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Research Article

Sharp Large Deviation for the Energy of α-Brownian Bridge

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We study the sharp large deviation for the energy of α-Brownian bridge. The full expansion of the tail probability for energy is obtained by the change of measure.

1. Introduction

We consider the following α-Brownian bridge:

$$dX_t = -\frac{\alpha}{T-t}X_t dt + dW_t, \quad X_0 = 0, \tag{1}$$

where W is a standard Brownian motion, $t \in [0,T)$, $T \in (0,\infty)$, and the constant $\alpha > 1/2$. Let P_α denote the probability distribution of the solution $\{X_t, t \in [0,T)\}$ of (1). The α-Brownian bridge is first used to study the arbitrage profit associated with a given future contract in the absence of transaction costs by Brennan and Schwartz [1].

α-Brownian bridge is a time inhomogeneous diffusion process which has been studied by Barczy and Pap [2, 3], Jiang and Zhao [4], and Zhao and Liu [5]. They studied the central limit theorem and the large deviations for parameter estimators and hypothesis testing problem of α-Brownian bridge. While the large deviation is not so helpful in some statistics problems since it only gives a logarithmic equivalent for the deviation probability, Bahadur and Ranga Rao [6] overcame this difficulty by the sharp large deviation principle for the empirical mean. Recently, the sharp large deviation principle is widely used in the study of Gaussian quadratic forms, Ornstein-Uhlenbeck model, and fractional Ornstein-Uhlenbeck (cf. Bercu and Rouault [7], Bercu et al. [8], and Bercu et al. [9, 10]).

In this paper we consider the sharp large deviation principle (SLDP) of energy S_t, where

$$S_t = \int_0^t \frac{X_s^2}{(s-T)^2} ds. \tag{2}$$

Our main results are the following.

Theorem 1. *Let $\{X_t, t \in [0,T)\}$ be the process given by the stochastic differential equation (1). Then $\{S_t/\lambda_t, t \in [0,T)\}$ satisfies the large deviation principle with speed λ_t and good rate function $I(\cdot)$ defined by the following:*

$$I(x) = \begin{cases} \dfrac{1}{8x}\left((2\alpha_0 - 1)x - 1\right)^2, & \text{if } x > 0; \\ +\infty, & \text{if } x \le 0, \end{cases} \tag{3}$$

where $\lambda_t = \log(T/(T-t))$.

Theorem 2. *$\{S_t/\lambda_t, t \in [0,T)\}$ satisfies SLDP; that is, for any $c > 1/(2\alpha - 1)$, there exists a sequence $b_{c,k}$ such that, for any $p > 0$, when t approaches T enough,*

$$P(S_t \ge c\lambda_t) = \frac{\exp\{-I(c)\lambda_t + H(a_c)\}}{\sqrt{2\pi}a_c\beta_t} \times \left(1 + \sum_{k=1}^{p} \frac{b_{c,k}}{\lambda_t} + O\left(\frac{1}{\lambda_t^{p+1}}\right)\right), \tag{4}$$

where

$$\sigma_c^2 = 4c^2, \qquad \beta_t = \sigma_c\sqrt{\lambda_t},$$
$$a_c = \frac{(1-2\alpha)^2c^2-1}{8c^2}, \tag{5}$$
$$H(a_c) = -\frac{1}{2}\log\left(\frac{1-(1-2\alpha)c}{2}\right).$$

The coefficients $b_{c,k}$ may be explicitly computed as functions of the derivatives of L and H (defined in Lemma 3) at point a_c. For example, $b_{c,1}$ is given by

$$b_{c,1} = \frac{1}{\sigma_c^2}\left(-\frac{h_2}{2}-\frac{h_1^2}{2}+\frac{l_4}{8\sigma_c^2}+\frac{l_3h_1}{2\sigma_c^2}\right.$$
$$\left.-\frac{5l_3^2}{24\sigma_c^4}+\frac{h_1}{a_c}-\frac{l_3}{2a_c\sigma_c^2}-\frac{1}{a_c^2}\right), \tag{6}$$

with $l_k = L^{(k)}(a_c)$, and $h_k = H^{(k)}(a_c)$.

2. Large Deviation for Energy

Given $\alpha > 1/2$, we first consider the following logarithmic moment generating function of S_t; that is,

$$L_t(u) := \log\mathbb{E}_\alpha\exp\left\{u\int_0^t\frac{X_s^2}{(s-T)^2}ds\right\}, \quad \forall\lambda\in\mathbb{R}. \tag{7}$$

And let

$$\mathscr{D}_{L_t} := \{u\in\mathbb{R},\ L_t(u) < +\infty\} \tag{8}$$

be the effective domain of L_t. By the same method as in Zhao and Liu [5], we have the following lemma.

Lemma 3. *Let $\mathscr{D}_L$ be the effective domain of the limit L of L_t; then for all $u\in\mathscr{D}_L$, one has*

$$\frac{L_t(u)}{\lambda_t} = L(u)+\frac{H(u)}{\lambda_t}+\frac{R(u)}{\lambda_t}, \tag{9}$$

with

$$L(u) = -\frac{1-2\alpha-\varphi(u)}{4},$$
$$H(\lambda) = -\frac{1}{2}\log\left\{\frac{1}{2}(1+h(u))\right\}, \tag{10}$$
$$R(u) = -\frac{1}{2}\log\left\{1+\frac{1-h(u)}{1+h(u)}\exp\{2\varphi(u)\lambda_t\}\right\},$$

where $\varphi(u) = -\sqrt{(1-2\alpha)^2-8u}$ and $h(u) = (1-2\alpha)/\varphi(u)$. Furthermore, the remainder $R(u)$ satisfies

$$R(u) = O_{t\to T}\left(\exp\{2\varphi(u)\lambda_t\}\right). \tag{11}$$

Proof. By Itô's formula and Girsanov's formula (see Jacob and Shiryaev [11]), for all $u\in\mathscr{D}_L$ and $t\in[0,T)$,

$$\log\frac{dP_\alpha}{dP_\beta}\Big|_{[0,t]}$$
$$= (\alpha-\beta)\int_0^t\frac{X_s}{s-T}dX_s-\frac{\alpha^2-\beta^2}{2}\int_0^t\frac{X_s^2}{(s-T)^2}ds, \tag{12}$$
$$\int_0^t\frac{X_s}{s-T}dX_s$$
$$= \frac{1}{2}\left(\frac{X_t^2}{(t-T)}+\int_0^t\frac{X_s^2}{(s-T)^2}ds-\log\left(1-\frac{t}{T}\right)\right).$$

Therefore,

$$L_t(u) = \log\mathbb{E}_\beta\left(\exp\left\{u\int_0^t\frac{X_s^2}{(s-T)^2}ds\right\}\frac{dP_\alpha}{dP_\beta}\Big|_{[0,t]}\right)$$
$$= \log\mathbb{E}_\beta\exp\left\{\frac{\alpha-\beta}{2(t-T)}X_t^2-\frac{\alpha-\beta}{2}\log\left(1-\frac{t}{T}\right)\right.$$
$$+\frac{1}{2}\left(\beta^2-\alpha^2+\alpha-\beta+2u\right)$$
$$\left.\times\int_0^t\frac{X_s^2}{(s-T)^2}ds\right\}. \tag{13}$$

If $4u\le(1-2\alpha)^2$, we can choose β such that $(\beta-1/2)^2-(\alpha-1/2)^2+2u=0$. Then

$$L_t(u) = -\frac{1-2\alpha-\varphi(\lambda)}{4}\lambda_t$$
$$-\frac{1}{2}\log\left\{\frac{1}{2}(1+h(u))\right\} \tag{14}$$
$$-\frac{1}{2}\log\left\{1+\frac{1-h(u)}{1+h(u)}\exp\{2\varphi(u)\lambda_t\}\right\},$$

where $\varphi(u) = -\sqrt{(1-2\alpha)^2-8u}$, and $h(u) = (1-2\alpha)/\varphi(u)$. Therefore,

$$\frac{L_t(u)}{\lambda_t} = -\frac{1-2\alpha-\varphi(u)}{4}$$
$$-\frac{1}{2\lambda_t}\log\left\{\frac{1}{2}(1+h(u))\right\}$$
$$-\frac{1}{2\lambda_t}\log\left\{1+\frac{1-h(u)}{1+h(u)}\exp\{2\varphi(u)\lambda_t\}\right\} \tag{15}$$
$$= L(u)+\frac{H(u)}{\lambda_t}+\frac{R(u)}{\lambda_t}.$$

□

Proof of Theorem 1. From Lemma 3, we have

$$L(u) = \lim_{t\to T}\frac{L_t(u)}{\lambda_t} = \frac{1-2\alpha-\varphi(u)}{4}, \tag{16}$$

and $L(\cdot)$ is steep; by the Gärtner-Ellis theorem (Dembo and Zeitouni [12]), S_t/λ_t satisfies the large deviation principle with speed λ_t and good rate function $I(\cdot)$ defined by the following:

$$I(x) = \begin{cases} \frac{1}{8x}((2\alpha-1)x-1)^2, & \text{if } x > 0; \\ +\infty, & \text{if } +x \le 0. \end{cases} \tag{17}$$

□

Remark 4. Theorem 1 can also be obtained by using Theorem 1 in Zhao and Liu [5].

3. Sharp Large Deviation for Energy

For $c > 1/(2\alpha-1)$, let

$$a_c = \frac{(1-2\alpha)^2c^2-1}{8c^2}, \qquad \sigma_c^2 = L''(a_c) = 4c^3,$$
$$H(a_c) = -\frac{1}{2}\log(1-(1-2\alpha)c). \tag{18}$$

Then

$$\begin{aligned} &P(S_t \ge c\lambda_t) \\ &= \int_{S_t \ge c\lambda_t} \exp\{L(a_c) - ca_c\lambda_t + ca_c\lambda_t - a_cS_t\}\, dQ_t \\ &= \exp\{L(a_c) - ca_c\lambda_t\}\, \mathbb{E}_Q \exp\left\{-a_c\beta_tU_tI_{\{U_t\ge 0\}}\right\} = A_tB_t, \end{aligned} \tag{19}$$

where $\mathbb{E}_Q$ is the expectation after the change of measure

$$\frac{dQ_t}{dP} = \exp\{a_cS_t - L_t(a_c)\},$$
$$U_t = \frac{S_t - c\lambda_t}{\beta_t}, \qquad \beta_t = \sigma_c\sqrt{\lambda_t}. \tag{20}$$

By Lemma 3, we have the following expression of A_t.

Lemma 5. *For all $c > 1/(2\alpha-1)$, when t approaches T enough,*

$$A_t = \exp\{-I(c)\lambda_t + H(a_c)\}(1 + O((T-t)^c)). \tag{21}$$

For B_t, one gets the following.

Lemma 6. *For all $c > 1/(2\alpha-1)$, the distribution of U_t under Q_t converges to $N(0,1)$ distribution. Furthermore, there exists a sequence ψ_k such that, for any $p > 0$ when t approaches T enough,*

$$B_t = \frac{1}{a_c\sigma_c\sqrt{2\pi\lambda_t}}\left(1 + \sum_{k=1}^{p}\frac{\psi_k}{\lambda_t^k} + O\left(\lambda_t^{-(p+1)}\right)\right). \tag{22}$$

Proof of Theorem 2. The theorem follows from Lemma 5 and Lemma 6. □

It only remains to prove Lemma 6. Let $\Phi_t(\cdot)$ be the characteristic function of U_t under Q_t; then we have the following.

Lemma 7. *When t approaches T, Φ_t belongs to $L^2(\mathbb{R})$ and, for all $u \in \mathbb{R}$,*

$$\begin{aligned} \Phi_t(u) = &\exp\left\{-\frac{iu\sqrt{\lambda_t}c}{\sigma_c}\right\} \\ &\times \exp\left\{\left(L_t\left(a_c + \frac{iu}{\beta_t}\right) - L_t(a_c)\right)\right\}. \end{aligned} \tag{23}$$

Moreover,

$$B_t = \mathbb{E}_Q \exp\left\{-a_c\beta_tU_tI_{\{U_t\ge 0\}}\right\} = C_t + D_t, \tag{24}$$

with

$$\begin{aligned} C_t &= \frac{1}{2\pi a_c\beta_t}\int_{|u|\le s_t}\left(1 + \frac{iu}{a_c\beta_t}\right)^{-1}\Phi_t(u)\, du, \\ D_t &= \frac{1}{2\pi a_c\beta_t}\int_{|u|> s_t}\left(1 + \frac{iu}{a_c\beta_t}\right)^{-1}\Phi_t(u)\, du, \\ |D_t| &= O\left(\exp\left\{-D\lambda_t^{1/3}\right\}\right), \end{aligned} \tag{25}$$

where

$$s_t = s\left(\log\left(\frac{T}{T-t}\right)\right)^{1/6}, \tag{26}$$

for some positive constant s, and D is some positive constant.

Proof. For any $u \in \mathbb{R}$,

$$\begin{aligned} \Phi_t(u) &= \mathbb{E}\left(\exp\{iuU_t\}\exp\{a_cS_t - L_t(a_c)\}\right) \\ &= \exp\left\{-\frac{iu\sqrt{\lambda_t}c}{\sigma_c}\right\} \\ &\quad\times \exp\left\{\left(L_t\left(a_c + \frac{iu}{\beta_t}\right) - L_t(a_c)\right)\right\}. \end{aligned} \tag{27}$$

By the same method as in the proof of Lemma 2.2 in [7] by Bercu and Rouault, there exist two positive constants τ and κ such that

$$|\Phi_t(u)|^2 \le \left(1 + \frac{\tau u^2}{\lambda_t}\right)^{-(\kappa/2)\lambda_t}; \tag{28}$$

therefore, $\Phi_t(\cdot)$ belongs to $L^2(\mathbb{R})$, and by Parseval's formula, for some positive constant s, let

$$s_t = s\left(\log\left(\frac{T}{T-t}\right)\right)^{1/6}; \tag{29}$$

we get

$$\begin{aligned} B_t = &\frac{1}{2\pi a_c\beta_t}\int_{|u|\le s_t}\left(1 + \frac{iu}{a_c\beta_t}\right)^{-1}\Phi_t(u)\, du + \frac{1}{2\pi a_c\beta_t} \\ &\times \int_{|u|> s_t}\left(1 + \frac{iu}{a_c\beta_t}\right)^{-1}\Phi_t(u)\, du \end{aligned} \tag{30}$$

$$= :C_t + D_t, \tag{31}$$

$$|D_t| = O\left(\exp\left\{-D\lambda_t^{1/3}\right\}\right), \tag{32}$$

where D is some positive constant. □

Proof of Lemma 6. By Lemma 3, we have

$$\frac{L_t^{(k)}(a_c)}{\lambda_t} = L^{(k)}(a_c) + \frac{H^{(k)}(a_c)}{\lambda_t} + \frac{O\left(\lambda_t^k (T-t)^{-2c}\right)}{\lambda_t}. \quad (33)$$

Noting that $L'(a_c) = 0$, $L''(a_c) = \sigma_c^2$ and

$$\frac{L''(a_c)}{2}\left(\frac{iu}{\beta_t}\right)^2 \lambda_t = -\frac{u^2}{2}, \quad (34)$$

for any $p > 0$, by Taylor expansion, we obtain

$$\begin{aligned}\log \Phi_t(u) = & -\frac{u^2}{2} + \lambda_t \sum_{k=3}^{2p+3}\left(\frac{iu}{\beta_t}\right)^k \frac{L^{(k)}(a_c)}{k!} \\ & + \sum_{k=1}^{2p+1}\left(\frac{iu}{\beta_t}\right)^k \frac{H^{(k)}(a_c)}{k!} \\ & + O\left(\frac{\max\left(1, |u|^{2p+4}\right)}{\lambda_t^{p+1}}\right);\end{aligned} \quad (35)$$

therefore, there exist integers $q(p)$, $r(p)$ and a sequence $\varphi_{k,l}$ independent of p; when t approaches T, we get

$$\begin{aligned}\Phi_t(u) = & \exp\left\{-\frac{u^2}{2}\right\}\left(1 + \frac{1}{\sqrt{\lambda_t}}\sum_{k=0}^{2p}\sum_{l=k+1}^{q(p)}\frac{\varphi_{k,l}u^l}{\lambda_t^{k/2}}\right. \\ & \left. + O\left(\frac{\max\left(1, |u|^{r(p)}\right)}{\lambda_t^{p+1}}\right)\right),\end{aligned} \quad (36)$$

where O is uniform as soon as $|u| \le s_t$.

Finally, we get the proof of Lemma 6 by Lemma 7 together with standard calculations on the $N(0,1)$ distribution. □

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