Research Article

A Characterization of Projective Special Unitary Group $U_3(7)$ by nse

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Let G a group and $\omega(G)$ be the set of element orders of G . Let $k \in \omega(G)$ and let s_k be the number of elements of order k in G . Let $\text{nse}(G) = \{s_k \mid k \in \omega(G)\}$. In Khatami et al. and Liu's works, $L_3(2)$ and $L_3(4)$ are uniquely determined by $\text{nse}(G)$. In this paper, we prove that if G is a group such that $\text{nse}(G) = \text{nse}(U_3(7))$, then $G \cong U_3(7)$.

1. Introduction

A finite group G is called a simple K_n -group if G is a simple group with $|\pi(G)| = n$. In 1987, J. G. Thompson posed a very interesting problem related to algebraic number fields as follows (see [1]).

Thompson's Problem. Let $T(G) = \{(n, s_n) \mid n \in \omega(G) \text{ and } s_n \in \text{nse}(G)\}$, where s_n is the number of elements with order n . Suppose that $T(G) = T(H)$. If G is a finite solvable group, is it true that H is also necessarily solvable?

It is easy to see that if G and H are of the same order type, then

$$\text{nse}(G) = \text{nse}(H), \quad |G| = |H|. \quad (1)$$

The proof is as follows: let G be a group and M some simple K_i -group, where $i = 3, 4$; then $G \cong M$ if and only if $|G| = |M|$ and $\text{nse}(G) = \text{nse}(M)$ (see [2, 3]). And also the group A_{12} is characterizable by order and nse (see [4]). Recently, all sporadic simple groups are characterizable by nse and order (see [5]).

Comparing the sizes of elements of same order but disregarding the actual orders of elements in $T(G)$ of Thompson's problem, whether it can characterize finite simple groups? Up to now, some groups especial for $L_2(q)$, where $q \in \{7, 8, 9, 11, 13\}$, can be characterized by only the set $\text{nse}(G)$ (see [6, 7]). The author has proved that the group $L_3(4)$ is

characterizable by nse (see [8]). In this paper, it is shown that the group $U_3(7)$ also can be characterized by nse.

Here, we introduce some notations which will be used. Let $a \cdot b$ denote the product of integer a by integer b . If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n . Let G be a group. The set of element orders of G is denoted by $\omega(G)$. Let $k \in \omega(G)$ and let s_k be the number of elements of order k in G . Let $\text{nse}(G) = \{s_k \mid k \in \omega(G)\}$. Let $\pi(G)$ denote the set of prime p such that G contains an element of order p . $L_n(q)$ denotes the projective special linear group of degree n over finite fields of order q . $U_n(q)$ denotes the projective special unitary group of degree n over finite fields of order q . The other notations are standard (see [9]).

2. Some Lemmas

Lemma 1 (see [10]). *Let G be a finite group and m a positive integer dividing $|G|$. If $L_m(G) = \{g \in G \mid g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2 (see [11]). *Let G be a finite group and let $p \in \pi(G)$ be odd. Suppose that P is a Sylow p -subgroup of G and $n = p^s m$ with $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is always a multiple of p^s .*

Lemma 3 (see [7]). *Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and $|G| \leq s(s^2 - 1)$.*

Lemma 4 (see [12, Theorem 9.3.1]). *Let G be a finite solvable group and $|G| = mn$, where $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m, n) = 1$. Let $\pi = \{p_1, \dots, p_r\}$ and let h_m be the number of Hall π -subgroups of G . Then, $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$ satisfies the following conditions for all $i \in \{1, 2, \dots, s\}$:*

- (1) $q_i^{\beta_i} \equiv 1 \pmod{p_j}$ for some p_j ;
- (2) the order of some chief factor of G is divided by $q_i^{\beta_i}$.

To prove $G \cong U_3(7)$, we need the structure of simple K_4 -groups.

Lemma 5 (see [13]). *Let G be a simple K_4 -group. Then, G is isomorphic to one of the following groups:*

- (1) A_7, A_8, A_9 , or A_{10} ;
- (2) M_{11}, M_{12} , or J_2 ;
- (3) one of the following:
 - (a) $L_2(r)$, where r is a prime and $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ with $a \geq 1, b \geq 1, c \geq 1$, and v is a prime greater than 3,
 - (b) $L_2(2^m)$, where $2^m - 1 = u$, $2^m + 1 = 3t^b$ with $m \geq 2, u, t$ are primes, $t > 3$, and $b \geq 1$,
 - (c) $L_2(3^m)$, where $3^m + 1 = 4t$, $3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b$, $3^m - 1 = 2u$, with $m \geq 2, u, t$ are odd primes, $b \geq 1$, and $c \geq 1$,
- (4) one of the following 28 simple groups: $L_2(16), L_2(25), L_2(49), L_2(81), L_3(4), L_3(5), L_3(7), L_3(8), L_3(17), L_4(3), S_4(4), S_4(5), S_4(7), S_4(9), S_6(2), O_8^+(2), G_2(3), U_3(4), U_3(5), U_3(7), U_3(8), U_3(9), U_4(3), U_5(2), Sz(8), Sz(32), {}^2D_4(2)$, or ${}^2F_4(2)$.

Lemma 6. *Let G be a simple K_4 -group and $43 \parallel |G| \cdot 2^7 \cdot 3 \cdot 7^3 \cdot 43$. Then, $G \cong U_3(7)$.*

Proof. From Lemma 5(1-2), order consideration rules out these cases.

So, we consider Lemma 5(3). We will deal with this with the following cases.

Case 1. Consider $G \cong L_2(r)$, where $r \in \{3, 7, 43\}$.

- (i) Let $r = 3$; then $|\pi(r^2 - 1)| = 1$, which contradicts $|\pi(q^2 - 1)| = 3$.
- (ii) Let $r = 7$; then $|\pi(r^2 - 1)| = 2$, which contradicts $|\pi(q^2 - 1)| = 3$.
- (iii) Let $r = 43$; then $|\pi(r^2 - 1)| = 4$, which contradicts $|\pi(q^2 - 1)| = 3$.

Case 2. Consider $G \cong L_2(2^m)$, where $u \in \{3, 7, 43\}$.

- (i) Let $u = 3$; then $m = 2$ and so $5 = 3t^b$. But the equation has no solution in $\mathbb{N}$, which is a contradiction.
- (ii) Let $u = 7$; then $m = 3$ and $2^3 + 1 = 3t^b$. Thus, $t = 3$ and $b = 1$. But $t > 3$, a contradiction.

- (iii) Let $u = 43$; then the equation $2^m - 1 = 43$ has no solution in $\mathbb{N}$, which is a contradiction.

Case 3. Consider $G \cong L_2(3^m)$.

We will consider the case by the following two subcases.

Subcase 3.1. Consider $3^m + 1 = 4t$ and $3^m - 1 = 2u^c$.

We can suppose that $t \in \{3, 7, 43\}$.

Let $t = 3, 7, 43$; the equation $3^m + 1 = 4t$ has no solution.

Subcase 3.2. Consider $3^m + 1 = 4t^b$ and $3^m - 1 = 2u$.

We can suppose that $u \in \{3, 7, 43\}$.

Let $u = 3, 5, 7$; then the equation $3^m - 1 = 2u$ has no solution in $\mathbb{N}$, which is a contradiction.

In review of Lemma 5(4), $G \cong U_3(7)$.

This completes the proof of the lemma. $\square$

3. Proof of Theorem

Let G be a group such that $\text{nse}(G) = \text{nse}(U_3(7))$, and let s_n be the number of elements of order n . By Lemma 3, we have that G is finite. We note that $s_n = k\phi(n)$, where k is the number of cyclic subgroups of order n . Also we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 1 and the above discussion, we have

$$\begin{aligned} \phi(m) & \mid s_m, \\ m & \mid \sum_{d \mid m} s_d. \end{aligned} \quad (2)$$

Theorem 7. *Let G be a group with $\text{nse}(G) = \text{nse}(U_3(7)) = \{1, 2107, 92708, 101136, 117468, 117992, 202272, 235984, 471968, 539392, 809088, 943936, 1843968\}$, where $U_3(7)$ is the projective special unitary group of degree 3 over field of order 7. Then, $G \cong U_3(7)$.*

Proof. We prove the theorem by first proving that $\pi(G) \subseteq \{2, 3, 7, 43\}$, second showing that $|G| = |U_3(7)|$, and so $G \cong U_3(7)$.

By (2), $\pi(G) \subseteq \{2, 3, 5, 7, 17, 43\}$. If $m > 2$, then $\phi(m)$ is even, and then $s_2 = 2107, 2 \in \pi(G)$.

In the following, we prove that $17 \notin \pi(G)$. If $17 \in \pi(G)$, then by (2), $s_{17} = 539392$. If $2 \cdot 17 \in \omega(G)$, then $s_{2 \cdot 17} \notin \text{nse}(G)$. Therefore, $2 \cdot 17 \notin \omega(G)$. Now we consider Sylow 17-subgroup P_{17} acts fixed-point-freely on the set of elements of order 2 of G ; then $|P_{17}| \mid s_2$, which is a contradiction. Hence, we have $\pi(G) \subseteq \{2, 3, 5, 7, 43\}$. Furthermore, by (2) $s_3 = 92708, 117992, 471968, s_5 = 235984, s_7 = 117648$, and $s_{43} = 1843968$.

If $2^a \in \omega(G)$, then $\phi(2^a) = 2^{a-1} \mid s_{2^a}$, and so $0 \leq a \leq 9$. If $a = 9$, then by (2), $s_{2^9} \notin \text{nse}(G)$, and so $0 \leq a \leq 8$.

If $3^a \in \omega(G)$, then $1 \leq a \leq 3$.

If $5^a \in \omega(G)$, then $a = 1$.

If $7^a \in \omega(G)$, then $1 \leq a \leq 5$. If $7^5 \in \omega(G)$, then by (2), $s_{7^5} \notin \text{nse}(G)$. So, $1 \leq a \leq 4$.

If $43^a \in \omega(G)$, then $1 \leq a \leq 2$. If $43^2 \in \omega(G)$, then by (2), $s_{43^2} \notin \text{nse}(G)$. Thus, $a = 1$.

In the following, we first prove that $5 \notin \pi(G)$, then consider the proper set of $\{2, 3, 7, 43\}$, and so $\pi(G) = \{2, 3, 7, 43\}$. Finally, we get the desired result by [2].

To remove the “bad” prime 5, we should prove that $43 \in \pi(G)$.

Since $2 \in \pi(G)$ and $\exp(P_2) = 2, 4, 8, 16, 32, 64, 128, 256$, then by Lemma 1, $|P_2| \mid 1 + s_2 + \dots + s_{2^8}$ and $|P_2| \mid 2^{11}$ (this case occurs only when $s_4 = 92708, s_8 = s_{16} = 101136, s_{32} = s_{64} = s_{128} = 809088$, and $s_{256} = 539392$).

If $3 \in \pi(G)$, then $\exp(P_3) = 3, 9, 27$.

(i) If $s_3 = 92708$, then $s_{3^2} = 117648$. It follows that $s_{3^3} \notin \text{nse}(G)$. So we have $|P_3| \mid 1 + s_3 + s_{3^2}$ and so, $|P_3| \mid 3^4$.

(ii) If $s_3 = 117992$, then $s_{3^2} = 202272$ or 809088 .

(a) Let $s_{3^2} = 202272$; then $3^3 \mid 1 + s_3 + s_{3^2} + s_{3^3}$, and so $s_{3^3} = 117648$. Then, $|P_3| \mid 1 + s_3 + s_{3^2} + s_{3^3}$, and so $|P_3| \mid 3^3$.

(b) Let $s_{3^2} = 809088$; then $3^3 \mid 1 + s_3 + s_{3^2} + s_{3^3}$, but $s_{3^3} \notin \text{nse}(G)$. Then, $|P_3| \mid 3^2$.

(iii) If $s_3 = 471968$, then $s_{3^2} = s_{3^3} = 117648$, and so $|P_3| \mid 3^3$.

If $5 \in \pi(G)$ and $\exp(P_5) = 5$, then by Lemma 1, $|P_5| \mid 1 + s_5$, and so $|P_5| = 5$.

If $7 \in \pi(G)$ and $\exp(P_7) = 7, 7^2, 7^3, 7^4$, then by Lemma 1, $|P_7| \mid 1 + s_7 + s_{7^2} + s_{7^3} + s_{7^4}$. By (2), $s_{7^2} = 101136, 202272, 809088, 1843968$.

(i) $s_{7^2} = 101136, 202272$ or 809088 , and then $7^3 \mid 1 + s_7 + s_{7^2} + s_{7^3}$, but $s_{7^3} \notin \text{nse}(G)$. Then, $|P_7| \mid 7^2$.

(ii) $s_{7^2} = 1843968$, and then $s_{7^3} = s_{7^4} = 1843968$. So, $|P_7| \mid 7^4$.

If $43 \in \pi(G)$ and $\exp(P_{43}) = 43$, then by Lemma 1, $|P_{43}| \mid 1 + s_{43}$, and so $|P_{43}| = 43$.

Assume that $43 \notin \pi(G)$. We deal with it with the following cases.

Case A. Let $7 \in \pi(G)$. We know that $\exp(P_7) = 7, 7^2, 7^3, 7^4$.

(i) Let $\exp(P_7) = 7$. Then by Lemma 1, $|P_7| \mid 1 + s_7$, and so $|P_7| \mid 7^6$. If $|P_7| = 7$, then since $n_7 = s_7/\phi(7)$, $43 \in \pi(G)$, which is a contradiction. If $|P_7| = 7^2$, then we can assume that $\pi(G) \subseteq \{2, 3, 5, 7\}$, $5663616 + 92708k_1 + 101136k_2 + 117468k_3 + 117992k_4 + 202272k_5 + 235984k_6 + 471968k_7 + 539392k_8 + 809088k_9 + 943936k_{10} + 1843968k_{11} = 2^m \cdot 3^n \cdot 5^p \cdot 7^2$, where $k_1, \dots, k_{11}, m, n$, and p are nonnegative integers and $0 \leq \sum_{i=1}^{11} k_i \leq 154$. Since $5663616 \leq |G| = 2^m \cdot 3^n \cdot 5^p \cdot 7^2 \leq 5663616 + 154 \cdot 1843968$, then since m is at most 11, the equation has no solution in $\mathbb{N}$. If $|P_7| = 7^3$, then similarly, $(m, n, p) = (7, 3, 1), (8, 3, 1), (9, 3, 1), (10, 3, 1), (11, 3, 1), (9, 2, 1), (10, 2, 1), (11, 2, 1)$. Let $|G| = 2^7 \cdot 3^3 \cdot 5 \cdot 7^3$. Then by Sylow's theorem, the number of Sylow 5-subgroups of G is 1, 6, 16, 21, 36, 56, 96, 126, 196, 216, 336, 441, 576, 686, 756, 896, 1176, 2016, 2646, 3136, 3456, 3528, 4116, 7056, 9261, 10976, 12096, 18816, 24696, 42336, 65856, and 395136, and so the number

of elements of order 5 is at most 4, 24, 64, 84, 144, 224, 384, 504, 784, 864, 1344, 1764, 2304, 2744, 3024, 3584, 4704, 8064, 10584, 12544, 13824, 14112, 16464, 28224, 37044, 43904, 48384, 75264, 98784, 169344, 263424, and 1580544, but none of which belongs to $\text{nse}(G)$, which is a contradiction. Similarly, we can rule out the other cases “ $(m, n, p) = (8, 3, 1), (9, 3, 1), (9, 2, 1)$.” Similarly, for $|P_7| \geq 7^4$, we can get a contradiction as the case “ $|P_7| = 7^3$.”

(ii) Let $\exp(P_7) = 7^2$. Then by Lemma 1, $|P_7| \mid 1 + s_7 + s_{7^2}$ for $s_{7^2} \in \{101136, 202272, 809088, 1843968\}$, and so $|P_7| \mid 7^2$. Since $n_7 = s_{7^2}/\phi(7^2)$ for $s_{7^2} \in \{101136, 202272, 809088\}$, $43 \in \pi(G)$, which is a contradiction. If $s_{7^2} = 1843968/42 = 43904$, by Sylow's theorem, $n_7 = 7k + 1$ for some integer k , but the equation has no solution in $\mathbb{N}$, which is a contradiction.

(iii) Let $\exp(P_7) = 7^3$. Then by Lemma 1, $|P_7| \mid 1 + s_7 + s_{7^2} + s_{7^3}$ and so $|P_7| \mid 7^4$. If $|P_7| = 7^3$, then since $n_7 = s_{7^3}/\phi(7^3) = 6272$, by Sylow's theorem, $n_7 = 7k + 1$ for some integer k , but the equation has no solution in $\mathbb{N}$, which is a contradiction. If $|P_7| = 7^4$, then we can assume that $\pi(G) \subseteq \{2, 3, 5, 7\}$, $5663616 + 92708k_1 + 101136k_2 + 117468k_3 + 117992k_4 + 202272k_5 + 235984k_6 + 471968k_7 + 539392k_8 + 809088k_9 + 943936k_{10} + 1843968k_{11} = 2^m \cdot 3^n \cdot 5^p \cdot 7^4$, where $k_1, \dots, k_{11}, m, n$, and p are nonnegative integers and $0 \leq \sum_{i=1}^{11} k_i \leq 163$. Since $5663616 \leq |G| = 2^m \cdot 3^n \cdot 5^p \cdot 7^4 \leq 5663616 + 163 \cdot 1843968$, then $(m, n, p) = (5, 3, 1), (6, 3, 1), (7, 3, 1), (8, 3, 1), (9, 3, 1), (6, 2, 1), (7, 2, 1), (8, 2, 1), (9, 2, 1), (10, 2, 1), (11, 2, 1), (8, 1, 1), (9, 1, 1), (10, 1, 1),$ and $(10, 1, 1)$. Let $|G| = 2^5 \cdot 3^3 \cdot 5 \cdot 7^4$. Then by Sylow's theorem, the number of Sylow 5-subgroups of G is 1, 6, 16, 21, 36, 56, 96, 126, 196, 216, 336, 441, 686, 756, 1176, 2401, 2016, 2646, 4116, 7056, 9261, 10976, 14406, 18816, 24696, 28416, 42336, 86436, 148176, and 230496. It follows that the number of elements of order 5 is 4, 24, 64, 84, 144, 224, 384, 504, 784, 864, 1344, 1764, 2744, 3024, 4704, 9604, 8064, 10584, 16464, 28224, 37044, 43904, 57624, 75264, 98784, 113664, 169344, 345744, 592704, and 921984, but none of which belongs to $\text{nse}(G)$. Similarly, we can rule the other cases.

(iv) Let $\exp(P_7) = 7^4$. Then by Lemma 1, $|P_7| \mid 1 + s_7 + s_{7^2} + s_{7^3} + s_{7^4}$, and so $|P_7| \mid 7^4$. Since $n_7 = s_{7^4}/\phi(7^4) = 896$, by Sylow's theorem, $n_7 = 7k + 1$ for some integer k , but the equation has no solution in $\mathbb{N}$, which is a contradiction.

Case B. Let $5 \in \pi(G)$. We know that $\exp(P_5) = 5$. Then by Lemma 1, $|P_5| \mid 1 + s_5$, and so $|P_5| = 5$. Since $n_5 = s_5/\phi(5)$, then $43 \in \pi(G)$, which is a contradiction.

Case C. Let $3 \in \pi(G)$. We know that $\exp(P_3) = 3, 9, 27$.

(i) Let $\exp(P_3) = 3$. Then by Lemma 1, $|P_3| \mid 1 + s_3$ and so $|P_3| \mid 3^2$. Since $n_3 = s_3/\phi(3)$, $43 \in \pi(G)$, which is a contradiction. Therefore, $|P_3| = 3^2$. From cases A and B, and by assumption “ $7 \notin \pi(G)$ ”, we can assume that

$\pi(G) = \{2, 3\}$. Then, $5663616 + 92708k_1 + 101136k_2 + 117468k_3 + 117992k_4 + 202272k_5 + 235984k_6 + 471968k_7 + 539392k_8 + 809088k_9 + 943936k_{10} + 1843968k_{11} = 2^m \cdot 9$, where $k_1, \dots, k_{11}$ and m are nonnegative integers and $0 \leq \sum_{i=1}^{11} k_i \leq 7$. Since $5663616 \leq |G| = 2^m \cdot 9 \leq 5663616 + 7 \cdot 1843968$, then since m is at most 11, the equation has no solution in $\mathbb{N}$.

- (ii) Let $\exp(P_3) = 9$. Then by Lemma 1, $|P_3| \mid 1 + s_3 + s_{3^2}$ and so $|P_3| \mid 3^4$. If $|P_3| = 9$, then $43 \in \pi(G)$, which is a contradiction. If $|P_3| = 27, 81$, then similarly as the case “ $\exp(P_3) = 3$ and $|P_3| = 9$,” the equation has no solution in $\mathbb{N}$ since m is at most 11.
- (iii) Let $\exp(P_3) = 27$. Then by Lemma 1, $|P_3| \mid 1 + s_3 + s_{3^2} + s_{3^3}$, and so $|P_3| = 27$. Since $n_3 = s_{3^3}/\phi(3^3)$, then $43 \in \pi(G)$, which is a contradiction.

Therefore, $43 \in \pi(G)$. We prove that $5 \notin \pi(G)$.

Suppose to the contrary that $5 \in \pi(G)$. If $5.43 \in \omega(G)$, sets P and Q are Sylow 43-subgroups of G ; then P and Q are conjugate in G , and so $C_G(P)$ and $C_G(Q)$ are also conjugate in G . Therefore, we have $s_{5.43} = \phi(5.43) \cdot n_{43} \cdot k$, where k is the number of cyclic subgroups of order 5 in $C_G(P_{43})$. Since $n_{43} = s_{43}/\phi(43) = 1843968/42,7375872 \mid s_{5.43}$ and so $s_{5.43} = 7375872t$ for some integer t , the equation has no solution in $\mathbb{N}$ since $s_{5.43} \in \text{nse}(G)$. Therefore, $5.43 \notin \omega(G)$; it follows that the Sylow 5-subgroup of G acts fixed-point-freely on the set of elements of order 43, and so $|P_5| \mid s_{43}$, which is a contradiction. Thus, $5 \notin \pi(G)$.

Therefore, we have that $\{2, 43\} \subseteq \pi(G) \subseteq \{2, 3, 7, 43\}$.

Case A ($\pi(G) = \{2, 43\}$). Therefore, $5663616 + 92708k_1 + 101136k_2 + 117468k_3 + 117992k_4 + 202272k_5 + 235984k_6 + 471968k_7 + 539392k_8 + 809088k_9 + 943936k_{10} + 1843968k_{11} = 2^m \cdot 9$, where $k_1, \dots, k_{11}$ and m are nonnegative integers and $0 \leq \sum_{i=1}^{11} k_i \leq 20$. Since $5663616 \leq |G| = 2^m \cdot 43 \leq 5663616 + 20 \cdot 1843968$, then since m is at most 11, the equation has no solution in $\mathbb{N}$.

Case B ($\pi(G) = \{2, 3, 43\}$). As $\exp(P_{43}) = 43$, by Lemma 1, $|P_{43}| \mid 1 + s_{43}$ and so $|P_{43}| = 43$. Since $n_{43} = s_{43}/\phi(43)$, $7 \in \pi(G)$, which is a contradiction.

Case C ($\pi(G) = \{2, 7, 43\}$). Since $\exp(P_{43}) = 43$, then by Lemma 1, $|P_{43}| \mid 1 + s_{43}$, and so $|P_{43}| = 43$. Since $n_{43} = s_{43}/\phi(43)$, $3 \in \pi(G)$, which is a contradiction.

Case D ($\pi(G) = \{2, 3, 7, 43\}$). In the following, we first show that $|G| = 2^7 \cdot 3 \cdot 7^3 \cdot 43$, $|G| = 2^7 \cdot 3 \cdot 7^4 \cdot 43$, $|G| = 2^8 \cdot 3 \cdot 7^3 \cdot 43$, $|G| = 2^8 \cdot 3 \cdot 7^4 \cdot 43$, $|G| = 2^9 \cdot 3 \cdot 7^3 \cdot 43$, or $|G| = 2^9 \cdot 3 \cdot 7^4 \cdot 43$, and second prove that $G \cong U_3(7)$.

Step 1 ($|G| = 2^7 \cdot 3 \cdot 7^3 \cdot 43$ or $|G| = 2^8 \cdot 3 \cdot 7^3 \cdot 43$). We have known that $|P_{43}| = 43$.

If $3.43 \in \omega(G)$, sets P and Q are Sylow 43-subgroups of G ; then P and Q are conjugate in G , and so $C_G(P)$ and $C_G(Q)$ are also conjugate in G . Therefore, we have $s_{3.43} = \phi(3.43) \cdot n_{43} \cdot k$, where k is the number of cyclic subgroups of order 3 in $C_G(P_{43})$. Since $n_{43} = s_{43}/\phi(43) = 1843968/42,3687936 \mid s_{3.43}$, so $s_{3.43} = 3687936t$ for some integer t . Since $s_{3.43} \in \text{nse}(G)$,

the equation has no solution. Therefore, $3.43 \notin \omega(G)$; it follows that the Sylow 3-subgroup P_3 of G acts fixed-point-freely on the set of elements of order 43, $|P_3| \mid s_{43}$, and so $|P_3| = 3$. Similarly, $3.7, 2.43 \notin \omega(G)$ and $|P_7| \mid 7^3$, $|P_2| \mid 2^8$.

Therefore, we can assume that $|G| = 2^m \cdot 3 \cdot 7^p \cdot 43$. Since $5663616 = 2^7 \cdot 3 \cdot 7^3 \cdot 43 \leq |G| = 2^m \cdot 3 \cdot 7^p \cdot 43$, then $|G| = 2^7 \cdot 3 \cdot 7^3 \cdot 43$, or $|G| = 2^8 \cdot 3 \cdot 7^3 \cdot 43$.

Step 2 ($G \cong U_3(7)$). First prove that there is no group such that $|G| = 2^7 \cdot 3 \cdot 7^3 \cdot 43$ or $|G| = 2^8 \cdot 3 \cdot 7^3 \cdot 43$ and $\text{nse}(G) = \text{nse}(U_3(7))$. Then by [2], $G \cong U_3(7)$.

Let $|G| = 2^8 \cdot 3 \cdot 7^3 \cdot 43$ and $\text{nse}(G) = \text{nse}(U_3(7))$.

If G is soluble, set H is a Hall $\{3, 7, 43\}$ -subgroup of G , and all the Hall $\{3, 7, 43\}$ -subgroups of G are conjugate, so the number of Hall $\{3, 7, 43\}$ -subgroups of G is $|G : N_G(H)| \mid 2^8$. Let $n_{43}(H) = 3^{a_1+a_2+\dots+a_r} \cdot 7^{b_1+b_2+\dots+b_s}$ by Lemma 4, $3^{a_i} \equiv 1 \pmod{43}$, $7^{b_j} \equiv 1 \pmod{43}$, $i = 1, \dots, r$, $j = 1, 2, \dots, s$, where $a_1, \dots, a_r, b_1, \dots, b_s$ are nonnegative integers and $a_1 + \dots + a_r \leq 1$, $b_1 + \dots + b_s \leq 4$. Hence, $n_{43}(H) = 1$. So $42 \leq s_{43}(G) \leq 10752$ and $42 \mid s_{43}(G)$, but $s_{43}(G) \notin \text{nse}(G)$, which is a contradiction. So G is insoluble.

Therefore, G has a normal series $1 \triangleleft K \triangleleft L \triangleleft G$ such that L/K is isomorphic to a simple K_i -group with $i = 3, 4$ as 9 and 1849 do not divide the order of G .

If L/K is isomorphic to a simple K_3 -group, from [14], $L/K \cong L_2(7)$. Then, $|G/L| \mid 2^5 \cdot 7^3 \cdot 43$. Let $A/K := C_{G/K}(L/K)$. Then, $A/K \cap L/K = 1$. It is easy to see that $G/K/A/K \leq \text{Aut}(L/K) = \text{SL}(2, 7)$, and so $G/A \leq \text{SL}(2, 7)$. Since $A/K, L/K \triangleleft G/K$, $A/K \times L/K \leq G/K$. Therefore, $|L/K| \mid |G/A|$, and so $G/K \cong L_2(7)$ or $\text{SL}(2, 7)$. That is, $|A| = 2^5 \cdot 7^3 \cdot 43$ or $2^4 \cdot 7^3 \cdot 43$. By Sylow's theorem, $n_{43}(A) = 1$. Since $A \triangleleft G$, we have that $n_7(A) = n_7(G)$, and so $s_7(G) = 42$, which contradicts $s_7(G) \in \text{nse}(G)$.

Hence, G is isomorphic to a simple K_4 -group; then by Lemma 6, $L/K \cong U_3(7)$. So $G/A \leq \text{Aut}(U_3(7))$. Therefore, $G/A \cong U_3(7)$ or $G/A \cong 2 \cdot U_3(7)$.

If $G/A \cong U_3(7)$, then order consideration $|A| = 2$. It follows that A is a normal subgroup generated by a 2-central element of G . So, there exists an element of order 2.43, which is a contradiction.

If $G/A \cong 2 \cdot U_3(7)$, then $A = 1$. But $\text{nse}(2 \cdot U_3(7)) \neq \text{nse}(U_3(7))$, which is a contradiction.

Therefore, $|G| = 2^7 \cdot 3 \cdot 7^3 \cdot 43 = |U_3(7)|$, and by assumption, $\text{nse}(G) = \text{nse}(U_3(7))$; then by [2], $G \cong U_3(7)$.

This completes the proof of the theorem. $\square$

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