Research Article

A New Upper Bound for $\|A^{-1}\|$ of a Strictly $\alpha$-Diagonally Dominant $M$-Matrix

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1. Introduction

The estimation for the bound for the norm $\|A^{-1}\|$ of a real invertible $n \times n$ matrix $A$ is important in numerical analysis, so many researchers were devoted to studying this kind of problems. For example, Varah [1] discussed the bound for the infinity norm $\|A^{-1}\|_\infty$ of a strictly diagonally dominant matrix $A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$ and obtained the following estimation:

$$\|A^{-1}\|_\infty \leq \max_i \left\{ \frac{1}{|a_{ii}| - \sum_{j=1, j \neq i}^{|a_{ij}|}} \right\}, \quad i \in N. \quad (1)$$

After that Varga [2] extended the result of [1] to $H$-matrices. Evidently, the upper bound for $\|A^{-1}\|_\infty$ in (1) only involves the entries in the matrix $A$. If the diagonal dominance of $A$ is weak, that is, $\min |a_{ii}| - \sum_{j=1, j \neq i} |a_{ij}|$ is small, then the bound given by (1) may be large. For this reason, some authors were devoted to improving the result of (1). Recently, Cheng and Huang [3] presented a more compacted upper bound for a strictly diagonally dominant $M$-matrix

$$\|A^{-1}\|_\infty \leq \frac{1}{a_{11} (1 - u_1 d_1)} + \sum_{i=2}^n \left[ \frac{1}{a_{ii} (1 - u_i d_i)} \right]^{i-1} \left( \frac{1}{1 - u_j f_j} \right),$$

and then Wang [4] further improved this bound and gave the following result:

$$\|A^{-1}\|_\infty \leq \frac{1}{a_{11} (1 - u_1 d_1)} + \sum_{i=2}^n \left[ \frac{1}{a_{ii} (1 - u_i d_i)} \right]^{i-1} \left( \frac{1}{1 - u_j f_j} \right),$$

where notations in (2) and (3) have the same meanings as those used in this paper, which will be shown later.

In this paper, we present a new upper bound $\|A^{-1}\|_\infty$ of a strictly diagonally dominant matrix $A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$, which is better than that obtained by Wang, and a new lower bound of the smallest eigenvalue $\lambda_{\min}(A)$ of $A$ is also obtained. Furthermore, an upper bound for $\|A^{-1}\|$ of a real strictly $\alpha$-diagonally dominant $M$-matrix is shown.
a positive eigenvalue of \( A \) related to the Perron eigenvalue of the nonnegative matrix \( A^{-1} \). If \( q(A) \) denotes the minimum of the real parts of the eigenvalues of \( A \), that is, \( q(A) = \alpha - \rho(B) \), then \( q(A) = 1/\rho(A^{-1}) \). For further properties of the \( M \)-matrix \( A \), we refer the readers to [5–7].

An \( n \times n \) matrix \( A = (a_{ij}) \) is called a strictly diagonally dominant matrix if \( |a_{ii}| > \sum_{j \neq i} |a_{ij}| \) for \( i \in N \). Let

\[
R_i(A) = \sum_{j=1, j \neq i}^n |a_{ij}|, \quad r_i(A) = \sum_{j=1}^n |a_{ij}|
\]

\[
d_i = \frac{1}{|a_{ii}|} \sum_{j=1, j \neq i}^n |a_{ij}|, \quad J(A) = \{ i \in N : d_i < 1 \},
\]

\[
u_i = \frac{1}{|a_{ii}|} \sum_{j=1}^n |a_{ij}|
\]

\[
l_k = \max_{k \leq n} \frac{\sum_{i,k,l \in J} |a_{ij}|}{|a_{ik}|}, \quad l_n = u_n = 0,
\]

\[
w_{ij} = \frac{|a_{ij}|}{|a_{ij}| - \sum_{k \neq j} |a_{ik}|}, \quad i \neq j, \quad j < k \leq n,
\]

\[
w_i = \max_{j \neq i} \{ w_{ij} \}, \quad C_i(A) = \sum_{j=1, j \neq i}^n |a_{ij}|,
\]

\[
m_j = \frac{|a_{ij}| + \sum_{k \neq j} |a_{ik}| w_{ki}}{|a_{ij}|}, \quad i \neq j, \quad j < k \leq n,
\]

where \( N \) is the set of positive integers. For an \( n \times n \) matrix \( A \), the principal matrix of \( A \) formed by rows and columns with indices between \( n_1 \) and \( n_2 \) is denoted by \( A^{[n_1,n_2]} \).

**Definition 1** (see [8]). \( A \in R^{n \times n} \) is weakly chained diagonally dominant if, for all \( i \in N \), \( d_i \leq 1 \) and \( J(A) \neq \emptyset \) and for all \( i \in N \), \( i \neq J(A) \), there exist indices \( i_1, i_2, \ldots, i_k \) in \( N \) with \( a_{i_1,i_2} \neq 0, 0 \leq r \leq k - 1 \), where \( i_0 = i \) and \( i_k \in J(A) \).

**Definition 2** (see [9]). Let \( A \in R^{n \times n} \), \( A \) is strictly diagonally dominant if \( J(A) = N \).

Obviously, if \( A \in R^{n \times n} \) is a strictly diagonally dominant matrix, then \( A \) is a weakly chained diagonally dominant matrix.

**Definition 3** (see [9]). \( A \in R^{n \times n} \) is an \( L \)-matrix if, for all \( i, j \in N \) with \( i \neq j \), \( a_{ij} \leq 0 \) and \( a_{ii} > 0 \).

**Definition 4** (see [10]). Let \( A \in R^{n \times n} \); if there exist \( \alpha \in [0, 1] \), such that

\[
|a_{ii}| \geq \alpha R_i(A) + (1 - \alpha) C_i(A),
\]

for all \( i \in N \), then \( A \) is said to be an \( \alpha \)-diagonal dominant matrix, denoted by \( D^\alpha_n \).

**Remark 5.** By Definition 4, we know that \( A \) is just a diagonal dominant matrix while \( \alpha = 1 \).

**Definition 6.** If all the inequalities in (5) strictly hold, then \( A \) is said to be strictly \( \alpha \)-diagonal dominant matrix (SD\(^\alpha\)_n).

### 2. Estimation for an Upper Bound for \( \|A^{-1}\|_\infty \) of Strictly Diagonally Dominant \( M \)-Matrix

We state some lemmas before giving a new upper bound for \( \|A^{-1}\|_\infty \).

**Lemma 7** (see [3]). Let \( A = (a_{ij}) \) be an \( n \times n \) weakly chained diagonally dominant \( M \)-matrix, \( B = A^{2n}, A^{-1} = (\alpha_j)_{i,j=1}^n \), and \( B^{-1} = (\beta_j)_{i,j=1}^n \). Then, for \( i, j = 1, 2, \ldots, n \),

\[
\alpha_{i1} = \frac{1}{\Delta},
\]

\[
\alpha_{i1} = \frac{\sum_{k=2}^n \beta_{k1}}{\beta_{i1}}, \quad \alpha_{ij} = \frac{\sum_{k=2}^n \beta_{kj} - \alpha_{jk}}{\beta_{ij}},
\]

where

\[
\Delta = a_{i1} - \sum_{k=2}^n \beta_{ki} a_{i1} > 0.
\]

Furthermore, if \( J(A) = N \), then \( \Delta \geq a_{i1}(1 - d_i l_1) \geq a_{i1}(1 - d_i) \).

**Lemma 8** (see [11]). A weakly chained diagonally dominant \( L \)-matrix is a nonsingular \( M \)-matrix.

**Lemma 9** (see [11]). Let \( A = (a_{ij}) \) be an \( n \times n \) weakly chained diagonally dominant \( M \)-matrix; then \( B = A^{2n} \) is an \((n-1)\times(n-1)\) weakly chained diagonally dominant \( M \)-matrix; that is, \( B^{-1} = (\beta_{ij}) \) exists and \( \beta_{ij} \geq 0 \) (\( i, j = 2, 3, \ldots, n \)).

**Lemma 10** (see [11]). Let \( A = (a_{ij}) \) be an \( n \times n \) weakly chained diagonally dominant \( M \)-matrix, \( A^{-1} = (\alpha_{ij}) \). Then, for \( i \neq j \),

\[
\alpha_{ij} \leq d_i \alpha_{jj} \leq \alpha_{ij}.
\]

**Lemma 11** (see [11]). Let \( A = (a_{ij}) \) be an \( n \times n \) row strictly diagonally dominant \( M \)-matrix; then

\[
\Delta \geq a_{i1}(1 - d_i l_1) > a_{i1}(1 - d_i) > 0.
\]

**Lemma 12** (see [2]). Let \( A = (a_{ij}) \) be an \( n \times n \) row strictly diagonally dominant \( M \)-matrix; then, for \( A^{-1} = (\alpha_{ij})_{i,j=1}^n \), we have

\[
\frac{1}{a_{ij}} \leq \alpha_{ij} \leq \frac{1}{a_{ij} - \sum_{j \neq i} |a_{ij}| m_{ji}}.
\]
Lemma 13 (see [1]). Let $A = (a_{ij})$ be an $n \times n$ weakly chained diagonally dominant $M$-matrix, $A^{-1} = (\alpha_{ij})$, and $q = q(A)$, $N = 1, 2, \ldots, n$. Then

$$q \leq \min_{i \in N} |a_{ii}|, \quad q \leq \max_{i \in N} \left\{ \sum_{j \in N} a_{ij} \right\}, \quad q \geq \min_{i \in N} \left\{ \sum_{j \in N} a_{ij} \right\},$$

$$\frac{1}{M} \leq q \leq \frac{1}{m}$$

where

$$M = \max_{i \in N} \left\{ \sum_{j \in N} a_{ij} \right\} = \|A^{-1}\|_{\infty}, \quad m = \min_{i \in N} \left\{ \sum_{j \in N} a_{ij} \right\}.$$  \hspace{1cm} (11)

Now we give an upper bound for $\|A^{-1}\|_{\infty}$ and $q(A)$ of a strictly diagonally dominant $M$-matrix $A$ by the following theorem.

Theorem 14. Let $A = (a_{ij})$ be an $n \times n$ row strictly diagonally dominant $M$-matrix, $A^{-1} = (\alpha_{ij})$. Then

$$\|A^{-1}\|_{\infty} \leq \frac{1}{a_{11} - \sum_{k=2}^{n} |a_{1k}| m_{k1}} + \sum_{j=2}^{n} \left[ \frac{1}{a_{ij} - \sum_{k=2}^{n} |a_{ik}| m_{k1}} \right] m_{k1} \prod_{j=1}^{i-1} \left( 1 - u_{j} l_{j} \right).$$  \hspace{1cm} (13)

Proof. We prove this theorem by induction.

(1) Let $r_{j} = \sum_{i=1}^{n} a_{ij}, B = A^{(2,n)}, M_{A} = \|A^{-1}\|_{\infty}$, and $M_{B} = \|B^{-1}\|_{\infty}$. Then

$$M_{A} = \max \left\{ r_{i} : i \in N \right\},$$

$$M_{B} = \max \left\{ \sum_{j=2}^{n} \beta_{ji} : 2 \leq i \leq n \right\}.$$  \hspace{1cm} (14)

By Lemmas 7, 11, and 12, we know that

$$r_{i} = \alpha_{i1} + \sum_{j=2}^{n} \alpha_{ij}$$

$$= \frac{1}{\Delta} + \sum_{k=2}^{n} \frac{1}{\Delta} \left( -\alpha_{k1} \right)$$

$$= \frac{1}{\Delta} \left( 1 + \sum_{k=2}^{n} \left( -\alpha_{k1} \right) \sum_{j=2}^{n} \beta_{kj} \right)$$

$$\leq \frac{1}{\Delta} \left( 1 + a_{11} \cdot d_{1} \cdot M_{B} \right) \leq \frac{1}{\Delta} + \frac{d_{1} M_{B}}{1 - d_{1} l_{1}}$$

$$\leq \frac{1}{\Delta} + \frac{M_{B}}{1 - d_{1} l_{1}}$$

$$\leq \frac{1}{a_{11} - \sum_{k=2}^{n} |a_{1k}| m_{k1}} + \frac{M_{B}}{1 - d_{1} l_{1}}.$$  \hspace{1cm} (15)

Let $2 \leq i \leq n$. By (8) and the second equality in (6), we have

$$\sum_{k=2}^{n} \beta_{ik} (-\alpha_{k1}) = \Delta \alpha_{i1} \leq \Delta d_{i} \alpha_{i1} = d_{i} < 1.$$  \hspace{1cm} (16)

From (8) with $2 \leq j \leq n$, we have

$$\alpha_{ij} \leq \beta_{ij} + \alpha_{1j} d_{i} < \beta_{ij} + \alpha_{1j}.$$  \hspace{1cm} (17)

Thus, for $2 \leq i \leq n$, we obtain

$$r_{i} = \alpha_{i1} + \sum_{j=2}^{n} \alpha_{ij}$$

$$\leq d_{i} \alpha_{i1} + \sum_{j=2}^{n} \left( \beta_{ij} + \alpha_{1j} d_{i} \right)$$

$$= d_{i} \alpha_{i1} + \sum_{j=2}^{n} \beta_{ij} + \sum_{j=2}^{n} \alpha_{1j} d_{i}$$

$$\leq r_{i} d_{i} + \sum_{j=2}^{n} \beta_{ij}$$

$$\leq r_{i} d_{i} + \sum_{j=2}^{n} \beta_{ij}$$

$$\leq r_{i} d_{i} + \sum_{j=2}^{n} \beta_{ij}$$

$$\leq r_{i} l_{1} + M_{B}$$

$$\leq \left\{ \frac{1}{\Delta} + \frac{d_{i} M_{B}}{1 - d_{1} l_{1}} \right\} l_{1} + M_{B}$$

$$\leq \frac{l_{1}}{\Delta} + \frac{d_{i} M_{B}}{1 - d_{1} l_{1}} + M_{B}$$

$$\leq \frac{1}{\Delta} + \frac{M_{B}}{1 - d_{1} l_{1}}$$

$$\leq \frac{1}{a_{11} - \sum_{k=2}^{n} |a_{1k}| m_{k1}} + \frac{M_{B}}{1 - d_{1} l_{1}}.$$  \hspace{1cm} (18)

So by (15) and (18), we get

$$\|A^{-1}\|_{\infty} \leq \frac{1}{a_{11} - \sum_{k=2}^{n} |a_{1k}| m_{k1}} + \frac{M_{B}}{1 - d_{1} l_{1}}.$$  \hspace{1cm} (19)

(2) Applying induction with respect to $k$ of $A^{(k,n)}$ in (19) finishes the proof. \hspace{1cm} \Box

From Theorem 14 and Lemma 13, the following theorem can be obtained easily.

Theorem 15. Let $A = (a_{ij})$ be an $n \times n$ row strictly diagonally dominant $M$-matrix. Then the smallest eigenvalue of $A$ is

$$q(A) \geq \left\{ \frac{1}{a_{11} - \sum_{k=2}^{n} |a_{1k}| m_{k1}} + \frac{M_{B}}{1 - d_{1} l_{1}} \right\}^{-1}.$$  \hspace{1cm} (20)
Theorem 16. Let $A = (a_{ij})$ be an $n \times n$ row strictly diagonally dominant $M$-matrix. Then the bound in (13) is sharper than that in (3), that is,

$$
\frac{1}{a_{ii} - \sum_{k=2}^{n} |a_{ik}| m_{ki}}
$$

$$
+ \sum_{i=2}^{n} \left[ \frac{1}{a_{ii} - \sum_{k \neq i, j \leq n} |a_{ik}| m_{ki}} \prod_{j=1}^{i-1} \frac{1}{1 - u_{ij}} \right]
$$

$$
\leq \frac{1}{a_{ii} (1 - u_{ii})} + \sum_{i=2}^{n} \left[ \frac{1}{a_{ii} (1 - u_{ii})} \prod_{j=1}^{i-1} \frac{1}{1 - u_{ij}} \right].
$$

(21)

Proof. Since $A$ is a strictly diagonally dominant matrix, $0 \leq d_k < 1, m_k \leq d_k < 1$, and $1 \leq j \leq n - 1$, then we have

$$
\frac{1}{a_{ii} - \sum_{k=2}^{n} |a_{ik}| m_{ki}} \leq \frac{1}{a_{ii} (1 - u_{ii})}.
$$

(22)

The results follow Lemma 12. Inequality (21) shows that the bound in (13) is better than that in (3).

For all $i$, $\max_{j \leq n} \{1/(a_{ii} - \sum_{k=2}^{n} |a_{ik}| m_{ki})\} < \max_{i \leq k \leq n} \{1/ a_{ii}(1 - u_{ii}d_i)\}$, we have

$$
\frac{1}{a_{ii} - \sum_{k=2}^{n} |a_{ik}| m_{ki}}
$$

$$
+ \sum_{i=2}^{n} \left[ \frac{1}{a_{ii} - \sum_{k \neq i, j \leq n} |a_{ik}| m_{ki}} \prod_{j=1}^{i-1} \frac{1}{1 - u_{ij}} \right]
$$

$$
< \frac{1}{a_{ii} (1 - u_{ii}d_i)} + \sum_{i=2}^{n} \left[ \frac{1}{a_{ii} (1 - u_{ii})} \prod_{j=1}^{i-1} \frac{1}{1 - u_{ij}} \right].
$$

(23)

With the help of the above discussions, we give the upper bound for $\|A^{-1}\|_\infty$ of a real strictly $\alpha$-diagonally dominant $M$-matrix.

3. Estimation for an Upper Bound for $\|A^{-1}\|_\infty$ of a Strictly $\alpha$-Diagonally Dominant $M$-Matrix

We show some notations and lemmas which are necessary to our conclusions.

Lemma 17 (see [12]). Let $A, B \in \mathbb{R}_{\text{non}}^{n \times n}$, $A - B$ be nonsingular, then

$$(A - B)^{-1} = A^{-1} + A^{-1}B(I - A^{-1}B)^{-1}A^{-1}. \quad (24)$$

Lemma 18. Let $A = (a_{ij}) \in \mathbb{R}_{\text{non}}^{n \times n}$ be a strictly diagonal dominant $M$-matrix. If $B = (b_{ij}) \in \mathbb{R}_{\text{non}}^{n \times n}$, with

$$
\|A^{-1}B\|_\infty \leq \max_{1 \leq i \leq n} \|B\|_\infty,
$$

(25)

and if

$$
\kappa_0 < \frac{1}{\|B\|_\infty},
$$

(26)

then $\|A^{-1}B\|_\infty < 1$, where

$$
\kappa_0 = \frac{1}{a_{11} - \sum_{k=2}^{n} |a_{1k}| m_{k1}}
$$

$$
+ \sum_{i=2}^{n} \left[ \frac{1}{a_{ii} - \sum_{k \neq i, j \leq n} |a_{ik}| m_{ki}} \prod_{j=1}^{i-1} \frac{1}{1 - u_{ij}} \right].
$$

(27)

Proof. By Theorem 14, we get

$$
\|A^{-1}B\|_\infty \leq \|A^{-1}\|_\infty \cdot \|B\|_\infty \leq \max_{1 \leq i \leq n} \kappa_0 \cdot \|B\|_\infty.
$$

(28)

It is easy to see that $\|A^{-1}B\|_\infty < 1$, if

$$
\kappa_0 < \frac{1}{\|B\|_\infty},
$$

(29)

where

$$
\kappa_0 = \frac{1}{a_{11} - \sum_{k=2}^{n} |a_{1k}| m_{k1}}
$$

$$
+ \sum_{i=2}^{n} \left[ \frac{1}{a_{ii} - \sum_{k \neq i, j \leq n} |a_{ik}| m_{ki}} \prod_{j=1}^{i-1} \frac{1}{1 - u_{ij}} \right].
$$

(30)

Lemma 19 (see [12]). If $\|A^{-1}\|_\infty < 1$, then $I - A$ is nonsingular and

$$
\|I - A\|_\infty < \frac{1}{1 - \|A\|_\infty}.
$$

(31)

Theorem 20. Let $A = (a_{ij}) \in \mathbb{R}_{\text{non}}^{n \times n}$ be a strictly $\alpha$-diagonal dominant matrix, $\alpha \in (0, 1]$, and $A$ be an $M$-matrix. If, for those $i \in N_1 \subset N$, $R_i(A) > C_i(A)$, and $\kappa_1 < 1/ \max_{1 \leq i \leq n} \alpha(R_i(A) - C_i(A))$, then

$$
\|A^{-1}\|_\infty \leq \frac{k_1}{1 - \kappa_1 \max_{1 \leq i \leq n} \alpha(R_i(A) - C_i(A))},
$$

(32)

where

$$
k_1 = \frac{1}{\beta_1 - \sum_{k=2}^{n} |a_{1k}| m_{k1}}
$$

$$
+ \sum_{i=2}^{n} \left[ \frac{1}{\beta_i - \sum_{k \neq i, j \leq n} |a_{ik}| m_{ki}} \prod_{j=1}^{i-1} \frac{1}{1 - u_{ij}} \right],
$$

$$
\beta_i = \max \{a_{ii}, a_{ii} + \alpha (R_i(A) - C_i(A))\}, \quad i = 1, 2, \ldots, n.
$$

(33)

Proof. Note that $R_i(A) > C_i(A)$. Then

$$
|a_{ii}| > (1 - \alpha)R_i(A) + \alpha C_i(A)
$$

$$
= R_i(A) - \alpha (R_i(A) - C_i(A)).
$$

(34)
So we can split $A$, such that $A = B - C$, where $B = (b_{ij})$ and $C = (c_{ij})$,

\[
\begin{align*}
b_{ij} &= \begin{cases} a_{ij} + \alpha(R_i(A) - C_i(A)) & i = j, R_i(A) > C_i(A) \\
a_{ij} & \text{others,}
\end{cases} \\
c_{ij} &= \begin{cases} \alpha(R_i(A) - C_i(A)) & i = j, R_i(A) > C_i(A) \\
0 & \text{others.}
\end{cases}
\end{align*}
\] (35)

We know $b_{ii} = a_{ii} + \alpha(R_i(A) - C_i(A)) > R_i(A) = R_i(B)$ and $A$ is an $M$-matrix. Thus, $B$ is a strictly diagonal dominant $M$-matrix; hence, $B^{-1} > 0$. Let $\beta_i = \max\{a_{ii}, a_{ii} + \alpha(R_i(A) - C_i(A))\}, i = 1, 2, \ldots, n$. If $\kappa_i < 1/\max_{1 \leq i \leq n}(R_i(A) - C_i(A))$, by Lemma 18, we get $\|B^{-1}\|_\infty \leq 1$. By Lemmas 17 and 19 and Theorem 14, we can obtain

\[
\begin{align*}
\|B^{-1}\|_\infty &\leq \frac{1}{b_{ii} - \sum_{k=2}^{n} |a_{ik}| m_{ki}} \\
&\quad + \sum_{j=2}^{n} \left[ \frac{1}{b_{ii} - \sum_{k=2}^{n} |a_{ik}| m_{ki}} \left( \frac{1}{1 - u_{ij}} \right)^{i-1} \right] \\
&\quad \leq \frac{1}{\beta_i - \sum_{k=2}^{n} |a_{ik}| m_{ki}} \\
&\quad + \sum_{j=2}^{n} \left[ \frac{1}{\beta_i - \sum_{k=2}^{n} |a_{ik}| m_{ki}} \left( \frac{1}{1 - u_{ij}} \right)^{i-1} \right].
\end{align*}
\] (36)

Let $\kappa_1 = 1/(\beta_1 - \sum_{k=2}^{n} |a_{ik}| m_{ki}) + \sum_{j=2}^{n} (1/(\beta_1 - \sum_{k=2}^{n} |a_{ik}| m_{ki})) [1/(1 - u_{ij})]^{i-1}$. Then

\[
\|B^{-1}C\|_\infty < \kappa_1 \max_{1 \leq i \leq n} |c_{ij}|
\] (37)

\[
< \kappa_1 \max_{1 \leq i \leq n} (R_i(A) - C_i(A)).
\]

Further, we have

\[
\|A^{-1}\|_\infty = \|(B - C)^{-1}\|_\infty
\]

\[
= \|B^{-1} + B^{-1}C(I - B^{-1}C)^{-1}B^{-1}\|_\infty
\]

\[
\leq \|B^{-1}\|_\infty + \|B^{-1}C\|_\infty \cdot \|(I - B^{-1}C)^{-1}\|_\infty \cdot \|B^{-1}\|_\infty
\]

\[
\leq \|B^{-1}\|_\infty + \frac{\|B^{-1}C\|_\infty}{1 - \|B^{-1}C\|_\infty} \|B^{-1}\|_\infty
\]

\[
= \frac{1}{1 - \|B^{-1}C\|_\infty} \|B^{-1}\|_\infty
\]

\[
\leq \frac{\kappa_1}{1 - \kappa_1 \max_{1 \leq i \leq n} (R_i(A) - C_i(A))},
\] (38)

where

\[
\kappa_1 = \frac{1}{\beta_1 - \sum_{k=2}^{n} |a_{ik}| m_{ki}} + \sum_{j=2}^{n} \left[ \frac{1}{\beta_1 - \sum_{k=2}^{n} |a_{ik}| m_{ki}} \left( \frac{1}{1 - u_{ij}} \right)^{i-1} \right],
\] (39)

\[
\beta_i = \max\{a_{ii}, a_{ii} + \alpha(R_i(A) - C_i(A))\},
\]

\[
i = 1, 2, \ldots, n.
\]

The proof is complete. \hfill \square

4. Examples

We illustrate our results by the following two examples.

(1) Consider the bound for $\|A^{-1}\|_\infty$ of a strictly diagonal dominant matrix $A$, where

\[
A = \begin{pmatrix} 10 & -1 & -1 & -1 & -1 \\ -1 & 10 & -1 & -1 & -1 \\ -1 & -1 & 10 & -1 & -1 \\ -1 & -1 & -1 & 10 & -1 \\ -1 & -1 & -1 & -1 & 10 \end{pmatrix}.
\] (40)

Direct calculation by MATLAB R2010a gives

\[
\|A^{-1}\|_\infty = 0.1669,
\]

\[
\|A^{-1}\|_\infty \leq 214.0217 \text{ (by Theorem 3.3 in [8])}
\]

\[
\|A^{-1}\|_\infty \leq 175.9183 \text{ (by (2))}
\] (41)

\[
\|A^{-1}\|_\infty \leq 9.2041 \text{ (by (3))}
\]

\[
\|A^{-1}\|_\infty \leq 6.5634 \text{ (by Theorem 14 (13))}.
\]

It is obvious that the bound of Theorem 14 of this paper is better than other known ones. Furthermore, we can estimate $q(A)$ by Theorem 15.

(2) Consider the bound for $\|A^{-1}\|_\infty$ of a strictly $\alpha$-diagonal dominant matrix $A$ for $\alpha = 0.5$,

\[
A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -0.5 & 0 & 2 \end{pmatrix},
\] (42)

\[
A^{-1} = \begin{pmatrix} 0.8889 & 0.4444 & 0.6667 \\ 0.5556 & 0.7778 & 0.6667 \\ 0.2222 & 0.1111 & 0.6667 \end{pmatrix}.
\]

Note that

\[
\|A^{-1}\|_\infty \approx 2.
\] (43)

We know that $A$ is not a strictly diagonal dominant matrix, and the bound of $\|A^{-1}\|_\infty$ cannot be obtained by (2) or (3), but it can be estimated by (32) in Theorem 20.
Split the matrix $A$ such that $A = B - C$, where $B = (b_{ij})$ and $b_{11} = a_{11} + \alpha (R_1(A) - C_1(A)) = 2 + 0.5 \times (2 - 1.5) = 2.25$, $b_{22} = a_{22} + \alpha (R_2(A) - C_2(A)) = 2 + 0.5 \times (2 - 1) = 2.5$. Then

$$B = \begin{pmatrix} 2.25 & -1 & -1 \\ -1 & 2.5 & -1 \\ -0.5 & 0 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (44)$$

The bound for $\|A^{-1}\|_\infty$ can be estimated by (13) in Theorem 14 and (32) in Theorem 20 as follows:

$$\|A^{-1}\|_\infty \leq 11.4259. \quad (45)$$

Conflict of Interests

There is no conflict of interests regarding the publication of this paper.

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References


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