Research Article

Some New Sets and Topologies in Ideal Topological Spaces

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An ideal topological space is a triplet $(X, \tau, \mathfrak{I})$, where X is a nonempty set, τ is a topology on X , and $\mathfrak{I}$ is an ideal of subsets of X . In this paper, we introduce L^* -perfect, R^* -perfect, and C^* -perfect sets in ideal spaces and study their properties. We obtained a characterization for compatible ideals via R^* -perfect sets. Also, we obtain a generalized topology via ideals which is finer than τ using R^* -perfect sets on a finite set.

1. Introduction and Preliminaries

The contributions of Hamlett and Jankovic [1–4] in ideal topological spaces initiated the generalization of some important properties in general topology via topological ideals. The properties like decomposition of continuity, separation axioms, connectedness, compactness, and resolvability [5–9] have been generalized using the concept of ideals in topological spaces.

By a space (X, τ) , we mean a topological space X with a topology τ defined on X on which no separation axioms are assumed unless otherwise explicitly stated. For a given point x in a space (X, τ) , the system of open neighborhoods of x is denoted by $N(x) = \{U \in \tau : x \in U\}$. For a given subset A of a space (X, τ) , $cl(A)$ and $int(A)$ are used to denote the closure of A and interior of A , respectively, with respect to the topology.

A nonempty collection of subsets of a set X is said to be an ideal on X , if it satisfies the following two conditions: (i) If $A \in \mathfrak{I}$ and $B \subseteq A$, then $B \in \mathfrak{I}$; (ii) If $A \in \mathfrak{I}$ and $B \in \mathfrak{I}$, then $A \cup B \in \mathfrak{I}$. An ideal topological space (or ideal space) $(X, \tau, \mathfrak{I})$ means a topological space (X, τ) with an ideal $\mathfrak{I}$ defined on X . Let (X, τ) be a topological space with an ideal $\mathfrak{I}$ defined on X . Then for any subset A of X , $A^*(\mathfrak{I}, \tau) = \{x \in X/A \cap U \notin \mathfrak{I} \text{ for every } U \in N(x)\}$ is called the local function of A with respect to $\mathfrak{I}$ and τ . If there is no ambiguity, we will write $A^*(\mathfrak{I})$ or simply A^* for $A^*(\mathfrak{I}, \tau)$. Also, $cl^*(A) = A \cup A^*$

defines a Kuratowski closure operator for the topology $\tau^*(\mathfrak{I})$ (or simply τ^*) which is finer than τ . An ideal $\mathfrak{I}$ on a space (X, τ) is said to be codense ideal if and only if $\tau \cap \mathfrak{I} = \{\emptyset\}$. X^* is always a proper subset of X . Also, $X = X^*$ if and only if the ideal is codense.

Lemma 1 (see [3]). Let (X, τ) be a space with $\mathfrak{I}_1$ and $\mathfrak{I}_2$ being ideals on X , and let A and B be two subsets on X . Then

- (i) $A \subseteq B \Rightarrow A^* \subseteq B^*$;
- (ii) $\mathfrak{I}_1 \subseteq \mathfrak{I}_2 \Rightarrow A^*(\mathfrak{I}_2) \subseteq A^*(\mathfrak{I}_1)$;
- (iii) $A^* = cl(A^*) \subseteq cl(A)$ (A^* is a closed subset of $cl(A)$);
- (iv) $(A^*)^* \subseteq A^*$;
- (v) $(A \cup B)^* = A^* \cup B^*$;
- (vi) $A^* - B^* = (A - B)^* - B^* \subseteq (A - B)^*$;
- (vii) for every $I \in \mathfrak{I}$, $(A \cup I)^* = A^* = (A - I)^*$.

Definition 2 (see [3]). Let (X, τ) be a space with an ideal $\mathfrak{I}$ on X . One says that the topology τ is compatible with the ideal $\mathfrak{I}$, denoted by $\tau \sim \mathfrak{I}$, if the following holds, for every $A \subseteq X$: if for every $x \in A$, there exists a $U \in N(x)$ such that $U \cap A \in \mathfrak{I}$, then $A \in \mathfrak{I}$.

Definition 3. A subset A of an ideal space $(X, \tau, \mathfrak{I})$ is said to be

- (i) τ^* -closed [3] if $A^* \subseteq A$,
- (ii) * -dense-in-itself [10] if $A \subseteq A^*$,
- (iii) I -open [11] if $A \subseteq \text{int}(A^*)$,
- (iv) almost I -open [12] if $A \subseteq \text{cl}(\text{int}(A^*))$,
- (v) I -dense [7] if $A^* = X$,
- (vi) almost strong β - I -open [13] if $A \subseteq \text{cl}^*(\text{int}(A^*))$,
- (vii) * -perfect [10] if $A = A^*$,
- (viii) regular I -closed [14] if $A = (\text{int}(A))^*$,
- (ix) an f_I -set [15] if $A \subseteq (\text{int}(A))^*$.

Theorem 4 ([3]). Let (X, τ) be a space with an ideal $\mathfrak{I}$ on X . Then the following are equivalent.

- (i) $\tau \sim \mathfrak{I}$.
- (ii) If A has a cover of open sets each of whose intersection with A is $\mathfrak{I}$, then A is in $\mathfrak{I}$.
- (iii) For every $A \subseteq X$, $A \cap A^* = \emptyset \Rightarrow A \in \mathfrak{I}$.
- (iv) For every $A \subseteq X$, $A - A^* \in \mathfrak{I}$.
- (v) For every τ^* -closed subset A , $A - A^* \in \mathfrak{I}$.
- (vi) For every $A \subseteq X$, if A contains no nonempty subset B with $B \subseteq B^*$, then $A \in \mathfrak{I}$.

2. L^* -Perfect, R^* -Perfect, and C^* -Perfect Sets

In this section, we define three collections of subsets $\mathfrak{L}$, $\mathfrak{R}$ and $\mathfrak{C}$ in an ideal space and study some of their properties.

Definition 5. Let $(X, \tau, \mathfrak{I})$ be an ideal topological space. A subset A of X is said to be

- (i) L^* -perfect if $A - A^* \in \mathfrak{I}$,
- (ii) R^* -perfect if $A^* - A \in \mathfrak{I}$,
- (iii) C^* -perfect if A is both L^* -perfect and R^* -perfect.

The collection of L^* -perfect sets, R^* -perfect sets, and C^* -perfect sets in $(X, \tau, \mathfrak{I})$ is denoted by $\mathfrak{L}$, $\mathfrak{R}$, and $\mathfrak{C}$, respectively.

Remark 6. (i) If $\mathfrak{I} = \{\emptyset\}$, then

$$\begin{aligned}\mathfrak{L} &= \{A \subseteq X : A \subseteq A^* = \text{cl}(A)\} \\ &= \text{the collection of all } ^*\text{-dense-in-itself sets} \\ &= \wp(X),\end{aligned}$$

$$\begin{aligned}\mathfrak{R} &= \{A \subseteq X : A^* \subseteq A\} \\ &= \text{the collection of all } \tau^*\text{-closed sets} \\ &= \{A \subseteq X : A = A^* = \text{cl}(A)\} \\ &= \text{the collection of all closed sets in } (X, \tau) \\ &= \text{the collection of all } ^*\text{-perfect sets,} \\ \mathfrak{C} &= \{A \subseteq X : A^* \subseteq A, A \subseteq A^*\} \\ &= AX : A = A^* = \text{cl}(A) \\ &= \text{the collection of all closed sets in } (X, \tau) \\ &= \text{the collection of all } ^*\text{-perfect sets.}\end{aligned}\tag{1}$$

(ii) If $\mathfrak{I} = \mathfrak{P}(X)$, then $\mathfrak{L} = \mathfrak{R} = \wp(X)$.

(iii) If $\tau \sim \mathfrak{I}$, then $\mathfrak{L} = \wp(X)$ (by Theorem 4(iv)).

Remark 7. Every * -perfect set is both R^* -perfect and L^* -perfect (and hence C^* -perfect). In Proposition 15, we proved that every members of an ideal is both R^* -perfect and L^* -perfect (and hence C^* -perfect). But any nonempty member of an ideal is not a * -perfect set. Hence R^* -perfect and L^* -perfect sets (and hence C^* -perfect sets) need not be * -perfect.

Proposition 8. If a subset A of an ideal space $(X, \tau, \mathfrak{I})$ is C^* -perfect, then $A\Delta A^* \in \mathfrak{I}$.

Proof. Since A is both L^* -perfect and R^* -perfect, $A - A^* \in \mathfrak{I}$ and $A^* - A \in \mathfrak{I}$. By the finite additive property of ideals, $(A - A^*) \cup (A^* - A) \in \mathfrak{I}$. Hence $A\Delta A^* \in \mathfrak{I}$. $\square$

Proposition 9. In an ideal space $(X, \tau, \mathfrak{I})$, every τ^* -closed set is R^* -perfect.

Proof. Let A be a τ^* -closed set. Therefore, $A^* \subseteq A$. Hence $A^* - A = \emptyset \in \mathfrak{I}$. Therefore, A is an R^* -perfect set. $\square$

Corollary 10. In an ideal space $(X, \tau, \mathfrak{I})$,

- (i) X and $\emptyset$ are R^* -perfect sets,
- (ii) every τ -closed set is R^* -perfect,
- (iii) for any subset A of an ideal topological space $(X, \tau, \mathfrak{I})$, $\text{cl}(A)$, A^* , $\text{cl}^*(A)$ are R^* -perfect sets,
- (iv) every regular- I -closed set is R^* -perfect.

Proof. The proof follows from Proposition 9. $\square$

The following example shows that the converses of Proposition 9 and Corollary 10(iv) are not true.

Example 11. Let $(X, \tau, \mathfrak{I})$ be an ideal space with $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, and $\mathfrak{I} = \{\emptyset, \{b\}\}$. The set $\{a, c\}$ is R^* -perfect set which is not a τ^* -closed set and hence not a regular- I -closed set.

Proposition 12. If a subset A of an ideal topological space $(X, \tau, \mathfrak{I})$ is such that $A \in \mathfrak{I}$, then A is C^* -perfect.

Proof. Since $A \in \mathfrak{I}, A^* = \phi$. Then $A - A^* = A \in \mathfrak{I}$ and $A^* - A = \phi \in \mathfrak{I}$. Hence A is both an L^* -perfect and R^* -perfect set. $\square$

Corollary 13. Let A be a subset of an ideal space $(X, \tau, \mathfrak{I})$. Consider the following.

- (i) If $A \in \mathfrak{I}$, then every subset of A is a C^* -perfect set.
- (ii) If A is R^* -perfect, then $A^* - A$ is C^* -perfect.
- (iii) If A is an L^* -perfect set, then $A - A^*$ is a C^* -perfect set.
- (iv) If A is C^* -perfect, then $A \Delta A^*$ is a C^* -perfect set.

Proof. The proof follows from Proposition 12. $\square$

Corollary 14. Let (X, τ) be a space with an ideal $\mathfrak{I}$ on X such that $\tau \sim \mathfrak{I}$. Then for every $A \subseteq X$,

- (i) if $A \cap A^* = \phi$, then A is C^* -perfect;
- (ii) $A - A^*$ is C^* -perfect;
- (iii) if A contains nonempty subset B with $B \subseteq B^*$, then A is C^* -perfect.

Proof. Follows from Theorem 4 and Proposition 12. $\square$

Proposition 15. In an ideal space $(X, \tau, \mathfrak{I})$, every * -dense-in-itself set is an L^* -perfect set.

Proof. Let A be a * -dense-in-itself set of X . Then $A \subseteq A^*$. Therefore, $A - A^* = \phi \in \mathfrak{I}$. Hence A is an L^* -perfect set. $\square$

Corollary 16. In an ideal space $(X, \tau, \mathfrak{I})$,

- (i) every I -dense set is L^* -perfect,
- (ii) every I -open set is L^* -perfect,
- (iii) every almost strong β - I -open set is L^* -perfect,
- (iv) every almost I -open set is L^* -perfect,
- (v) every regular- I -closed set is L^* -perfect,
- (vi) every f_I -set is L^* -perfect.

Proof. Since all the above sets are * -dense-in-itself, by Proposition 15, these sets are L^* -perfect. $\square$

Remark 17. The members of the ideal of an ideal space are L^* -perfect, but the nonempty members of the ideal are not * -dense-in-itself. Therefore, the converses of the above Corollary and Proposition 15 need not to be true.

Proposition 18. In an ideal space $(X, \tau, \mathfrak{I})$,

- (i) empty set is an L^* -perfect set,
- (ii) X is an L^* -perfect set if the ideal is codense.

Proof. (i) Since $\phi - \phi^* = \phi \in \mathfrak{I}$, the empty set is an L^* -perfect set. (ii) We know that $X = X^*$ if and only if the ideal $\mathfrak{I}$ is codense. Then $X - X^* = \phi \in \mathfrak{I}$. Hence the result follows. $\square$

3. Main Results

In this section, we prove that finite union and intersection of R^* -perfect sets are again R^* -perfect set. Using these results, we obtain a new topology for the finite topological spaces which is finer than τ^* -topology.

In Ideal spaces, usually $A \subseteq B$ implies $A^* \subseteq B^*$. We observe that there are some sets A and B such that $A \subseteq B$ but $A^* = B^*$.

Example 19. Let $(X, \tau, \mathfrak{I})$ be an ideal space with $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a, c\}, \{d\}, \{a, c, d\}\}$, $\mathfrak{I} = \{\phi, \{c\}, \{d\}, \{c, d\}\}$. Here the sets $A = \{a\}$ and $B = \{a, b\}$ are such that $A \subseteq B$, but $A^* = B^* = \{a, b, c\}$.

Proposition 20. Let $(X, \tau, \mathfrak{I})$ be an ideal space. Let A and B be two subsets of X such that $A \subseteq B$ and $A^* = B^*$; then

- (i) B is R^* -perfect if A is R^* -perfect;
- (ii) A is L^* -perfect if B is L^* -perfect.

Proof. (i) Let A be an R^* -perfect set. Then $A^* - A \in \mathfrak{I}$. Now, $B^* - B = A^* - B \subseteq A^* - A$. By heredity property of ideals, $B^* - B \in \mathfrak{I}$. Hence B is R^* -perfect.

(ii) Let B be an L^* -perfect set. Then $B - B^* \in \mathfrak{I}$. Now, $A - A^* = A - B^* \subseteq B - B^*$. By heredity property of ideals, $A - A^* \in \mathfrak{I}$. Hence A is L^* -perfect. $\square$

Corollary 21. Let $(X, \tau, \mathfrak{I})$ be an ideal space. Let A and B be two subsets of X such that $A \subseteq B \subseteq cl^* A$; then

- (i) B is R^* -perfect if A is R^* -perfect,
- (ii) A is L^* -perfect if B is L^* -perfect.

Proof. Since $A \subseteq B \subseteq cl^* A$, $A^* \subseteq B^* \subseteq (cl^* A)^* = A^*$. Hence $A^* = B^*$. Therefore, the result follows from Proposition 20. $\square$

Proposition 22. Let A be a subset of an ideal topological space $(X, \tau, \mathfrak{I})$ such that A is L^* -perfect set and $A \cap A^*$ is R^* -perfect; then both A and $A \cap A^*$ are C^* -perfect.

Proof. Since A is L^* -perfect, $A - A^* \in \mathfrak{I}$. By Lemma 1(vii), for every $I \in \mathfrak{I}$, $(A \cup I)^* = A^* = (A - I)^*$. Therefore, $(A \cup (A - A^*))^* = A^* = (A - (A - A^*))^*$. This implies $A^* = (A \cap A^*)^*$. Therefore, we have $A \cap A^* \subseteq A$ with $(A \cap A^*)^* = A^*$. By Proposition 20, A is R^* -perfect if $A \cap A^*$ is R^* -perfect and $A \cap A^*$ is L^* -perfect if A is L^* -perfect set. Hence A is R^* -perfect and $A \cap A^*$ is L^* -perfect. $\square$

Proposition 23. If a subset A of an ideal topological space $(X, \tau, \mathfrak{I})$ is R^* -perfect set and A^* is L^* -perfect, then $A \cap A^*$ is L^* -perfect.

Proof. Since A is R^* -perfect, $A^* - A \in \mathfrak{I}$. By Lemma 1(vii), for every $I \in \mathfrak{I}$, $(A \cup I)^* = A^* = (A - I)^*$. Therefore, $(A^* \cup (A^* - A))^* = A^{**} = (A^* - (A^* - A))^*$. This implies $A^{**} = (A \cap A^*)^*$. Therefore, we have $A \cap A^* \subseteq A^*$ with $(A \cap A^*)^* = A^{**}$. By Proposition 20, $A \cap A^*$ is L^* -perfect if A^* is L^* -perfect set. Hence $A \cap A^*$ is L^* -perfect. $\square$

Proposition 24. If A and B are R^* -perfect sets, then $A \cup B$ is an R^* -perfect set.

Proof. Let A and B be R^* -perfect sets. Then $A^* - A \in \mathfrak{F}$ and $B^* - B \in \mathfrak{F}$. By finite additive property of ideals, $(A^* - A) \cup (B^* - B) \in \mathfrak{F}$. Since $(A^* \cup B^*) - (A \cup B) \subseteq (A^* - A) \cup (B^* - B)$, by heredity property $(A^* \cup B^*) - (A \cup B) \in \mathfrak{F}$. Hence $(A \cup B)^* - (A \cup B) \in \mathfrak{F}$. This proves the result. $\square$

Corollary 25. Finite union of R^* -perfect sets is an R^* -perfect set.

Proof. The proof follows from Proposition 24. $\square$

Proposition 26. If A and B are L^* -perfect sets, then $A \cup B$ is an L^* -perfect set.

Proof. Since A and B are L^* -perfect sets, $A - A^* \in \mathfrak{F}$ and $B - B^* \in \mathfrak{F}$. Hence by finite additive property of ideals, $(A - A^*) \cup (B - B^*) \in \mathfrak{F}$. Since $(A \cup B) - (A \cup B)^* = (A \cup B) - (A^* \cup B^*) \subseteq (A - A^*) \cup (B - B^*)$, by heredity property $(A \cup B) - (A \cup B)^* \in \mathfrak{F}$. This proves that $A \cup B$ is an L^* -perfect set. $\square$

Corollary 27. Finite union of L^* -perfect sets is an L^* -perfect sets.

Proof. The proof follows from Proposition 26. $\square$

Proposition 28. If A and B are R^* -perfect sets, then $A \cap B$ is an R^* -perfect set.

Proof. Suppose that A and B are R^* -perfect sets. Then $A^* - A \in \mathfrak{F}$ and $B^* - B \in \mathfrak{F}$. By finite additive property of ideals, $(A^* - A) \cup (B^* - B) \in \mathfrak{F}$. Since $(A^* \cap B^*) - (A \cap B) \subseteq (A^* - A) \cup (B^* - B)$, by heredity property $(A^* \cap B^*) - (A \cap B) \in \mathfrak{F}$. Also $(A \cap B)^* - (A \cap B) \subseteq (A^* \cap B^*) - (A \cap B) \in \mathfrak{F}$. This proves the result. $\square$

Corollary 29. Finite intersection of R^* -perfect sets is an R^* -perfect set.

Proof. The proof follows from Proposition 28. $\square$

Proposition 30. Finite union of C^* -perfect sets is a C^* -perfect set.

Proof. From Corollaries 27 and 29, finite union of C^* -perfect sets is a C^* -perfect set. $\square$

Proposition 31. If $(X, \tau, \mathfrak{F})$ is an ideal topological space with X being finite, then the collection $\mathfrak{R}$ is a topology which is finer than the topology of τ^* -closed sets.

Proof. By Corollary 10, X and ϕ are R^* -perfect sets. By Corollary 25, finite union of R^* -perfect sets is an R^* -perfect set, and by Corollary 29, finite intersection of R^* -perfect sets is R^* -perfect. Hence the collection $\mathfrak{R}$ is a topology if X is finite. Also, by Proposition 9 every τ^* -closed set is an R^* -perfect set. Hence the topology $\mathfrak{R}$ is finer than the topology of τ^* -closed sets if X is finite. $\square$

Proposition 32. In an ideal space $(X, \tau, \mathfrak{F})$, $\{\tau^*$ -closed sets $\} \cup \mathfrak{F} \subseteq \mathfrak{R}$.

Proof. The proof follows from Propositions 9 and 12. $\square$

The following example shows that $\{\tau^*$ -closed sets $\} \cup \mathfrak{F} \neq \mathfrak{R}$.

Example 33. Let $(X, \tau, \mathfrak{F})$ be an ideal space with $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$, and $\mathfrak{F} = \{\emptyset, \{b\}\}$. The collection of τ^* -closed sets is $\{\emptyset, X, \{b\}, \{b, c\}\}$ and $\mathfrak{R} = \{\emptyset, X, \{b\}, \{b, c\}, \{a, c\}\}$.

Now, $\{\tau^*$ -closed sets $\} \cup \mathfrak{F} = \{\emptyset, X, \{b\}, \{b, c\}\} \neq \mathfrak{R}$.

Proposition 34. Let $(X, \tau, \mathfrak{F})$ be an ideal space and $A \subseteq X$. The set A is R^* -perfect if and only if $F \subseteq A^* - A$ in X implies that $F \in \mathfrak{F}$.

Proof. Assume that A is an R^* -perfect set. Then $A^* - A \in \mathfrak{F}$. By heredity property of ideals, every set $F \subseteq A^* - A$ in X is also in $\mathfrak{F}$. Conversely assume that $F \subseteq A^* - A$ in X implies that $F \in \mathfrak{F}$. Since $A^* - A$ is a subset of itself, by assumption $A^* - A \in \mathfrak{F}$. Hence A is R^* -perfect. $\square$

Proposition 35. Let $(X, \tau, \mathfrak{F})$ be an ideal space and $A \subseteq X$. The set A is L^* -perfect if and only if $F \subseteq A - A^*$ in X implies that $F \in \mathfrak{F}$.

Proof. Assume that A is an L^* -perfect set. Then $A - A^* \in \mathfrak{F}$. By heredity property of ideals, every set $F \subseteq A - A^*$ in X is also in $\mathfrak{F}$. Conversely, assume that $F \subseteq A - A^*$ in X implies that $F \in \mathfrak{F}$. Since $A - A^*$ is a subset of itself, by assumption $A - A^* \in \mathfrak{F}$. Hence A is L^* -perfect. $\square$

Proposition 36. Let (X, τ) be a topological space and $A \subseteq X$. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two ideals on X with $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$. Then A is R^* -perfect with respect to $\mathfrak{F}_2$ if it is R^* -perfect with respect to $\mathfrak{F}_1$.

Proof. Since $\mathfrak{F}_1 \subseteq \mathfrak{F}_2$, $A^*(\mathfrak{F}_2) \subseteq A^*(\mathfrak{F}_1)$ by Lemma 1(ii). Let A be R^* -perfect with respect to $\mathfrak{F}_1$. Then $A^*(\mathfrak{F}_1) - A \in \mathfrak{F}_1$. Also, $A^*(\mathfrak{F}_2) - A \subseteq A^*(\mathfrak{F}_1) - A$. Hence by heredity property of ideals, $A^*(\mathfrak{F}_2) - A \in \mathfrak{F}_1 \subseteq \mathfrak{F}_2$. Therefore A is R^* -perfect with respect to $\mathfrak{F}_2$. $\square$

Theorem 37. Let (X, τ) be a space with an ideal $\mathfrak{F}$ on X . Then the following are equivalent.

- (i) $\tau \sim \mathfrak{F}$.
- (ii) If A has a cover of open sets each of whose intersection with A is $\mathfrak{F}$, then A is in $\mathfrak{F}$.
- (iii) If $A \subseteq X$, then $A \cap A^* = \phi \Rightarrow A \in \mathfrak{F}$.
- (iv) If $A \subseteq X$, then $A - A^* \in \mathfrak{F}$.
- (v) If $A \subseteq X$ and A is R^* -perfect set, then $A \Delta A^* \in \mathfrak{F}$.
- (vi) For every τ^* -closed subset A , $A - A^* \in \mathfrak{F}$.

(vii) For every $A \subseteq X$, if A contains no nonempty subset B with $B \subseteq B^*$, then $A \in \mathfrak{F}$.

Proof. To prove this theorem, it is enough to prove (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi). Others follow from Theorem 4. (iv) $\Rightarrow$ (v) follows from Proposition 8. Suppose that $A\Delta A^* \in \mathfrak{F}$. Since $A - A^* \subseteq A\Delta A^*$, by heredity property $A - A^* \in \mathfrak{F}$. Hence (v) $\Rightarrow$ (vi). $\square$

4. R^* -Topology

By Corollary 10 and Proposition 28, we observe that the collection $\mathfrak{R}$ satisfies the conditions of being a basis for some topology and it will be called as $R_c^*(\tau, \mathfrak{F})$. We define $R^*(\tau, \mathfrak{F}) = \{A \subseteq X/X - A \in R_c^*(\tau, \mathfrak{F})\}$ on a nonempty set X . Clearly, $R^*(\tau, \mathfrak{F})$ is a topology if the set X is finite. The members of the collection $R^*(\tau, \mathfrak{F})$ will be called R^* -open sets. If there is no confusion about the topology τ and the ideal $\mathfrak{F}$, then we call $R^*(\tau, \mathfrak{F})$ as R^* -topology when X is finite.

Definition 38. A subset A of an ideal topological space $(X, \tau, \mathfrak{F})$ is said to be R^* -closed if it is a complement of an R^* -open set.

Definition 39. Let A be a subset of an ideal topological space $(X, \tau, \mathfrak{F})$. One defines R^* -interior of the set A as the largest R^* -open set contained in A . One will denote R^* -interior of a set A by $R^* - \text{int}(A)$.

Definition 40. Let A be a subset of an ideal topological space $(X, \tau, \mathfrak{F})$. A point $x \in A$ is said to be an R^* -interior point of the set A if there exists an R^* -open set U of x such that $x \in U \subseteq A$.

Definition 41. Let $(X, \tau, \mathfrak{F})$ be an ideal space and $x \in X$. One defines R^* -neighborhood of x as an R^* -open set containing x . One denotes the set of all R^* -neighborhoods of x by $R^* - N(x)$.

Proposition 42. In an ideal space $(X, \tau, \mathfrak{F})$, every τ^* -open set is an R^* -open set.

Proof. Let A be a τ^* -open set. Therefore, $X - A$ is a τ^* -closed set. That implies that $X - A$ is an R^* -closed set. Hence A is an R^* -open set. $\square$

Corollary 43. The topology $R^*(\tau, \mathfrak{F})$ on a finite set X is finer than the topology $\tau^*(\tau, \mathfrak{F})$.

Proof. The proof follows from Proposition 42. $\square$

Corollary 44. For any subset A of an ideal topological space $(X, \tau, \mathfrak{F})$, $\text{int}(A)$ is an R^* -open set.

Proof. The proof follows from Proposition 42. $\square$

Remark 45. (i) Since every open set is an R^* -open set, every neighborhood U of a point $x \in X$ is an R^* -neighborhood of x .

(ii) If $x \in X$ is an interior point of a subset A of X , then x is an R^* -interior point of A .

(iii) From (ii), we have $\text{int}(A) \subseteq \text{int}^*(A) \subseteq R^* - \text{int}(A)$, where $\text{int}^*(A)$ denotes interior of A with respect to the topology τ^* .

Theorem 46. Let A and B be subsets of an ideal space $(X, \tau, \mathfrak{F})$ with X being finite. Then the following properties hold.

- (i) $R^* - \text{int}(A) = \cup\{U : U \subseteq A \text{ and } U \text{ is an } R^* - \text{open set}\}.$
- (ii) $R^* - \text{int}(A)$ is the largest R^* -open set of X contained in A .
- (iii) A is R^* -open if and only if $A = R^* - \text{int}(A)$.
- (iv) $R^* - \text{int}(R^* - \text{int}(A)) = R^* - \text{int}(A)$.
- (v) If $A \subseteq B$, then $R^* - \text{int}(A) \subseteq R^* - \text{int}(B)$.

Proof. The proof follows from Definitions 39, 40, and 41. $\square$

Definition 47. Let A be a subset of an ideal topological space $(X, \tau, \mathfrak{F})$. One defines R^* -closure of the set A as the smallest R^* -closed set containing A . One will denote R^* -closure of a set A by $R^* - \text{cl}(A)$.

Remark 48. For any subset A of an ideal topological space $(X, \tau, \mathfrak{F})$, $R^* - \text{cl}(A) \subseteq \text{cl}^*(A) \subseteq \text{cl}(A)$.

Theorem 49. Let A and B be subsets of an ideal space $(X, \tau, \mathfrak{F})$ where X is finite. Then the following properties hold:

- (i) $R^* - \text{cl}(A) = \cap\{F : A \subseteq F \text{ and } F \text{ is } R^* - \text{closed set}\}.$
- (ii) A is R^* -closed if and only if $A = R^* - \text{cl}(A)$
- (iii) $R^* - \text{cl}(R^* - \text{cl}(A)) = R^* - \text{cl}(A)$
- (iv) If $A \subseteq B$, then $R^* - \text{cl}(A) \subseteq R^* - \text{cl}(B)$.

Proof. The proof follows from Definition 47. $\square$

Theorem 50. Let A be a subset of an ideal space $(X, \tau, \mathfrak{F})$. Then the following properties hold:

- (i) $R^* - \text{int}(X - A) = X - R^* - \text{cl}(A);$
- (ii) $R^* - \text{cl}(X - A) = X - R^* - \text{int}(A).$

Proof. The proof follows from Definitions 38, 39, and 47. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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