Research Article

Maximum Likelihood Estimator of AUC for a Bi-Exponentiated Weibull Model

Fazhe Chang¹ and Lianfen Qian^{1,2}

¹ Department of Mathematical Sciences, Florida Atlantic University, Boca Raton, FL 33431, USA

² College of Mathematics and Information Science, Wenzhou University, Zhejiang 325035, China

Correspondence should be addressed to Lianfen Qian; lqian@fau.edu

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For a Bi-Exponentiated Weibull model, the authors obtain a general AUC formula and derive the maximum likelihood estimator of AUC and its asymptotic property. A simulation study is carried out to illustrate the finite sample size performance.

1. ROC Curve and AUC for a Bi-Exponentiated Weibull Model

In medical science, a diagnostic test result called a biomarker [1, 2] is an indicator for disease status of patients. The accuracy of a medical diagnostic test is typically evaluated by sensitivity and specificity. Receiver Operating Characteristic (ROC) curve is a graphical representation of the relationship between sensitivity and specificity. The area under the ROC curve (AUC) is an overall performance measure for the biomarker. Hence the main issue in assessing the accuracy of a diagnostic test is to estimate the ROC curve and its AUC.

Suppose that there are two groups of study subjects: diseased and nondiseased. Let T be a continuous biomarker. Assume that the larger T is, the more likely a subject is diseased. That is, a subject is classified as positive or diseased status if $T > c$ and as negative or nondiseased status otherwise, where c is a cutoff point. Let D be the disease status: $D = 1$ represents diseased population while $D = 0$ represents nondiseased population. Sensitivity of T is defined as the probability of being correctly classified as disease status and specificity as the probability of being correctly classified as nondisease status. That is,

$$\begin{aligned} \text{sensitivity}(c) &= P(T > c \mid D = 1), \\ \text{specificity}(c) &= P(T < c \mid D = 0). \end{aligned} \quad (1)$$

Let $X = \{T \mid D = 1\}$ and $Y = \{T \mid D = 0\}$ be the biomarkers for diseased and nondiseased subjects with survival functions S_1 and S_0 , respectively. Then

$$\begin{aligned} \text{sensitivity}(c) &= S_1(c), \\ \text{specificity}(c) &= 1 - S_0(c). \end{aligned} \quad (2)$$

The ROC function (curve) is defined as

$$\text{ROC}(t) = S_1(S_0^{-1}(t)), \quad \text{where } t = S_0(c). \quad (3)$$

Notice that ROC curve is a monotonic increasing curve in a unit square starting at $(0, 0)$ and ending at $(1, 1)$. A good biomarker has a very concave down ROC curve.

We say a bivariate random vector (X, Y) is a Bi-model if X and Y are independent and are from the same distribution family but with different parameters. Pepe [3] summarizes ROC analysis and notices that among Bi-model, Bi-normal model has been the most popular one. The ROC curve method traces back to Green and Swets [4] in the signal detection theory. Bi-normal model assumes the existence of a monotonic increasing transformation such that the biomarker can be transformed into normally distributed for both diseased and nondiseased patients. Pepe [3, Results 4.1 and 4.4] shows that the ROC curve of a biomarker is invariant under a monotone increasing transformation. Hence, a good estimator of ROC curve should satisfy this invariance property.

The literature for ROC analysis under Bi-normal model is extensive and the majority of the results are for a continuous biomarker. For a continuous biomarker, Fushing and Turnbull [5] obtained the estimators and their asymptotic properties of the estimated ROC curve and AUC using the empirical based nonparametric and semiparametric approaches for Bi-normal model. Metz et al. [6] studied properties of maximum likelihood estimator via discretization over finite support points. Zou and Hall [7] developed an MLE algorithm to estimate ROC curve under an unspecific or a specific monotonic transformation. Gu et al. [8] introduced a nonparametric method based on the Bayesian bootstrap technique to estimate ROC curve and compared the new method with several different semiparametric and nonparametric methods. Pepe and Cai [9] considered the ROC curve as the distribution of placement value and then estimated the ROC curve by maximizing the pseudolikelihood function of the estimated placement values. All the above methods require complicated computations. Zhou and Lin [10] proposed a semi-parametric maximum likelihood (ML) estimator of ROC which satisfies the property of invariance. Their method largely reduced the number of nuisance parameters and therefore the computational complexity.

In reality, the distribution of the biomarker may be skewed and the normality assumption is not reasonable. Under this situation, other models such as generalized exponential (GE), Weibull (WE), and gamma, to name a few, provide a reasonable alternative; see Faraggi et al. [11]. Another issue is that in the presence of two different biomarkers, the one with uniformly higher ROC curve is better. However, when two ROC curves cross, we cannot simply compare and find which one is better. Many different numerical indices have been proposed to summarize and compare ROC curves of different biomarkers [12–15]. The most popular summary index is AUC, defined as

$$\text{AUC} = \int_0^1 \text{ROC}(t) dt = P(X > Y). \quad (4)$$

The estimation of $\text{AUC} = P(X > Y)$ is also of great importance in engineering reliability, typically, in stress-strength model. For example, if X is the strength of a system which is subject to a stress Y , then $P(X > Y)$ is a measure of a system performance. The system fails if and only if at any time the applied stress is greater than its strength. During the past twenty years, many results have been obtained on the estimation of the probability under the different parametric models. Kotz et al. [16] give a comprehensive summarization about stress-strength model.

Many research results have been obtained for the estimation of AUC under popular survival models: Bi-exponential model [17], Bi-normal model [5, 18], Bi-gamma model [19], Bi-Burr type X [20–23], Bi-generalized exponential [24], Bi-Weibull model [25]. However, all the popular survival models such as generalized exponential (GE), Weibull (WE), and Gamma do not allow nonmonotonic failure rates which often occur in real practice. Furthermore, the common nonmonotonic failure rate in engineering and biological science involves bathtub shapes.

To overcome the aforementioned drawback, many models such as generalized gamma [26], generalized F distribution [27], two families [28], a four-parameter family [29], and a three-parameter (IDB) family [30] have been proposed for modeling nonmonotone failure-rate data. Mudholkar and Srivasta [31] proposed an exponentiated Weibull (EW) model, an extension of both GE and WE models. Mudholkar and Hutson [32] studied some properties for the model. EW model not only includes distributions with unimodal and bathtub failure rates but also allows for a broader class of monotonic hazard rates. EW model may fit better than GE for some actual data. For instance, Khedhairi et al. [33] conclude that generalized Rayleigh model (a special case of EW model) is the best among the three models compared: exponential, generalized exponential, and generalized Rayleigh model.

In the research of estimation of $P(X > Y)$ under the different survival models, the most recent result in Baklizi [34] focuses the two-parameter Weibull model, which is a special case of our Bi-EW model. Baklizi assumes a common shape parameter and different scale parameter which is also studied in Kundu and Gupta [25]. In contrast to their studies, here we study the estimation of $P(X > Y)$ under the Bi-EW model which shares one shape parameter only. Since EW includes Weibull as a submodel, our results extend the ones in Baklizi [34] and Kundu and Gupta [25], as shown in Theorem 1. On the other hand, Baklizi [34] also studied Bayes estimator, which we have not studied. It will be a future project.

In this paper, we adopt AUC as a main summary index of ROC when comparing biomarkers under a Bi-Exponentiated Weibull (Bi-EW) model. We obtain a precise formula for the AUC and derive the MLE of AUC and its asymptotic distribution. The remaining of this paper is organized as follows. Section 2 introduces the Bi-EW model and obtains the theoretical AUC formula. Section 3 studies the maximum likelihood estimation of AUC under the Bi-EW model and derives the asymptotic normality of the estimated AUC. Section 4 reports the simulation results to compare the accuracy of the estimated parameters and estimated AUC for different sample sizes and various parameter settings in terms of absolute relative biases (ARB) and square root of mean square error (RMSE). The conclusion is given in Section 5.

2. A Bi-EW Model and Its AUC

A random variable X has an Exponentiated Weibull (EW) distribution if its distribution function is defined as

$$F(x; \alpha, \beta, \lambda) = \left(1 - e^{-\lambda x^\beta}\right)^\alpha, \quad x > 0, \alpha > 0, \beta > 0, \lambda > 0. \quad (5)$$

Denote $X \sim \text{EW}(\alpha, \beta, \lambda)$. Mudholkar and Srivasta [31] and Qian [35] show that this distribution allows both monotonic and bathtub shaped hazard rates. The latter shape is useful in reality. In this section, we first derive a general moment formula for the distribution and then obtain AUC formula for a Bi-EW model with a common shape parameter.

2.1. General Moment Formula of Exponentiated Weibull. To fully understand a new family of distributions, it is important to derive its summary properties such as moment formulas.

Theorem 1. Let $X \sim EW(\alpha, \beta, \lambda)$. One has

$$\begin{aligned} E[X^{i\beta}(\ln X)^j] &= (-1)^j \frac{\alpha}{\lambda^i \beta^j} \sum_{l=0}^j \binom{j}{l} \Gamma^{(l)}(i+1) \\ &\quad \times \sum_{k=0}^{\infty} (-1)^k \binom{\alpha-1}{k} \frac{[\ln \lambda(k+1)]^{j-l}}{(k+1)^{i+1}}, \end{aligned} \quad (6)$$

where $i, j = 0, 1, 2, 3, \dots$ and $\Gamma(i) = \int_0^{\infty} e^{-t} t^{i-1} dt$.

Proof. By the definition of the EW model, we have

$$\begin{aligned} E[X^{i\beta}(\ln X)^j] &= \alpha \lambda \beta \int_0^{\infty} x^{i\beta} (\ln x)^j (1 - e^{-\lambda x^\beta})^{\alpha-1} \\ &\quad \times e^{-\lambda x^\beta} x^{\beta-1} dx \\ &= \alpha \lambda \beta \sum_{k=0}^{\infty} (-1)^k \binom{\alpha-1}{k} \\ &\quad \times \int_0^{\infty} x^{(i+1)\beta-1} e^{-\lambda(k+1)x^\beta} (\ln x)^j dx. \end{aligned} \quad (7)$$

By change of variables $t = \lambda(k+1)x^\beta$, we have

$$dx = \frac{1}{\beta} \frac{t^{(1/\beta)-1}}{[\lambda(k+1)]^{1/\beta}} dt, \quad x^{(i+1)\beta-1} = \frac{t^{i+1-1/\beta}}{[\lambda(k+1)]^{i+1-1/\beta}}. \quad (8)$$

Then the integral part in (7) is equal to

$$\begin{aligned} &\int_0^{\infty} e^{-t} \left(\frac{\ln t - \ln \lambda(k+1)}{\beta} \right)^j \frac{1}{\beta} \frac{t^i}{[\lambda(k+1)]^{i+1}} dt \\ &= \frac{1}{\beta^{j+1} [\lambda(k+1)]^{i+1}} \int_0^{\infty} e^{-t} t^i [\ln t - \ln \lambda(k+1)]^j dt \\ &= \frac{1}{\beta^{j+1} [\lambda(k+1)]^{i+1}} \sum_{l=0}^j \binom{j}{l} [-\ln \lambda(k+1)]^{j-l} \\ &\quad \times \int_0^{\infty} e^{-t} t^i (\ln t)^l dt \\ &= \frac{1}{\beta^{j+1} [\lambda(k+1)]^{i+1}} \sum_{l=0}^j \binom{j}{l} [-\ln \lambda(k+1)]^{j-l} \Gamma^{(l)}(i+1). \end{aligned} \quad (9)$$

Therefore, we have

$$\begin{aligned} E[X^{i\beta}(\ln X)^j] &= \alpha \lambda \beta \sum_{k=0}^{\infty} (-1)^k \binom{\alpha-1}{k} \frac{1}{\beta^{j+1} [\lambda(k+1)]^{i+1}} \\ &\quad \times \sum_{l=0}^j \binom{j}{l} [-\ln \lambda(k+1)]^{j-l} \Gamma^{(l)}(i+1) \\ &= (-1)^j \frac{\alpha}{\lambda^i \beta^j} \sum_{l=0}^j (-1)^l \binom{j}{l} \Gamma^{(l)}(i+1) \\ &\quad \times \sum_{k=0}^{\infty} \binom{\alpha-1}{k} \frac{[\ln \lambda(k+1)]^{j-l}}{(k+1)^{i+1}}. \end{aligned} \quad (10)$$

□

Remark 2. Theorem 1 covers the following results as special cases.

Case 1. If $j = 0$, and $i\beta = k$, then it reduces to the formula in Choudhury [36], the simple moments of EW family. One has

$$\begin{aligned} \mu_k = E(X^k) &= \alpha \lambda^{-k/\beta} \Gamma\left(\frac{k}{\beta} + 1\right) \\ &\quad \times \sum_{i=0}^{\alpha-1} (-1)^i \binom{\alpha-1}{i} \frac{1}{(i+1)^{k/(\beta+1)}}. \end{aligned} \quad (11)$$

Moreover, if $k\beta^{-1} = r$ is positive integer, then

$$\mu_k = \alpha(-\lambda)^{-r} \left[\frac{d^r}{ds^r} B(s, \alpha) \right] \Big|_{s=1}, \quad (12)$$

where $B(s, \alpha) = \int_0^1 x^{s-1} (1-x)^{\alpha-1} dx$.

Case 2. If $j = 0$, and $\beta = 1$, it reduces to the moments of Generalized exponential family in Kunda and Gupta [24].

Case 3. If $j = 0$, $\alpha = 1$, and $i\beta = n$, it reduces to the moments of Weibull family in Kunda and Gupta [25].

2.2. AUC under a Bi-EW Model. In this subsection, we derive the general formula of AUC under a Bi-EW model with a common shape parameter. That is, we assume that X and Y are independent and with the following EW distribution functions:

$$\begin{aligned} F_1(x; \alpha_1, \beta, \lambda_1) &= \left(1 - e^{-\lambda_1 x^\beta}\right)^{\alpha_1}, \quad x > 0, \alpha_1 > 0, \beta > 0, \\ &\quad \lambda_1 > 0, \\ F_0(y; \alpha_2, \beta, \lambda_2) &= \left(1 - e^{-\lambda_2 y^\beta}\right)^{\alpha_2}, \quad y > 0, \alpha_2 > 0, \beta > 0, \\ &\quad \lambda_2 > 0, \end{aligned} \quad (13)$$

where the two distribution functions share a common shape parameter β .

Theorem 3. Suppose that (X, Y) follows the Bi-EW model with the above distribution functions. Then

$$AUC = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \binom{\alpha_2}{i} \binom{\alpha_1}{j} (-1)^{i+j-1} \frac{\lambda_1 j}{\lambda_1 j + \lambda_2 i}. \quad (14)$$

Proof. By Taylor expansion, we have

$$\begin{aligned} (1 - e^{-\lambda_2 x^\beta})^{\alpha_2} &= \sum_{i=0}^{\infty} \binom{\alpha_2}{i} (-e^{-\lambda_2 x^\beta})^i, \\ (1 - e^{-\lambda_1 x^\beta})^{\alpha_1} &= \sum_{j=0}^{\infty} \binom{\alpha_1}{j} (-e^{-\lambda_1 x^\beta})^j. \end{aligned} \quad (15)$$

Thus,

$$\begin{aligned} AUC &= \int_0^{\infty} F_0(x; \alpha_2, \beta, \lambda_2) dF_1(x; \alpha_1, \beta, \lambda_1) \\ &= \int_0^{\infty} \sum_{i=0}^{\infty} \binom{\alpha_2}{i} (-e^{-\lambda_2 x^\beta})^i \\ &\quad \times \sum_{j=1}^{\infty} \binom{\alpha_1}{j} j (-e^{-\lambda_1 x^\beta})^{j-1} e^{-\lambda_1 x^\beta} \lambda_1 \beta x^{\beta-1} dx \\ &= \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \int_0^{\infty} \binom{\alpha_2}{i} \binom{\alpha_1}{j} j (-1)^{i+j-1} \lambda_1 e^{-(\lambda_2 i + \lambda_1 j)x^\beta} dx^\beta \\ &= \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \binom{\alpha_2}{i} \binom{\alpha_1}{j} (-1)^{i+j-1} \lambda_1 j \int_0^{\infty} e^{-(\lambda_2 i + \lambda_1 j)x^\beta} dx^\beta \\ &= \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \binom{\alpha_2}{i} \binom{\alpha_1}{j} (-1)^{i+j-1} \frac{\lambda_1 j}{\lambda_1 j + \lambda_2 i}. \end{aligned} \quad (16)$$

□

Remark 4. (i) Note that the exact expression of AUC is independent of the common parameter β . This is similar to the cases in Bi-WE and Bi-GE models; see Kundu and Gupta [24, 25].

(ii) When $\alpha_1 = \alpha_2 = 1$, (14) reduces to the AUC for a Bi-GE model.

(iii) When $\lambda_1 = \lambda_2$, (14) reduces to the AUC for a Bi-WE model. To see (iii), we actually prove the following corollary.

Corollary 5. For $\alpha_1 > 0$ and $\alpha_2 > 0$, one has

$$\sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \binom{\alpha_2}{i} \binom{\alpha_1}{j} (-1)^{i+j-1} \frac{j}{j+i} = \frac{\alpha_1}{\alpha_1 + \alpha_2}. \quad (17)$$

Proof. Denote

$$g(s) = \frac{\partial}{\partial s} \int_0^1 \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \binom{\alpha_2}{i} \binom{\alpha_1}{j} (-1)^{i+j-1} t^{i+j-1} s^j dt. \quad (18)$$

Then, on one hand we have

$$\begin{aligned} g(s) &= \frac{\partial}{\partial s} \int_0^1 \sum_{i=0}^{\infty} \binom{\alpha_2}{i} (-1)^i t^{i-1} \sum_{j=1}^{\infty} \binom{\alpha_1}{j} (-1)^{j-1} (ts)^j dt \\ &= \frac{\partial}{\partial s} \int_0^1 \frac{1}{t(1-t)^{\alpha_2} (1-(1-ts)^{\alpha_1})} dt \\ &= \int_0^1 \frac{1}{t} (1-t)^{\alpha_2} (1-ts)^{\alpha_1-1} \alpha_1 t dt \\ &= \alpha_1 \int_0^1 (1-t)^{\alpha_2} (1-ts)^{\alpha_1-1} dt. \end{aligned} \quad (19)$$

On the other hand, we have

$$\begin{aligned} g(s) &= \frac{\partial}{\partial s} \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \binom{\alpha_2}{i} \binom{\alpha_1}{j} (-1)^{i+j-1} s^j \frac{t^{i+j}}{i+j} \\ &= \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \binom{\alpha_2}{i} \binom{\alpha_1}{j} (-1)^{i+j-1} \frac{j}{i+j} s^{j-1} t^{i+j}. \end{aligned} \quad (20)$$

Let $s = 1$; then (19) implies that

$$g(1) = \frac{\alpha_1}{\alpha_1 + \alpha_2}, \quad (21)$$

while (20) implies that

$$g(1) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \binom{\alpha_2}{i} \binom{\alpha_1}{j} (-1)^{i+j-1} \frac{j}{i+j}. \quad (22)$$

This completes the proof of the corollary. □

3. Maximum Likelihood Estimator of AUC and Its Asymptotic Property under the Bi-EW Model

To estimate AUC under the Bi-EW model, we adopt the plug in method. That is, we first obtain the maximum likelihood estimator of the model parameters and then plug it into (14) to get the MLE of AUC.

3.1. Maximum Likelihood Estimator of AUC under the Bi-EW Model. Let $X_1, X_2, \dots, X_m$ and $Y_1, Y_2, \dots, Y_n$ be independent random samples from $X \sim F_1$ and $Y \sim F_0$, respectively.

Denote the model parameter vector by $\theta = (\theta_1, \dots, \theta_5) = (\alpha_1, \alpha_2, \beta, \lambda_1, \lambda_2)$. Then the log-likelihood function is

$$\begin{aligned} l(\theta) = & m \ln \alpha_1 + m \ln \lambda_1 + m \ln \beta + n \ln \alpha_2 + n \ln \lambda_2 \\ & + n \ln \beta + (\alpha_1 - 1) \sum_{i=1}^m \ln \left(1 - e^{-\lambda_1 x_i^\beta} \right) \\ & + (\alpha_2 - 1) \sum_{j=1}^n \ln \left(1 - e^{-\lambda_2 y_j^\beta} \right) \\ & - \sum_{i=1}^m \lambda_1 x_i^\beta + (\beta - 1) \sum_{i=1}^m \ln x_i \\ & - \sum_{j=1}^n \lambda_2 y_j^\beta + (\beta - 1) \sum_{j=1}^n \ln y_j. \end{aligned} \quad (23)$$

We take derivatives with respect to the parameters to get the following score equations:

$$\begin{aligned} \frac{\partial l}{\partial \alpha_1} &= \frac{m}{\alpha_1} + \sum_{i=1}^m \ln \left(1 - e^{-\lambda_1 x_i^\beta} \right) = 0, \\ \frac{\partial l}{\partial \alpha_2} &= \frac{n}{\alpha_2} + \sum_{j=1}^n \ln \left(1 - e^{-\lambda_2 y_j^\beta} \right) = 0, \\ \frac{\partial l}{\partial \lambda_1} &= \frac{m}{\lambda_1} - \sum_{i=1}^m x_i^\beta + (\alpha_1 - 1) \sum_{i=1}^m \frac{e^{-\lambda_1 x_i^\beta} x_i^\beta}{1 - e^{-\lambda_1 x_i^\beta}} = 0, \\ \frac{\partial l}{\partial \lambda_2} &= \frac{n}{\lambda_2} - \sum_{j=1}^n y_j^\beta + (\alpha_2 - 1) \sum_{j=1}^n \frac{e^{-\lambda_2 y_j^\beta} y_j^\beta}{1 - e^{-\lambda_2 y_j^\beta}} = 0, \\ \frac{\partial l}{\partial \beta} &= \frac{m+n}{\beta} - \lambda_1 \sum_{i=1}^m x_i^\beta \ln x_i - \lambda_2 \sum_{j=1}^n y_j^\beta \ln y_j \\ &+ \sum_{i=1}^m \ln x_i + \sum_{j=1}^n \ln y_j \\ &+ (\alpha_1 - 1) \sum_{i=1}^m \frac{e^{-\lambda_1 x_i^\beta} \lambda_1 x_i^\beta \ln x_i}{1 - e^{-\lambda_1 x_i^\beta}} \\ &+ (\alpha_2 - 1) \sum_{j=1}^n \frac{e^{-\lambda_2 y_j^\beta} \lambda_2 y_j^\beta \ln y_j}{1 - e^{-\lambda_2 y_j^\beta}} = 0. \end{aligned} \quad (24)$$

The MLE $\hat{\theta}$ of θ is the numerical solution of the above score equations. Plugging $\hat{\theta}$ into (14), we obtain the maximum likelihood estimator of AUC as below:

$$\widehat{\text{AUC}} = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \binom{\hat{\alpha}_2}{i} \binom{\hat{\alpha}_1}{j} (-1)^{i+j-1} \frac{\hat{\lambda}_1 j}{\hat{\lambda}_1 j + \hat{\lambda}_2 i}. \quad (25)$$

3.2. Asymptotic Normality of the Estimated AUC. In this section, we obtain the asymptotic distribution of $\hat{\theta} = (\hat{\alpha}_1, \hat{\alpha}_2,$

$\hat{\beta}, \hat{\lambda}_1, \hat{\lambda}_2)$ and hence, by the continuous mapping theorem, the asymptotic distribution of the estimated AUC. We need to introduce some notation. Let

$$\begin{aligned} \mu_{j,k}^X &= E \left(X^{\beta(j+1)} (\ln X)^k \right), \\ \mu_{j,k}^Y &= E \left(Y^{\beta(j+1)} (\ln Y)^k \right), \quad \text{for } j, k = 0, 1, 2, \dots, \\ \mathbf{A}(\theta) &= \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} & 0 \\ 0 & a_{22} & a_{23} & 0 & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & 0 & a_{43} & a_{44} & 0 \\ 0 & a_{52} & a_{53} & 0 & a_{55} \end{pmatrix}. \end{aligned} \quad (26)$$

Then

$$\begin{aligned} a_{11} &= \frac{1}{\alpha_1^2}, & a_{22} &= \frac{1}{\alpha_2^2}, \\ a_{13} &= a_{31} = -\lambda_1 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-\lambda_1 (1+i))^j}{j!} \mu_{j,1}^Y, \\ a_{23} &= a_{32} = -\frac{1}{\sqrt{p}} \lambda_2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-\lambda_2 (1+i))^j}{j!} \mu_{j,1}^Y, \\ a_{25} &= a_{52} = -\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-\lambda_2 (1+i))^j}{j!} \mu_{j,0}^Y, \\ a_{33} &= \lambda_1 \mu_{0,2}^X + \frac{1}{\sqrt{p}} \lambda_2 \mu_{0,2}^Y + \frac{1+p}{\sqrt{p} \beta^2} \\ &\quad - \lambda_1 (\alpha_1 - 1) \\ &\quad \times \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(1 - \lambda_1 (1+i))}{j!} (-\lambda_1 (1+i))^j \mu_{j,2}^X \\ &\quad - \frac{1}{\sqrt{p}} \lambda_2 (\alpha_2 - 1) \\ &\quad \times \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-\lambda_1 (1+i))}{j!} (1 - \lambda_2 (1+i))^j \mu_{j,2}^Y, \\ a_{34} &= a_{43} = \mu_{0,1}^X - (\alpha_1 - 1) \\ &\quad \times \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(1 - \lambda_1 (1+i))}{j!} (-\lambda_1 (1+i))^j \mu_{j,1}^X, \\ a_{35} &= a_{53} = \frac{1}{\sqrt{p}} \mu_{0,1}^Y - \frac{1}{\sqrt{p}} (\alpha_2 - 1) \\ &\quad \times \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(1 - \lambda_2 (1+i))}{j!} (-\lambda_2 (1+i))^j \mu_{j,1}^Y, \\ a_{44} &= \frac{1}{\lambda_1^2} + (\alpha_1 - 1) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(i+1)^{j+1}}{j!} (-\lambda_1)^j \mu_{j+1,0}^X, \\ a_{55} &= \frac{1}{\lambda_2^2} + (\alpha_2 - 1) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(i+1)^{j+1}}{j!} (-\lambda_2)^j \mu_{j+1,0}^Y. \end{aligned} \quad (27)$$

FIGURE 1: Histograms of the estimated AUC for (a) $(m, n) = (50, 50)$, (b) $(m, n) = (100, 100)$, (c) $(m, n) = (50, 100)$, and (d) $(m, n) = (100, 200)$.

We denote $D = \text{diag}(\sqrt{m}, \sqrt{n}, \sqrt{m}, \sqrt{m}, \sqrt{n})$. Notice that EW family satisfies all the regularity conditions almost everywhere. Now we are ready to state the following theorem.

Theorem 6. Suppose (X, Y) follows the Bi-EW model with the model parameter θ . If $m \rightarrow \infty$ and $n \rightarrow \infty$, but $m/n \rightarrow p$, for $0 < p < \infty$, then one has the following.

(a) The MLE $\hat{\theta}$ of θ is asymptotic normal:

$$D(\hat{\theta} - \theta) \rightarrow N_5(0, \mathbf{A}^{-1}(\theta)). \quad (28)$$

(b) Part (a), the continuous mapping theorem, and Delta method imply that

$$\sqrt{m}(\widehat{AUC} - AUC) \rightarrow N(0, B), \quad (29)$$

where

$$B = \left(\frac{\partial AUC}{\partial \theta} \right) \mathbf{A}^{-1}(\theta) \left(\frac{\partial AUC}{\partial \theta} \right)^T, \quad (30)$$

$$\frac{\partial AUC}{\partial \theta} = \left(\frac{\partial AUC}{\partial \alpha_1}, \frac{\partial AUC}{\partial \alpha_2}, 0, \frac{\partial AUC}{\partial \beta_1}, \frac{\partial AUC}{\partial \beta_2} \right).$$

4. A Simulation Study

In this section we present some results based on Monte Carlo simulation to check the performance of the MLE of parameters and AUC. We consider small to moderate sample sizes: $m, n = 50, 100$, and 200 . We set two scale parameters: $\lambda_1 = \lambda_2 = 1$. The common shape parameter is $\beta = 2$. The other shape parameters are $\alpha_1 = 2$ and $\alpha_2 = 1, 2, 3, 4$, respectively. The simulation is based on 1000 replicates.

Figure 1 illustrates the histograms of the estimated AUC under the sample sizes $(m, n) = (50, 50), (100, 100), (50, 100), (100, 200)$ when the true parameters are $\alpha_1 = 2, \alpha_2 = 4$,

TABLE 1: ARB and RMSE of five estimated parameters ($\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}, \hat{\lambda}_1$, and $\hat{\lambda}_2$) and $\widehat{AUC}$, when $\alpha_1 = 2, \beta = 2, \lambda_1 = 1$, and $\lambda_2 = 1$ and with different values of α_2 .

(m, n)	MLE	$\alpha_2 = 1$	$\alpha_2 = 2$	$\alpha_2 = 3$	$\alpha_2 = 4$
(50, 50)	$\hat{\alpha}_1$	0.3721 (1.6569)	0.35630 (2.211)	0.3607 (1.5814)	0.3995 (1.950)
	$\hat{\alpha}_2$	0.2556 (1.1068)	0.3645 (1.2601)	0.4204 (1.5814)	0.5547 (2.1276)
	$\hat{\beta}$	0.0801 (0.7525)	0.0675 (0.6655)	0.0615 (0.6679)	0.0525 (0.6243)
	$\hat{\lambda}_1$	0.0842 (0.5661)	0.0756 (0.5421)	0.961 (0.5442)	0.0979 (0.5683)
	$\hat{\lambda}_2$	0.0913 (0.5235)	0.0805 (0.5493)	0.0916 (0.560)	0.119 (0.6189)
	$\widehat{AUC}$	0.0309 (0.0648)	0.0187 (0.0734)	0.0074 (0.08)	0.0078 (0.0648)
(50, 100)	$\hat{\alpha}_1$	0.1872 (1.6205)	0.3035 (2.1506)	0.2333 (1.5885)	0.2205 (1.9536)
	$\hat{\alpha}_2$	0.0823 (0.4995)	0.2404 (1.8192)	0.2507 (2.6231)	0.2896 (2.1170)
	$\hat{\beta}$	0.0636 (0.5464)	0.0293 (0.5054)	0.0271 (0.4935)	0.0467 (0.5228)
	$\hat{\lambda}_1$	0.0348 (0.4380)	0.0985 (0.4777)	0.0859 (0.4421)	0.0653 (0.4614)
	$\hat{\lambda}_2$	0.0217 (0.3694)	0.0751 (0.4532)	0.0762 (0.4510)	0.0612 (0.4861)
	$\widehat{AUC}$	0.0072 (0.060)	0.005 (0.0591)	0.0037 (0.0565)	0.0058 (0.0424)
(100, 50)	$\hat{\alpha}_1$	0.1609 (1.3638)	0.1689 (1.5535)	0.1774 (1.3754)	0.1731 (1.4751)
	$\hat{\alpha}_2$	0.1379 (0.5709)	0.2031 (1.6370)	0.2773 (1.9342)	0.3209 (1.7746)
	$\hat{\beta}$	0.0467 (0.5281)	0.0455 (0.4998)	0.0320 (0.4814)	0.0425 (0.5079)
	$\hat{\lambda}_1$	0.0399 (0.4085)	0.0430 (0.4129)	0.0602 (0.3962)	0.0461 (0.4191)
	$\hat{\lambda}_2$	0.0709 (0.4078)	0.0524 (0.4300)	0.0824 (0.4498)	0.0668 (0.4889)
	$\widehat{AUC}$	0.0048 (0.0519)	0.0031 (0.0489)	0.0017 (0.0469)	0.0056 (0.0435)
(100, 100)	$\hat{\alpha}_1$	0.1205 (1.1180)	0.1322 (1.2528)	0.1396 (1.2052)	0.1276 (1.1109)
	$\hat{\alpha}_2$	0.0726 (0.4195)	0.1291 (1.2499)	0.1842 (2.1241)	0.1295 (2.0528)
	$\hat{\beta}$	0.0367 (0.4324)	0.0346 (0.4317)	0.0290 (0.4219)	0.0323 (0.4281)
	$\hat{\lambda}_1$	0.0318 (0.3521)	0.0354 (0.3687)	0.0469 (0.3720)	0.0369 (0.3644)
	$\hat{\lambda}_2$	0.029 (0.3286)	0.0323 (0.3739)	0.0593 (0.3962)	0.0452 (0.3954)
	$\widehat{AUC}$	0.0036 (0.0412)	0.003 (0.0458)	0.0018 (0.0331)	0.0031 (0.040)
(200, 200)	$\hat{\alpha}_1$	0.0668 (0.6871)	0.0569 (0.6666)	0.0700 (0.6368)	0.0602 (0.6553)
	$\hat{\alpha}_2$	0.0404 (0.2707)	0.0637 (0.6581)	0.0833 (1.1019)	0.0965 (0.7324)
	$\hat{\beta}$	0.0152 (0.2879)	0.0135 (0.2853)	0.0090 (0.2791)	0.0102 (0.2756)
	$\hat{\lambda}_1$	0.0214 (0.2487)	0.0215 (0.2457)	0.0324 (0.2399)	0.0217 (0.2430)
	$\hat{\lambda}_2$	0.0162 (0.2274)	0.0222 (0.2419)	0.0292 (0.2562)	0.0280 (0.2715)
	$\widehat{AUC}$	0.0021 (0.0256)	0.0033 (0.0267)	0.0031 (0.0270)	0.0013 (0.0264)

$\beta = 2$, and $\lambda_1 = \lambda_2 = 1$. Obviously, as the sample size increases, the histogram tends to be more symmetric bell shaped. Therefore, the MLE of AUC performs well for the moderate sample size.

We report the absolute relative biases (ARB) and the square root of mean squared errors (RMSEs) for the estimators of $\theta = (\theta_1, \dots, \theta_5)$ and AUC defined as below:

$$ARB(\hat{\theta}_j) = \frac{1}{1000} \sum_{i=1}^{1000} \left| \frac{\hat{\theta}_j^i - \theta_j}{\theta_j} \right|, \quad (31)$$

$$RMSE(\hat{\theta}_j) = \sqrt{\frac{1}{1000} \sum_{i=1}^{1000} (\hat{\theta}_j^i - \theta_j)^2}, \quad j = 1, 2, 3, 4, 5,$$

where $\hat{\theta}_j^i$ is the estimates of θ_j for the i th replicate.

Recall that

$$AUC = \int_0^\infty F_0(x; \alpha_2, \beta, \lambda_2) dF_1(x; \alpha_1, \beta, \lambda_1). \quad (32)$$

The estimator $\widehat{AUC}$ and true value AUC can be computed by Riemann sums

$$\sum_{x=0.0001}^b F_0(x; \hat{\alpha}_2, \hat{\beta}, \hat{\lambda}_2) f_1(x; \hat{\alpha}_1, \hat{\beta}, \hat{\lambda}_1) \Delta x, \quad (33)$$

$$\sum_{x=0.0001}^b F_0(x; \alpha_2, \beta, \lambda_2) f_1(x; \alpha_1, \beta, \lambda_1) \Delta x,$$

respectively. Here, we choose $b = 4$ and $\Delta x = 0.0001$. The interval $[0, 4]$ is divided evenly with each subinterval length 0.0001 and x takes the value of the right end of each subinterval. We have verified that the summation keeps stable for any larger value b and smaller value Δx .

Similarly,

$$ARB(\widehat{AUC}) = \frac{1}{1000} \sum_{i=1}^{1000} \left| \frac{\widehat{AUC}_i - AUC}{AUC} \right|,$$

TABLE 2: ARB and RMSE of five estimated parameters ($\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}, \hat{\lambda}_1$, and $\hat{\lambda}_2$) and $\widehat{AUC}$, when $\alpha_1 = 2$, $\alpha_2 = 3$, $\lambda_1 = 1$, and $\lambda_2 = 1$ and with different values of β .

(m, n)	MLE	$\beta = 2$	$\beta = 2.5$	$\beta = 3$
(50, 50)	$\hat{\alpha}_1$	0.1925 (1.7453)	0.1646 (1.7159)	0.1863 (1.7626)
	$\hat{\alpha}_2$	0.4685 (2.245)	0.3917 (1.9021)	0.4641 (2.348)
	$\hat{\beta}$	0.0554 (0.6402)	0.0912 (0.7801)	0.0558 (0.9117)
	$\hat{\lambda}_1$	0.1044 (0.5624)	0.0700 (0.5442)	0.0800 (0.5353)
	$\hat{\lambda}_2$	0.1048 (0.5779)	0.0746 (0.5624)	0.0939 (0.5719)
	$\widehat{AUC}$	0.0067 (0.0556)	0.0026 (0.0557)	0.0036 (0.0529)
(50, 80)	$\hat{\alpha}_1$	0.2538 (1.8319)	0.2472 (2.0245)	0.2775 (2.0910)
	$\hat{\alpha}_2$	0.2818 (2.0819)	0.2691 (2.245)	0.3046 (2.3255)
	$\hat{\beta}$	0.0401 (0.5415)	0.0557 (0.6907)	0.0475 (0.8629)
	$\hat{\lambda}_1$	0.0825 (0.4706)	0.0635 (0.4821)	0.0815 (0.4803)
	$\hat{\lambda}_2$	0.0724 (0.4790)	0.0489 (0.4931)	0.0741 (0.4952)
	$\widehat{AUC}$	0.0045 (0.0490)	0.0015 (0.0510)	0.0008 (0.050)
(80, 50)	$\hat{\alpha}_1$	0.1846 (1.6341)	0.2311 (1.7011)	0.2874 (1.7425)
	$\hat{\alpha}_2$	0.2515 (2.2201)	0.3733 (2.2412)	0.3969 (1.956)
	$\hat{\beta}$	0.0501 (0.4921)	0.0414 (0.701)	0.0339 (0.8101)
	$\hat{\lambda}_1$	0.0380 (0.4365)	0.0665 (0.4577)	0.0905 (0.4761)
	$\hat{\lambda}_2$	0.0443 (0.4681)	0.0952 (0.5278)	0.0962 (0.5221)
	$\widehat{AUC}$	0.0003 (0.0479)	0.0013 (0.0479)	0.0055 (0.0489)
(80, 80)	$\hat{\alpha}_1$	0.162 (1.3518)	0.1531 (1.3652)	0.1594 (1.3561)
	$\hat{\alpha}_2$	0.2096 (1.8697)	0.2047 (3.9241)	0.1655 (2.183)
	$\hat{\beta}$	0.0372 (0.4715)	0.0438 (0.5934)	0.0458 (0.7095)
	$\hat{\lambda}_1$	0.0464 (0.4011)	0.0404 (0.3996)	0.0433 (0.4048)
	$\hat{\lambda}_2$	0.0488 (0.4211)	0.0417 (0.4334)	0.0291 (0.4084)
	$\widehat{AUC}$	0.002 (0.0412)	0.0023 (0.0424)	0.003 (0.0435)

$$\text{RMSE}(\widehat{AUC}) = \sqrt{\frac{1}{1000} \sum_{i=1}^{1000} (\widehat{AUC}_i - \widehat{AUC})^2}, \quad (34)$$

where $\widehat{AUC}_i$ is the estimate of AUC for the i th replicate.

Tables 1 and 2 report the ARB and the RMSE for the estimators of $\alpha_1, \alpha_2, \beta, \lambda_1, \lambda_2$, and the AUC. Table 1 reports the sensitivity analysis against α_2 and sample sizes. In all cases, the two scale parameters have much smaller ARB and RMSE compared with shape parameters. The MLE of AUC behaves well. For two different shape parameters, it is always the common parameter β which behaves better than the other two shape parameters α_1 and α_2 as expected. For two shape parameters α_1 and α_2 , when $\alpha_2 > \alpha_1$, α_1 always behaves better than α_2 . Furthermore, as expected, the ARB and the RMSE decrease as the sample size increases. Table 2 reports the sensitivity analysis against the common shape parameter β . The results show the robustness of the MLEs for small sample sizes.

Our second simulations are to check the performance of MLE as the common shape parameter β increases. We choose the following sample sizes: $(m, n) = (50, 50), (80, 80), (50, 80), (80, 50)$. The two scale parameters are still set to 1, and $\alpha_1 = 2$, $\alpha_2 = 3$ and β takes three values: 2, 2.5 and 3. The results are reported in Table 2. Once again, one notices

that AUC has very small ARB and RMSE. Under all settings, α_2 has larger ARB and RMSE than that of α_1 when $\alpha_2 > \alpha_1$. When β increases, we do not find a noticeable change of ARB and RMSE of all parameters.

5. Conclusions

In summary, this paper considers an Exponentiated Weibull model with two shape parameters and one scale parameter. Firstly, we derive a general moment formula which extends many results in the literature. Secondly, we obtain the theoretical AUC formula for a Bi-Exponentiated Weibull model which shares a common shape parameter between the two groups. We have noticed that the formula of AUC is independent of the common shape parameter, which is consistent with existing known results for simpler models such as Bi-generalized exponential and Bi-Weibull. Thirdly, we derive the maximum likelihood estimator of AUC for the Bi-EW model with a common shape parameter and show the asymptotic normality of the estimators. Finally, we conduct a simulation study to illustrate the performance of our estimator for small to moderate sample sizes. The simulation results show that the MLE of AUC can behave very well under moderate sample sizes although the performance of the MLE of some shape parameters is somewhat unsatisfactory but still acceptable.

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