \(\mathcal{D}_L\) be the effective domain of the limit of \(L_t\); then for all \(u \in \mathcal{D}_L\), one has
\[\frac{L_t(u)}{\lambda_t} = L(u) + \frac{H(u)}{\lambda_t} + \frac{R(u)}{\lambda_t},\]
with
\[L(u) = -\frac{1 - 2\alpha - \varphi(u)}{4},\]
\[H(\lambda) = -\frac{1}{2} \log \left\{ \frac{1}{2} (1 + h(u)) \right\},\]
\[R(u) = -\frac{1}{2} \log \left\{ 1 + \frac{1 - h(u)}{1 + h(u)} \exp \left[ 2\varphi(u) \right] \lambda_t \right\},\]
where \(\varphi(u) = \sqrt{(1 - 2\alpha)^2 - 8u}\) and \(h(u) = (1 - 2\alpha)/\varphi(u)\). Furthermore, the remainder \(R(u)\) satisfies
\[R(u) = O_{1-T} \left( \exp \left[ 2\varphi(u) \lambda_t \right] \right).\]

**Proof.** By Itô’s formula and Girsanov’s formula (see Jacob and Shiryaev [11]), for all \(u \in \mathcal{D}_L\) and \(t \in [0, T]\),
\[\log \frac{dP_u}{dP_\beta} = (\alpha - \beta) \int_0^t \frac{X_s^2}{s - T} dX_s - \frac{\alpha^2 - \beta^2}{2} \int_0^t \frac{X_s^2}{(s - T)^2} ds,\]
\[+ \frac{1}{2} (\beta^2 - \alpha^2 + \alpha - \beta + 2u) \times \int_0^t \frac{X_s^2}{(s - T)^2} ds.\]

Therefore,
\[L_t(u) = \log E_\beta \exp \left\{ u \int_0^t \frac{X_s^2}{(s - T)^2} ds \right\} \log \left( 1 - \frac{t}{T} \right) + \frac{1}{2} (\beta^2 - \alpha^2 + \alpha - \beta + 2u) \times \int_0^t \frac{X_s^2}{(s - T)^2} ds.\]

If \(4u \leq (1 - 2\alpha)^2\), we can choose \(\beta\) such that \((\beta - 1/2)^2 - (\alpha - 1/2)^2 + 2u = 0\). Then
\[L_t(u) = \frac{1 - 2\alpha - \varphi(u)}{4} \lambda_t - \frac{1}{2} \log \left\{ \frac{1}{2} (1 + h(u)) \right\} - \frac{1}{2} \log \left\{ 1 + \frac{1 - h(u)}{1 + h(u)} \exp \left[ 2\varphi(u) \lambda_t \right] \right\},\]
where \(\varphi(u) = \sqrt{(1 - 2\alpha)^2 - 8u}\) and \(h(u) = (1 - 2\alpha)/\varphi(u)\). Therefore,
\[L_t(u) = \frac{1 - 2\alpha - \varphi(u)}{4} \lambda_t - \frac{1}{2} \log \left\{ \frac{1}{2} (1 + h(u)) \right\} - \frac{1}{2} \log \left\{ 1 + \frac{1 - h(u)}{1 + h(u)} \exp \left[ 2\varphi(u) \lambda_t \right] \right\}\]
\[= L(u) + \frac{H(u)}{\lambda_t} + \frac{R(u)}{\lambda_t}.\]

**Proof of Theorem 1.** From Lemma 3, we have
\[L(u) = \lim_{t \to T} \frac{L_t(u)}{\lambda_t} = \frac{1 - 2\alpha - \varphi(u)}{4}.\]
and \( L(\cdot) \) is steep; by the Gärtner-Ellis theorem (Dembo and Zeitouni [12]), \( S_t/\lambda_t \) satisfies the large deviation principle with speed \( \lambda_t \) and good rate function \( I(\cdot) \) defined by the following:

\[
I(x) = \begin{cases} 
\frac{1}{8x}((2\alpha - 1)x - 1)^2, & \text{if } x > 0; \\
+\infty, & \text{if } x \leq 0.
\end{cases}
\]  

(17)

\[\begin{align*}
\Phi_{\lambda}(u) = & \frac{1}{2\pi \sigma_c} \int_{-\infty}^{\infty} \exp \left\{ -\frac{\tau u^2}{2\sigma_c^2} \right\} \exp \left\{ \frac{\tau u^2}{2\sigma_c^2} - \frac{\tau u^2}{2\sigma_c^2} \right\} du \\
& \times \exp \left\{ \frac{\tau u^2}{2\sigma_c^2} - \frac{\tau u^2}{2\sigma_c^2} \right\}
\end{align*}\]  

(23)

Moreover,

\[B_t = \mathbb{E}_Q \exp \left( -a_\beta U_t 1_{\{U_t \geq 0\}} \right) = C_t + D_t,
\]  

(24)

with

\[
C_t = \frac{1}{2\pi a_\beta^2} \int_{|u| \leq s_t} \left( 1 + \frac{iu}{a_\beta} \right) \Phi_t(u)\, du,
\]  

(25)

\[
D_t = \frac{1}{2\pi a_\beta^2} \int_{|u| > s_t} \left( 1 + \frac{iu}{a_\beta} \right) \Phi_t(u)\, du,
\]  

(26)

where \( [D_t] = O(\exp \{-D\lambda_t^{1/3}\}) \),

for some positive constant \( s \), and \( D \) is some positive constant.

\begin{proof}

For any \( u \in \mathbb{R} \),

\[
\Phi_t(u) = \mathbb{E} \left( \exp \{iuS_t\} \exp \{a_\beta L_t(\sigma_t)\} \right)
\]  

(27)

\[
= \exp \left\{ -\frac{iu\sqrt{\lambda_t}}{\sigma_c} \right\}
\]  

By the same method as in the proof of Lemma 2.2 in [7] by Bercu and Rouault, there exist two positive constants \( \tau \) and \( \kappa \) such that

\[
|\Phi_t(u)|^2 \leq \left( 1 + \frac{\tau u^2}{\lambda_t} \right)^{-(\kappa/2)\lambda_t},
\]  

(28)

therefore, \( \Phi_t(\cdot) \) belongs to \( L^2(\mathbb{R}) \), and by Parseval's formula, for some positive constant \( s \), let

\[
s_t = s \left( \log \left( \frac{T}{T - t} \right) \right)^{1/6},
\]  

(29)

we get

\[
B_t = \frac{1}{2\pi a_\beta^2} \int_{|u| \leq s_t} \left( 1 + \frac{iu}{a_\beta} \right) \Phi_t(u)\, du + \frac{1}{2\pi a_\beta^2} \int_{|u| > s_t} \left( 1 + \frac{iu}{a_\beta} \right) \Phi_t(u)\, du
\]  

(30)

\[
|D_t| = O(\exp \{-D\lambda_t^{1/3}\}),
\]  

(32)

where \( D \) is some positive constant.
\end{proof}
Proof of Lemma 6. By Lemma 3, we have
\[ \frac{L^{(k)}(\alpha_t)}{\lambda_t} = L^{(k)}(\alpha_t) + \frac{H^{(k)}(\alpha_t)}{\lambda_t} + O\left(\frac{\lambda_t^2(T-t)^2}{\lambda_t}\right). \] (33)
Noting that \( L'(\alpha_t) = 0 \), \( L''(\alpha_t) = \sigma_t^2 \) and
\[ \frac{L''(\alpha_t)}{2\beta_t} \frac{u}{\beta_t} = -\frac{u^2}{2}, \] (34)
for any \( p > 0 \), by Taylor expansion, we obtain
\[ \log \Phi_t(u) = -\frac{u^2}{2} + \lambda_t \sum_{k=0}^{2p+3} \frac{(iu)^k}{k!} L^{(k)}(\alpha_t) \]
\[ + \sum_{k=1}^{2p+1} \left( \frac{iu}{\beta_t} \right)^k H^{(k)}(\alpha_t) \]
\[ + O\left( \max\left(1, \left|u\right|^{2p+4}\right) \right); \] (35)
therefore, there exist integers \( q(p), r(p) \) and a sequence \( \varphi_{k,l} \) independent of \( p \); when \( t \) approaches \( T \), we get
\[ \Phi_t(u) = \exp \left\{ -\frac{u^2}{2} \left( 1 + \frac{1}{\sqrt{\lambda_t}} \sum_{k=0}^{2p} \sum_{l=0}^{q(p)} \varphi_{k,l} u^l \right) \right\} \]
\[ + O\left( \max\left(1, \left|u\right|^{r(p)} \right) \right) \] (36)
where \( O \) is uniform as soon as \( |u| \leq s \).

Finally, we get the proof of Lemma 6 by Lemma 7 together with standard calculations on the \( N(0,1) \) distribution.

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