Research Article

Partitions of Natural Numbers with the Intersection Not Empty

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Let $\mathbb{N}$ be the set of nonnegative integers. For a given set $A \subset \mathbb{N}$ the representation functions $R_2(A, n)$, $R_3(A, n)$ are defined as the number of solutions of the equation $n = a + a'$, $a, a' \in A$ with condition $a < a'$, $a \leq a'$, respectively. In this paper, we prove that if $\mathbb{N} = A \cup B$ and $A \cap B = \{8k : k \in \mathbb{N}\}$, then $R_i(A, n) = R_i(B, n)$ cannot hold for all sufficiently large integers n where $i = 2, 3$.

1. Introduction

Let $\mathbb{N}$ be the set of nonnegative integers. For $A \subset \mathbb{N}$, let $R_1(A, n)$, $R_2(A, n)$, $R_3(A, n)$ denote the number of solutions of

$$\begin{aligned} a + a' &= n, \quad a, a' \in A, \\ a + a' &= n, \quad a, a' \in A, \quad a < a', \\ a + a' &= n, \quad a, a' \in A, \quad a \leq a', \end{aligned} \quad (1)$$

respectively. Sárközy asked ever whether there exist two sets A and B of nonnegative integers with infinite symmetric difference, that is:

$$\begin{aligned} |(A \cup B) \setminus (A \cap B)| &= \infty, \\ R_i(A, n) &= R_i(B, n), \quad n \geq n_0 \end{aligned} \quad (2)$$

for $i = 1, 2, 3$. As Dombi [1] has shown, the answer is negative for $i = 1$ by the simple observation that $R_1(A, n)$ is odd if and only if $n = 2a$ for some $a \in A$. For $i = 2$, Dombi presented a partition of the set of all positive integers into two disjoint subsets A and B such that $R_2(A, n) = R_2(B, n)$ for all sufficiently large n . For $i = 3$, Chen and Wang [2] constructed sets A, B such that $R_3(A, n) = R_3(B, n)$ for all integers $n \geq 3$, where A is the set of all those positive integers n such that the number of zeroes in the binary representation of $n - 1$ is even, and B is the set of all those positive integers n such that the number of zeroes in the binary representation of $n - 1$ is odd. For the other related results, the reader is referred to see [3–7].

It is natural to ask the following: for $i = 2, 3$, are there subsets $A, B \subset \mathbb{N}$ with $R_i(A, n) = R_i(B, n)$ for all large enough integers n such that $\mathbb{N} = A \cup B$ and $A \cap B \neq \emptyset$? Recently, the authors of this paper [8] obtained some results in this direction.

In this paper, we obtain the following results.

Theorem 1. *If $\mathbb{N} = A \cup B$ and $A \cap B = \{8k : k \in \mathbb{N}\}$, then $R_2(A, n) = R_2(B, n)$ cannot hold for all sufficiently large integers n .*

Theorem 2. *If $\mathbb{N} = A \cup B$ and $A \cap B = \{8k : k \in \mathbb{N}\}$, then $R_3(A, n) = R_3(B, n)$ cannot hold for all sufficiently large integers n .*

Remark 3. *For $r = 1, 2, 3, 4, 5, 6, 7$, similarly, we can prove that if $\mathbb{N} = A \cup B$ and $A \cap B = \{8k + r : k \in \mathbb{N}\}$, then $R_i(A, n) = R_i(B, n)$ cannot hold for all sufficiently large integers n , where $i = 2, 3$.*

2. Proof of Theorem 1

Proof. Suppose that there exist integer n_0 and sets $A, B \subset \mathbb{N}$ with $A \cup B = \mathbb{N}$ and $A \cap B = \{8k : k \in \mathbb{N}\}$ such that $R_2(A, n) = R_2(B, n)$ for all $n \geq n_0$. Without loss of generality, we may assume that $n_0 = 16N - 1$, N is a positive integer; then there exists a polynomial $p(x)$ of degree at most $16N - 2$ such that

$$\sum_{n=0}^{\infty} (R_2(A, n) - R_2(B, n)) x^n = p(x). \quad (3)$$

Let

$$\begin{aligned} f(x) &= \sum_{a \in A} x^a, & g(x) &= \sum_{b \in B} x^b, \\ \varepsilon_i &= \begin{cases} 1, & i \in A, \\ 0, & i \notin A. \end{cases} \end{aligned} \quad (4)$$

Then we have

$$\begin{aligned} f(x) &= \sum_{a \in A} x^a = \sum_{i=0}^{\infty} \varepsilon_i x^i, \\ \sum_{n=0}^{\infty} R_2(A, n) x^n &= \frac{1}{2} (f^2(x) - f(x^2)). \end{aligned} \quad (5)$$

Moreover,

$$\begin{aligned} g(x) &= \sum_{b \in B} x^b = \sum_{b \in (\mathbb{N} \setminus A) \cup (A \cap B)} x^b \\ &= \sum_{n=0}^{\infty} x^n - f(x) + \sum_{k=0}^{\infty} x^{8k} \\ &= \frac{1}{1-x} + \frac{1}{1-x^8} - f(x), \\ \sum_{n=0}^{\infty} R_2(B, n) x^n &= \frac{1}{2} (g^2(x) - g(x^2)) \\ &= \frac{1}{2} \left(\left(\frac{1}{1-x} + \frac{1}{1-x^8} - f(x) \right)^2 \right. \\ &\quad \left. - \left(\frac{1}{1-x^2} + \frac{1}{1-x^{16}} - f(x^2) \right) \right) \\ &= \frac{x}{(1-x)(1-x^2)} + \frac{x^8}{(1-x^8)(1-x^{16})} \\ &\quad + \frac{1}{(1-x)(1-x^8)} - \frac{f(x)}{1-x} - \frac{f(x)}{1-x^8} \\ &\quad + \frac{1}{2} f^2(x) + \frac{1}{2} f(x^2). \end{aligned} \quad (6)$$

Therefore,

$$\begin{aligned} \sum_{n=0}^{\infty} (R_2(A, n) - R_2(B, n)) x^n \\ &= \frac{f(x)}{1-x} + \frac{f(x)}{1-x^8} - f(x^2) - \frac{x}{(1-x)(1-x^2)} \\ &\quad - \frac{x^8}{(1-x^8)(1-x^{16})} - \frac{1}{(1-x)(1-x^8)}. \end{aligned} \quad (7)$$

Write

$$p(x)(1-x^8) = \sum_{i=0}^{16N+6} \alpha_i x^i. \quad (8)$$

By (3) and (7), we have

$$\begin{aligned} \left(\sum_{i=0}^7 x^i + 1 \right) f(x) \\ &= (1-x^8) f(x^2) + \frac{1+x+x^3+x^5+x^7}{1-x} \\ &\quad + \frac{x^8}{1-x^{16}} + p(x)(1-x^8). \end{aligned} \quad (9)$$

Writing both sides of (9) in power series, then we have

$$\begin{aligned} \sum_{i=0}^{\infty} 2\varepsilon_i x^i + \sum_{i=0}^{\infty} \varepsilon_i x^{i+1} + \cdots + \sum_{i=0}^{\infty} \varepsilon_i x^{i+7} \\ &= \sum_{i=0}^{\infty} \varepsilon_i x^{2i} - \sum_{i=0}^{\infty} \varepsilon_i x^{2i+8} + \sum_{i=0}^{\infty} x^i + \sum_{i=0}^{\infty} x^{i+1} + \sum_{i=0}^{\infty} x^{i+3} \\ &\quad + \sum_{i=0}^{\infty} x^{i+5} + \sum_{i=0}^{\infty} x^{i+7} + \sum_{i=0}^{\infty} x^{16i+8} + \sum_{i=0}^{16N+6} \alpha_i x^i. \end{aligned} \quad (10)$$

Comparing the coefficients of x^{16N+t} , $t = 7, 9, 11, 13, 15, 17, 19, 21, 23, 25$ on the both sides of (10), we have

$$2\varepsilon_{16N+t} + \varepsilon_{16N+t-1} + \cdots + \varepsilon_{16N+t-7} = 5, \quad t = 7, 9, \dots, 25. \quad (11)$$

Then we have

$$2\varepsilon_{16N+7} + \varepsilon_{16N+6} + \cdots + \varepsilon_{16N} = 5, \quad (12)$$

$$2\varepsilon_{16N+9} = \varepsilon_{16N+7} + \varepsilon_{16N+1}, \quad (13)$$

$$2\varepsilon_{16N+11} + \varepsilon_{16N+10} = \varepsilon_{16N+9} + \varepsilon_{16N+3} + \varepsilon_{16N+2}, \quad (14)$$

$$2\varepsilon_{16N+13} + \varepsilon_{16N+12} = \varepsilon_{16N+11} + \varepsilon_{16N+5} + \varepsilon_{16N+4}, \quad (15)$$

$$2\varepsilon_{16N+15} + \varepsilon_{16N+14} = \varepsilon_{16N+13} + \varepsilon_{16N+7} + \varepsilon_{16N+6}, \quad (16)$$

$$2\varepsilon_{16N+17} = \varepsilon_{16N+15} + \varepsilon_{16N+9}, \quad (17)$$

$$2\varepsilon_{16N+19} + \varepsilon_{16N+18} = \varepsilon_{16N+17} + \varepsilon_{16N+11} + \varepsilon_{16N+10}, \quad (18)$$

$$2\varepsilon_{16N+21} + \varepsilon_{16N+20} = \varepsilon_{16N+19} + \varepsilon_{16N+13} + \varepsilon_{16N+12}, \quad (19)$$

$$2\varepsilon_{16N+23} + \varepsilon_{16N+22} = \varepsilon_{16N+21} + \varepsilon_{16N+15} + \varepsilon_{16N+14}, \quad (20)$$

$$2\varepsilon_{16N+25} = \varepsilon_{16N+23} + \varepsilon_{16N+17}. \quad (21)$$

Since $A \cap B = \{8k : k \in \mathbb{N}\}$, we have $\varepsilon_{16N+m} = 1$, $m = 0, 8, 16, 24$. Then, by (12), $(\varepsilon_{16N+1}, \varepsilon_{16N+2},$

$\varepsilon_{16N+3}, \varepsilon_{16N+4}, \varepsilon_{16N+5}, \varepsilon_{16N+6}, \varepsilon_{16N+7}$ has the following 30 cases:

$$\begin{aligned}
 &(0, 0, 0, 0, 1, 1, 1), (0, 0, 0, 1, 0, 1, 1), (0, 0, 0, 1, 1, 0, 1), \\
 &(0, 0, 1, 0, 0, 1, 1), (0, 0, 1, 0, 1, 0, 1), (0, 0, 1, 1, 0, 0, 1), \\
 &(0, 0, 1, 1, 1, 0, 0), (0, 1, 0, 0, 0, 1, 1), (0, 1, 0, 0, 1, 0, 1), \\
 &(0, 1, 0, 1, 0, 0, 1), (0, 1, 0, 1, 1, 1, 0), (0, 1, 1, 0, 0, 0, 1), \\
 &(0, 1, 1, 0, 1, 1, 0), (0, 1, 1, 1, 0, 1, 0), (0, 1, 1, 1, 1, 0, 0), \\
 &(1, 0, 0, 0, 0, 1, 1), (1, 0, 0, 0, 1, 0, 1), (1, 0, 0, 1, 0, 0, 1), \\
 &(1, 0, 0, 1, 1, 1, 0), (1, 0, 1, 0, 0, 0, 1), (1, 0, 1, 0, 1, 1, 0), \\
 &(1, 0, 1, 1, 0, 1, 0), (1, 0, 1, 1, 1, 0, 0), (1, 1, 0, 0, 0, 0, 1), \\
 &(1, 1, 0, 0, 1, 1, 0), (1, 1, 0, 1, 0, 1, 0), (1, 1, 0, 1, 1, 0, 0), \\
 &(1, 1, 1, 0, 0, 1, 0), (1, 1, 1, 0, 1, 0, 0), (1, 1, 1, 1, 0, 0, 0).
 \end{aligned} \tag{22}$$

By (13), we have

$$\varepsilon_{16N+9} = \varepsilon_{16N+7} = \varepsilon_{16N+1} = 0, \tag{23}$$

or

$$\varepsilon_{16N+9} = \varepsilon_{16N+7} = \varepsilon_{16N+1} = 1. \tag{24}$$

Thus $(\varepsilon_{16N+2}, \varepsilon_{16N+3}, \varepsilon_{16N+4}, \varepsilon_{16N+5}, \varepsilon_{16N+6}, \varepsilon_{16N+7}, \varepsilon_{16N+9})$ has the following 10 cases:

$$\begin{aligned}
 &(0, 1, 1, 1, 1, 0, 0), (1, 0, 1, 1, 1, 0, 0), (1, 1, 0, 1, 1, 0, 0), \\
 &(1, 1, 1, 0, 1, 0, 0), (1, 1, 1, 1, 0, 0, 0), (0, 0, 0, 0, 1, 1, 1), \\
 &(0, 0, 0, 1, 0, 1, 1), (0, 0, 1, 0, 0, 1, 1), (0, 1, 0, 0, 0, 1, 1), \\
 &(1, 0, 0, 0, 0, 1, 1).
 \end{aligned} \tag{25}$$

By (14) and (25), we have $(\varepsilon_{16N+4}, \varepsilon_{16N+5}, \varepsilon_{16N+6}, \varepsilon_{16N+7}, \varepsilon_{16N+9}, \varepsilon_{16N+10}, \varepsilon_{16N+11})$ which has the following 8 cases:

$$\begin{aligned}
 &(1, 1, 1, 0, 0, 1, 0), (0, 1, 1, 0, 0, 0, 1), (1, 0, 1, 0, 0, 0, 1), \\
 &(1, 1, 0, 0, 0, 0, 1), (0, 0, 1, 1, 1, 1, 0), (0, 1, 0, 1, 1, 1, 0), \\
 &(1, 0, 0, 1, 1, 1, 0), (0, 0, 0, 1, 1, 0, 1).
 \end{aligned} \tag{26}$$

By (15) and (26), we have $(\varepsilon_{16N+6}, \varepsilon_{16N+7}, \varepsilon_{16N+9}, \varepsilon_{16N+10}, \varepsilon_{16N+11}, \varepsilon_{16N+12}, \varepsilon_{16N+13})$ which has the following 6 cases:

$$\begin{aligned}
 &(1, 0, 0, 1, 0, 0, 1), (1, 0, 0, 0, 1, 0, 1), (0, 0, 0, 0, 1, 1, 1), \\
 &(1, 1, 1, 1, 0, 0, 0), (0, 1, 1, 1, 0, 1, 0), (0, 1, 1, 0, 1, 1, 0).
 \end{aligned} \tag{27}$$

By (16) and (27), we have $(\varepsilon_{16N+9}, \varepsilon_{16N+10}, \varepsilon_{16N+11}, \varepsilon_{16N+12}, \varepsilon_{16N+13}, \varepsilon_{16N+14}, \varepsilon_{16N+15})$ which has the following 6 cases:

$$\begin{aligned}
 &(0, 1, 0, 0, 1, 0, 1), (0, 0, 1, 0, 1, 0, 1), (0, 0, 1, 1, 1, 1, 0), \\
 &(1, 1, 0, 0, 0, 0, 1), (1, 1, 0, 1, 0, 1, 0), (1, 0, 1, 1, 0, 1, 0).
 \end{aligned} \tag{28}$$

By (17) and (28), we have $(\varepsilon_{16N+10}, \varepsilon_{16N+11}, \varepsilon_{16N+12}, \varepsilon_{16N+13}, \varepsilon_{16N+14}, \varepsilon_{16N+15}, \varepsilon_{16N+17})$ which has the following 2 cases:

$$(0, 1, 1, 1, 1, 0, 0), (1, 0, 0, 0, 0, 1, 1). \tag{29}$$

By (18) and (29), we have $(\varepsilon_{16N+12}, \varepsilon_{16N+13}, \varepsilon_{16N+14}, \varepsilon_{16N+15}, \varepsilon_{16N+17}, \varepsilon_{16N+18}, \varepsilon_{16N+19})$ which has the following 2 cases:

$$(1, 1, 1, 0, 0, 1, 0), (0, 0, 0, 1, 1, 0, 1). \tag{30}$$

By (19) and (30), we have $(\varepsilon_{16N+14}, \varepsilon_{16N+15}, \varepsilon_{16N+17}, \varepsilon_{16N+18}, \varepsilon_{16N+19}, \varepsilon_{16N+20}, \varepsilon_{16N+21})$ which has the following 2 cases:

$$(1, 0, 0, 1, 0, 0, 1), (0, 1, 1, 0, 1, 1, 0). \tag{31}$$

By (20) and (31), we have $(\varepsilon_{16N+17}, \varepsilon_{16N+18}, \varepsilon_{16N+19}, \varepsilon_{16N+20}, \varepsilon_{16N+21}, \varepsilon_{16N+22}, \varepsilon_{16N+23})$ which has the following 2 cases:

$$(0, 1, 0, 0, 1, 0, 1), (1, 0, 1, 1, 0, 1, 0). \tag{32}$$

By (21), we have

$$\varepsilon_{16N+25} = \varepsilon_{16N+23} = \varepsilon_{16N+17} = 0, \tag{33}$$

or

$$\varepsilon_{16N+25} = \varepsilon_{16N+23} = \varepsilon_{16N+17} = 1, \tag{34}$$

which contradicts (32).

This completes the proof of Theorem 1. $\square$

3. Proof of Theorem 2

Proof. Suppose that there exist integer n_0 and sets $A, B \subset \mathbb{N}$ with $A \cup B = \mathbb{N}$ and $A \cap B = \{8k : k \in \mathbb{N}\}$ such that $R_3(A, n) = R_3(B, n)$ for all $n \geq n_0$. Without loss of generality, we may assume that $n_0 = 16N - 1$, N is a positive integer then there exists a polynomial $q(x)$ of degree at most $16N - 2$ such that

$$\sum_{n=0}^{\infty} (R_3(A, n) - R_3(B, n)) x^n = q(x). \tag{35}$$

Define $f(x), g(x)$, and ε_i as in the proof of Theorem 1. Then we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} R_3(A, n) x^n &= \frac{1}{2} (f^2(x) + f(x^2)), \\
 \sum_{n=0}^{\infty} R_3(B, n) x^n &= \frac{1}{2} (g^2(x) + g(x^2)) \\
 &= \frac{1}{2} \left(\left(\frac{1}{1-x} + \frac{1}{1-x^8} - f(x) \right)^2 \right. \\
 &\quad \left. + \left(\frac{1}{1-x^2} + \frac{1}{1-x^{16}} - f(x^2) \right) \right) \\
 &= \frac{1}{(1-x)(1-x^2)} + \frac{1}{(1-x^8)(1-x^{16})} \\
 &\quad + \frac{1}{(1-x)(1-x^8)} - \frac{f(x)}{1-x} - \frac{f(x)}{1-x^8} \\
 &\quad + \frac{1}{2} f^2(x) - \frac{1}{2} f(x^2).
 \end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned} & \sum_{n=0}^{\infty} (R_3(A, n) - R_3(B, n)) x^n \\ &= \frac{f(x)}{1-x} + \frac{f(x)}{1-x^8} + f(x^2) - \frac{1}{(1-x)(1-x^2)} \\ & \quad - \frac{1}{(1-x^8)(1-x^{16})} - \frac{1}{(1-x)(1-x^8)}. \end{aligned} \quad (37)$$

Write

$$q(x)(1-x^8) = \sum_{i=0}^{16N+6} \beta_i x^i. \quad (38)$$

By (35) and (37), we have

$$\begin{aligned} \left(\sum_{i=0}^7 x^i + 1 \right) f(x) &= (x^8 - 1) f(x^2) \\ & \quad + \frac{2 + x^2 + x^4 + x^6}{1-x} \\ & \quad + \frac{1}{1-x^{16}} + q(x)(1-x^8). \end{aligned} \quad (39)$$

We write both sides of (39) in power series; then we have

$$\begin{aligned} & \sum_{i=0}^{\infty} 2\varepsilon_i x^i + \sum_{i=0}^{\infty} \varepsilon_i x^{i+1} + \cdots + \sum_{i=0}^{\infty} \varepsilon_i x^{i+7} \\ &= \sum_{i=0}^{\infty} \varepsilon_i x^{2i+8} - \sum_{i=0}^{\infty} \varepsilon_i x^{2i} + \sum_{i=0}^{\infty} 2x^i + \sum_{i=0}^{\infty} x^{i+2} \\ & \quad + \sum_{i=0}^{\infty} x^{i+4} + \sum_{i=0}^{\infty} x^{i+6} + \sum_{i=0}^{\infty} x^{16i} + \sum_{i=0}^{16N+6} \beta_i x^i. \end{aligned} \quad (40)$$

Comparing the coefficients of x^{16N+t} , $t = 7, 9, 11, 13, 15, 17, 19, 21, 23, 25$ on the both sides of (40), we have

$$2\varepsilon_{16N+t} + \varepsilon_{16N+t-1} + \cdots + \varepsilon_{16N+t-7} = 5, \quad t = 7, 9, \dots, 25. \quad (41)$$

The remainder of the proof is the same as that of the proof of Theorem 1. We omit it here. $\square$

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