Research Article

On Connected $m$-HPK$(n_1, n_2, n_3, n_4)[K_t]$-Residual Graphs

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We define $m$-HPK$(n_1, n_2, n_3, n_4)[K_t]$-residual graphs in which HPK is a hyperplane complete graph. We extend P. Erdős, F. Harary, and M. Klável’s definition of plane complete residual graph to hyperplane and obtain the hyperplane complete residual graph. Further, we obtain the minimum order of $HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graphs and $m$-HPK$(n_1, n_2, n_3, n_4)[K_t]$-residual graphs. In addition, we obtain a unique minimal $HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graphs and a unique minimal $m$-HPK$(n_1, n_2, n_3, n_4)[K_t]$-residual graphs.

1. Introduction

A graph $G$ is said to be $F$-residual graph [1], if for every vertex $v$ in $G$, the graph obtained from $G$ by removing the closed neighborhood of $v$ is isomorphic to $F$. We inductively define a multiply $F$-residual graph by saying that $G$ is $m$–$F$-residual graph, if the removal of the closed neighborhood of any vertex of $G$ results in an $(m-1)$–$F$-residual graph, where of course a $1$–$F$-residual graph is simply said to be $F$-residual graph.

It is natural to ask what is the minimum number of vertices that an $m$–$K_n$-residual graph must contain? It is easy to prove that this number is $(m+1)n$ and that the only $m$–$K_n$-residual graph with this number of vertex is $(m+1)K_n$.

In [1], Erdős et al. show that a connected $K_n$-residual graph must have at least $2(n + 1)$ points, if $n \neq 2$. Furthermore, the Cartesian product $G = K_{n+1} \times K_2$ is the only such graph with $2(n + 1)$ points for $n \neq 2, 3, 4$. They complete the result by determining all connected $K_n$-residual graphs of minimal order for $n = 2, 3, 4$.

In [1], the following conjectures were stated.

Conjecture 1 (see [1]). If $n \neq 2$, then every connected $m$–$K_n$-residual graph has at least $\min\{2n(m + 1),(n + m)(m + 1)\}$ vertices.

Conjecture 2 (see [1]). For $n$ large, there is a unique smallest connected $m$–$K_n$-residual graph.

Theorem 3 (see [1]). (1) If $G$ is an $F$-residual graph, then for any vertex $u$ in $G$, the degree $d(u) = (v(G) − v(F) − 1$.

(2) Every $m$–$K_n$-residual graph has at least $(m+1)n$ vertices, and $(m+1)K_n$ is the only $m$–$K_n$-residual graph with $(m+1)n$ vertices.

(3) Every connected $K_n$-residual graph has at least $2n + 2$ vertices, if $n \neq 2$.

(4) If $n \neq 2$, then $G = K_{n+1} \times K_2$ is a connected $K_n$-residual graph of minimum order, and, except for $n = 3$ and $n = 4$, it is the only such graph.

We know that these supporting results are summarized in residual graph [7–16]. In this paper, we will study the important properties of hyperplane complete graph in relatin to the minimum order of hyperplane complete graph and the minimum graph of hyperplane complete graph. We will investigate Erdős, Harary, and Klável’s residual graph and obtain the minimum order of $HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graphs and $m$-HPK$(n_1, n_2, n_3, n_4)[K_t]$-residual graphs. And we also show a unique minimal connected $HPK(n_1, n_2, n_3, n_4)[K_t]$-residual graphs of order $(n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1)t$...
and a unique minimal connected $m$-HPK$(n_1, n_2, n_3, n_4)[K_t]$-residual graphs of order $(n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t$.

In general, we follow the notation in [1]. In particular, $v(G)$ is the number of vertices in a graph $G$, $N(u)$ is the closed neighborhood of the vertex $u$ in $V(G)$, and $N^*(u) = N(u) \cup \{u\}$ are called as neighborhood and closed neighborhood of $u$ in $G$.

### 2. On Connected HPK$(n_1, n_2, n_3, n_4)[K_t]$-Residual Graphs

**Definition 4.** A graph $G$ is a 3-dimensional hyperplane complete graph, if for every $x \in V(G)$, $V(G) = V_1 \times V_2 \times V_3 \times V_4$, $x = [(x_1, x_2, x_3, x_4) \mid x_i \in V_i, i = 1, 2, 3, 4]$, $|V_1| = n_1, i = 1, 2, 3, 4$. Two vertices $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ are adjacent to each other, if and only if $x \neq y$ and $x_k = y_k$ for some $k = 1, 2, 3, 4$. For convenience, it writes HPK$(n_1, n_2, n_3, n_4)$ instead of 3-dimensional hyperplane complete graphs.

**Definition 5.** Let $F = \text{HPK}(n_1, n_2, n_3, n_4)$, a graph $G$ is HPK$(n_1, n_2, n_3, n_4)$-residual graph, if for every $u \in G$, it is $G - N^*(u) \subseteq F$. A multiply HPK$(n_1, n_2, n_3, n_4)$-residual graph is $m$-HPK$(n_1, n_2, n_3, n_4)$-residual graph, if the removal of the closed neighborhood of any vertex $u$ in $G$ results in an $(m-1)$-HPK$(n_1, n_2, n_3, n_4)$-residual graph.

By Definitions 4 and 5, let $G = \text{HPK}(n_1, n_2, n_3, n_4)$ and $n = v(G) = n_1n_2n_3n_4$, if $n_1 = 1$ for some $i$, then $G$ is indeed a complete graph $K_n$. In fact, $\text{HPK}(m, l) = K_m \times K_l$, and $\text{HPK}(l)$ is an empty graph with $n$ vertices.

**Definition 6.** Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are simple graphs. In the composition [17], $G_1$ and $G_2$ are recorded as $G = G_1[G_2]$, $G_1[G_2] = V_1 \times V_2$, and $u = (u_1, u_2)$ is adjacent to $v = (v_1, v_2)$ in $G$, if and only if $u_1 = v_1, u_2 \neq v_2, u_2$ is adjacent to $v_2$ in $G_2$, but $u_1$ is adjacent to $v_1$ in $G_1$.

By Definitions 4, 5, and Theorem 3, we have the following.

**Lemma 7.** If $F$ is a connected noncomplete graph, then there is no disconnected $F$-residual graph.

**Proof.** Suppose that $G$ is an $F$-residual graph, if $G$ is disconnected, let $G = G_1 \cup G_2$, where $G_1$ is a connected component of the $G$. Then, for any $v \in G_1$, so

$$G - N^*(v) = (G_1 \cup G_2) - N^*_{G_1}(v)$$

$$= G_2 \cup (G_1 - N^*_{G_1}(v)) \subseteq F.$$  \hspace{1cm} (1)

Since $F$ is connected, so $V(G_1) - N^*_{G_1}(v) = \phi$ and $G_2 \subseteq F$. Thus, $G_2$ is connected, and $G_2$ is not a complete graph; hence, there are two vertices $u$ and $w$ in $G_2$, and $u$ is nonadjacent to $w$,

$$G - N^*(u) = (G_1 \cup G_2) - N^*_{G_1}(v) = G_1 \cup (G_2 - N^*_G(u)).$$  \hspace{1cm} (2)

Since $w \in G_2 - N^*_G(u)$. Hence, $G - N^*(u)$ is disconnected; however $G - N^*(u) \equiv F$ is connected, which is a contradiction. \hfill \square

**Theorem 8.** Assume that $\text{HPK}(n_1 + m), (n_2 + m), \ldots, (n_t + m)[K_t]$ is an $m$-HPK$(n_1, n_2, \ldots, n_t)[K_t]$-residual graph, particularly, $\text{HPK}(2, 2, \ldots, n + 1)[K_t]$ is a $K_{nt}$-residual graph.

**Proof.** The proof follows using Definitions 4, 5, and 6.

**Lemma 9.** Let $F = \text{HPK}(n_1, n_2, n_3, n_4)[K_t]$, $n_1, n_2, n_3, n_4 \geq 4$, $x, y, z, w$ are pairwise nonadjacent in $F$, then $|N^*(x) \cap N^*(y) \cap N^*(z) \cap N^*(w)| = 24t$.

**Proof.** By Definitions 5 and 6 and let $V(F) = V_1 \times V_2 \times V_3 \times V_4$, $V_i = 1, 2, \ldots, n_i, i = 1, 2, 3, 4, V_5 = 1, 2, \ldots, t$. Let $x = (x_1, x_2, x_3, x_4, x_5), y = (y_1, y_2, y_3, y_4, y_5), z = (z_1, z_2, z_3, z_4, z_5), x, y, z$ is pairwise nonadjacent in $F$, if and only if $x \neq y$ and $x_i = y_i$ for some $i = 1, 2, 3, 4$. Since $x, y, z, w$ is pairwise nonadjacent, then $x_1 \neq y_1 \neq z_1 \neq w_1, x_2 \neq y_2 \neq z_2 \neq w_2, x_3 \neq y_3 \neq z_3 \neq w_3, x_4 \neq y_4 \neq z_4 \neq w_4$. And $n_1, n_2, n_3, n_4 \geq 4$, hence,

$$N^*(x) \cap N^*(y) \cap N^*(z) \cap N^*(w) = \{(x_1, y_2, z_3, w_4, r), (x_1, y_2, w_3, z_4, r), (x_1, z_2, y_3, w_4, r), (x_1, z_2, w_3, y_4, r), (x_1, z_2, y_3, w_4, r), (x_1, z_2, w_3, y_4, r), (x_1, w_2, z_3, y_4, r), (x_1, w_2, y_3, z_4, r), (x_1, w_2, z_3, y_4, r), (x_1, w_2, y_3, z_4, r), (x_1, w_2, z_3, y_4, r), (x_1, w_2, y_3, z_4, r), (z_1, x_2, y_3, w_4, r), (z_1, x_2, w_3, y_4, r), (z_1, x_2, w_3, y_4, r), (z_1, y_2, x_3, w_4, r), (z_1, y_2, w_3, x_4, r), (z_1, y_2, w_3, x_4, r), (z_1, w_2, x_3, y_4, r), (z_1, w_2, y_3, x_4, r), (z_1, w_2, y_3, x_4, r), (z_1, w_2, y_3, x_4, r), (w_1, x_2, z_3, y_4, r), (w_1, x_2, y_3, z_4, r), (w_1, x_2, y_3, z_4, r), (w_1, x_2, y_3, z_4, r), (w_1, y_2, x_3, z_4, r), (w_1, y_2, z_3, x_4, r), (w_1, y_2, z_3, x_4, r), \{r \in V_5\}, \}.$$  \hspace{1cm} (3)

then $|N^*(x) \cap N^*(y) \cap N^*(z) \cap N^*(w)| = 24t$. \hfill \square

**Lemma 10.** Assume that $G$ is an $F$-residual graph, $F = \text{HPK}(n_1, n_2, n_3, n_4)[K_t]$, $n_1, n_2, n_3, n_4 \geq 4$, $x, y$ is pairwise nonadjacent in $G$, then $|N^*(x) \cap N^*(y)| = (n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1) - 2n_1n_2n_3n_4 + (n_1 - 1)(n_2 - 1)(n_3 - 1)(n_4 - 1)$.\hfill \square

**Proof.** The proof follows using Definitions 4, 5, and 6.

**Lemma 11.** Assume that $G$ is an $F$-residual graph, $F = \text{HPK}(n_1, n_2, n_3, n_4)[K_t]$, $n_1, n_2, n_3, n_4 \geq 4$, for any $u \in V(G)$, then $\delta(u) \geq (n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1) - n_1n_2n_3n_4t - 1$. \hfill \square
Proof. For any \( v \in V(G) \), and \( v \) is nonadjacent to \( u \) in \( G \), let \( F \) = \( G - N^+(v) \) \( \equiv \) \( HPK(n_1, n_2, n_3, n_4)[K_4] \), then \( u \in V(F) \), by Definitions 6, then

\[
\begin{align*}
|V(F) - N^+_F(u)| & = (n_1 - 1)(n_2 - 1)(n_3 - 1)(n_4 - 1)t, \\
|N^+_F(u)| & = n_1n_2n_3n_4t - (n_1 - 1)(n_2 - 1)(n_3 - 1)(n_4 - 1)t,
\end{align*}
\]

hence, \( d_F(u) = n_1n_2n_3n_4t - (n_1 - 1)(n_2 - 1)(n_3 - 1)(n_4 - 1)t - 1 \).

Since \( v, u, w \) is pairwise nonadjacent in \( G \), by Lemma 10, then

\[
|N^+(u) \cap N^+(v) \cap N^+(w)| \geq 6(n_1 + n_2 + n_3 + n_4)t, \text{ so}
\]

\[
d(u) = d_F(u) + |N^+(u) \cap N^+(v)| \geq [n_1n_2n_3n_4 - (n_1 - 1)(n_2 - 1)(n_3 - 1)(n_4 - 1)]
\]

\[
+ (n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1) - 2n_1n_2n_3n_4
\]

\[
+ (n_1 - 1)(n_2 - 1)(n_3 - 1)(n_4 - 1)t - 1
\]

\[
= [(n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1) - n_1n_2n_3n_4]t - 1.
\]

This completes the proof.

\[ \square \]

**Theorem 12.** Let \( G \) is \( HPK(n_1, n_2, n_3, n_4)[K_4] \)-residual graph, \( n_1, n_2, n_3, n_4 \geq 4 \), and \( v(G) \geq (n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1)t \).

Proof. For any \( u \in V(G) \), let \( G - N^+(u) = F \equiv HPK(n_1, n_2, n_3, n_4)[K_4] \), by Theorem 3 and Lemma 11, then

\[
v(G) = d(u) + 1 + v(F) \geq [(n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1) - n_1n_2n_3n_4]t
\]

\[
+ n_1n_2n_3n_4t = (n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1)t.
\]

\[ \square \]

**Theorem 13.** Let \( G \) is \( HPK(n_1, n_2, n_3, n_4)[K_4] \)-residual graph, \( n_1, n_2, n_3, n_4 \geq 4 \) and \( v(G) = (n_1 + 1)(n_2 + 1)(n_3 + 1)(n_4 + 1)t \), then \( G \equiv HPK(n_1 + 1, n_2 + 1, n_3 + 1, n_4 + 1)[K_4] \).

In order to prove Theorem 13, we have the following Lemma.

**Lemma 14.** Assume that \( F = HPK(n_1, n_2, n_3, n_4)[K_4] \), \( n_1, n_2, n_3, n_4 \geq 4 \), and \( x, y, z, w \) is pairwise nonadjacent in \( F \), let \( x = (x_1, x_2, x_3, x_4, x_5) \), \( y = (y_1, y_2, y_3, y_4, y_5) \), \( z = (z_1, z_2, z_3, z_4, z_5) \), \( w = (w_1, w_2, w_3, w_4, w_5) \), then

\[
(1) N(x) \cap N(y) \cap N(z) \cap N(w) \cap N^+(X) = \{(w_1, x_2, y_3, z_4, r \mid r \in V_2\}, \text{ and } X \text{ contains } x_2, y_3, z_4 \text{ 54 set intersection, and}
\]

\[
N^+(X) \equiv N(x_1, x_2, x_3, x_4, r) \cap N(x_1, x_2, z_3, x_4, r)
\]

\[
\cap N(x_1, x_2, z_3, r) \cap N(y_1, x_2, z_4, r) \cap N(y_1, x_2, y_4, r)
\]

\[
\cap N(y_1, z_1, x_3, z_4, r) \cap N(z_1, x_2, x_3, r)
\]

\[
\cap N(z_1, x_2, r) \cap N(z_1, z_2, z_3, r) \cap N(z_1, z_2, z_4, r)
\]

\[
\cap N(z_1, x_2, z_3, r) \cap N(z_1, x_2, z_4, r)
\]

\[
\cap N(x_1, x_2, y_3, r) \cap N(x_1, x_2, z_4, r)
\]

\[
(2) N(x) \cap N(y) \cap N(z) \cap N(w) \cap N^+(X) = \{(w_1, x_2, y_3, z_4, r \mid r \in V_2\}, \text{ and } X \text{ contains } x_2, y_3, z_4 \text{ 54 set intersection, and}
\]

\[
N^+(X) \equiv N(x_1, x_2, x_3, x_4, r) \cap N(x_1, x_2, z_3, x_4, r)
\]

\[
\cap N(x_1, x_2, z_3, r) \cap N(y_1, x_2, z_4, r) \cap N(y_1, x_2, y_4, r)
\]

\[
\cap N(y_1, z_1, x_3, z_4, r) \cap N(z_1, x_2, x_3, r)
\]

\[
\cap N(z_1, x_2, r) \cap N(z_1, z_2, z_3, r) \cap N(z_1, z_2, z_4, r)
\]

\[
\cap N(z_1, x_2, z_3, r) \cap N(z_1, x_2, z_4, r)
\]

\[
\cap N(x_1, x_2, y_3, r) \cap N(x_1, x_2, z_4, r)
\]

\[
N(x_1, x_2, y_3, z_4, r) \cap N(x_1, x_2, x_3, x_4, r)
\]

\[
\cap N(x_1, x_2, x_3, r) \cap N(x_1, x_2, z_4, r)
\]

\[
\cap N(x_1, x_2, z_3, r) \cap N(x_1, x_2, z_4, r)
\]

\[
\cap N(x_1, x_2, y_3, r) \cap N(x_1, x_2, z_4, r)
\]

\[
N(x_1, x_2, y_3, z_4, r) \cap N(x_1, x_2, x_3, x_4, r)
\]

\[
N(x_1, x_2, x_3, r) \cap N(x_1, x_2, z_4, r)
\]

\[
\cap N(x_1, x_2, z_3, r) \cap N(x_1, x_2, z_4, r)
\]

\[
\cap N(x_1, x_2, y_3, r) \cap N(x_1, x_2, z_4, r)
\]

\[
N(x_1, x_2, y_3, z_4, r) \cap N(x_1, x_2, x_3, x_4, r)
\]

\[
N(x_1, x_2, x_3, r) \cap N(x_1, x_2, z_4, r)
\]

\[
\cap N(x_1, x_2, z_3, r) \cap N(x_1, x_2, z_4, r)
\]
\[N(w_1, y_2, y_3, y_4, r) \cap N(y_1, w_2, y_3, y_4, r) \cap N(y_1, w_2, y_3, z_4, r) - N(y_1, z_2, z_3, z_4, r)
\]
\[= \{(w_1, w_2, x_3, x_4, r) \mid r \in V_3\}, \quad (8)\]

\[M = N(w_1, w_2, w_3, w_4, r) \cap N(x_1, x_2, x_3, x_4, r) \cap N(y_1, y_2, y_3, y_4, r) \cap N(y_1, y_2, y_3, z_4, r) - N(y_1, z_2, z_3, z_4, r)
\]
\[= \{(w_1, w_2, w_3, w_4, r) \mid r \in V_3\}, \quad (9)\]

**Proof.** Part (1) follows from Definition 6 and Lemma 9. Similarly,

\[N(x) \cap N(y) \cap N(z) \cap N(w) \cap N^*(Y)
\]
\[= \{(x_1, w_1, y_3, z_4, r) \mid 1 \leq r \leq t\}, \quad (10)\]

where \(N^*(Y)\) contains \(x_1, y_2, z_3, z_4\) 54 set intersection, \(N^*(Z)\) contains \(x_1, y_2, z_3, z_4\) 54 set intersection, and \(N^*(W)\) contains \(x_1, y_2, z_3, z_4\) 54 set intersection.

(2) Let \(v = (v_1, v_2, v_3, v_4, v_5) \in M\), by Definition 6, and \(v\) is not adjacent \(x, y, z\), hence, \(v_i \neq v_j \neq v_i, i = 1, 2, 3, 4,\) and \(v = (v_1, v_2, v_3, v_4, v_5)\) is adjacent to \((w_1, w_2, y_3, y_4, r)\), \((w_1, w_2, y_3, y_4, r)\) is adjacent to \((w_1, y_2, y_3, y_4, r)\), \((w_1, w_2, y_3, y_4, r)\) is adjacent to \((w_1, w_2, y_3, z_4, r)\), \((z_1, w_2, z_3, z_4, r)\), hence \(v_1 = w_1, v_2 = w_2\), because \(v\) is adjacent to \(x\), then \(x_1 \neq w_1 = v_1, x_2 \neq w_2 = v_2\), hence, \(v_3 = x_3, v_4 = x_4\), for any \(v_3 \in V_3\).

Part (3) follows from the proof of part (2). \(\square\)

**Proof of Theorem 13.** We are now ready to prove the following quantitative version of our theorem, we have four cases as follows.

**Case 1.** For any \(u \in V(G)\), let \(F_1 = G - N^*(u) = HPK(n_1, n_2, n_3, n_4)[K_4]\), by Definition 6, \(V(F_1) = \{(x_1, x_2, x_3, x_4) \mid 1 \leq x_i \leq n_i, i = 1, 2, 3, 4\}\), \(x = (x_1, x_2, x_3, x_4)\) is adjacent to \(y = (y_1, y_2, y_3, y_4)\) in \(F_1\), if and only if \(x = y\), for any \(x_1, y_1, i = 1, 2, 3, 4\).

**Case 2.** Let \(F_2 = G - N^*(n_1, n_2, n_3, n_4, 1) = HPK(n_1, n_2, n_3, n_4, 1)[K_4]\), by Lemma 14, using \(V(F_1) \cap V(F_2)\) in the name of the vertex of \(V(F_2) \cap V(F_2)\) in the naming of vertices. First, prove that \(u = (0, 0, 0, 0, 1)\), for any \(x, y \in V(F_1) \cap V(F_2), x = (x_1, x_2, x_3, x_4, 1), y = (y_1, y_2, y_3, y_4, 1), x \neq y, i = 1, 2, 3, 4.\) For convenience, \(N(x_1, x_2, x_3, x_4, 1)\) is the closed neighborhood of a vertex \((x_1, x_2, x_3, x_4, 1)\), \(N^*(x_1, x_2, x_3, x_4, 1) = N(x_1, x_2, x_3, x_4, 1) \cup (x_1, x_2, x_3, x_4, 1)\), by Lemma 14, let

\[N(u) \cap N(x) \cap N(y) \cap N(z) \cap N^*(X)
\]
\[= \{(0, x_3, z_3, z_4, r) \mid 1 \leq r \leq t\}, \quad (11)\]

\[N(u) \cap N(x) \cap N(y) \cap N(z) \cap N^*(Y)
\]
\[= \{(x_1, 0, y_3, z_4, r) \mid 1 \leq r \leq t\}, \quad (12)\]

\[N(u) \cap N(x) \cap N(y) \cap N(z) \cap N^*(Z)
\]
\[= \{(x_1, y_2, 0, z_4, r) \mid 1 \leq r \leq t\}, \quad (13)\]

\[N(u) \cap N(x) \cap N(y) \cap N(z) \cap N^*(W)
\]
\[= \{(x_1, y_2, z_3, z_4, 0) \mid 1 \leq r \leq t\}, \quad (14)\]

\[N(0, 0, 0, 1) \cap N(x_1, x_2, x_3, x_4, 1)
\]
\[\cap N(0, y_2, y_3, y_4, 1) \n\cap N(0, z_2, z_3, z_4, 1) \n\cap N(y_1, y_2, y_3, y_4, 1)
\]
\[\cap N(z_1, z_2, z_3, z_4, 1)
\]
\[\cap N(y_1, y_2, y_3, y_4, 1)
\]
\[\cap N(z_1, z_2, z_3, z_4, 1)
\]
\[= \{(0, x_3, z_3, z_4, r) \mid 1 \leq r \leq t\}, \quad (15)\]

\[N(0, 0, 0, 1) \cap N(x_1, x_2, x_3, x_4, 1)
\]
\[\cap N(0, y_2, y_3, y_4, 1) \n\cap N(0, z_2, z_3, z_4, 1) \n\cap N(y_1, y_2, y_3, y_4, 1)
\]
\[\cap N(z_1, z_2, z_3, z_4, 1)
\]
\[\cap N(y_1, y_2, y_3, y_4, 1)
\]
\[\cap N(z_1, z_2, z_3, z_4, 1)
\]
\[= \{(0, x_3, z_3, z_4, r) \mid 1 \leq r \leq t\}, \quad (16)\]

\[N(0, 0, 0, 1) \cap N(x_1, x_2, x_3, x_4, 1)
\]
\[\cap N(0, y_2, y_3, y_4, 1) \n\cap N(0, z_2, z_3, z_4, 1) \n\cap N(y_1, y_2, y_3, y_4, 1)
\]
\[\cap N(z_1, z_2, z_3, z_4, 1)
\]
\[\cap N(y_1, y_2, y_3, y_4, 1)
\]
\[\cap N(z_1, z_2, z_3, z_4, 1)
\]
\[= \{(0, x_3, z_3, z_4, r) \mid 1 \leq r \leq t\}, \quad (17)\]

\[N(0, 0, 0, 1) \cap N(x_1, x_2, x_3, x_4, 1)
\]
\[\cap N(0, y_2, y_3, y_4, 1) \n\cap N(0, z_2, z_3, z_4, 1) \n\cap N(y_1, y_2, y_3, y_4, 1)
\]
\[\cap N(z_1, z_2, z_3, z_4, 1)
\]
\[\cap N(y_1, y_2, y_3, y_4, 1)
\]
\[\cap N(z_1, z_2, z_3, z_4, 1)
\]
\[= \{(0, x_3, z_3, z_4, r) \mid 1 \leq r \leq t\}, \quad (18)\]
Lastly let \( \{ v \in V(F_2) \mid N_{F_2}^*(v) = N_{F_2}^*(u) \} = \{ (0,0,0,0,r) \mid 1 \leq r \leq t \} \), so far \( V(F_2) - V(F_1) \) vertex already all named, the uniqueness of the name. By Lemma 14 and Definition 6, then \( F_4 = HPK(n_1,n_2,n_3,n_4)[K_t] \), \( V(F_4) = \{ (x_1,x_2,x_3,x_4) \mid 0 \leq x_1 \leq x_2, x_3 = 1,2,3,4; 1 \leq x_4 \leq t \} \). Two vertices \( x = (x_1,x_2,x_3,x_4,x_5) \) and \( y = (y_1,y_2,y_3,y_4,y_5) \) are adjacent to each other, if and only if exist \( x \neq y \) and \( x_i = y_i \) for \( i = 1,2,3,4,5 \).

Case 3. Let \( F_3 = G - N^*(1,1,1,1,1) \equiv HPK(n_1,n_2,n_3,n_4)[K_t] \), to \( F_3 \) not named point naming, these point sets are composed of

\[
V(F_3) - V(F_1) - V(F_2)
\]

\[
= V(F_3) \cap (V(F_1) \cup V(F_2))
\]

\[
= V(F_3) \cap (N^*(n_1,n_2,n_3,n_4,1) \cap N^*(n_1,n_2,n_3,n_4,1) \cap N^*(n_1,n_2,n_3,n_4,1))
\]

\[
= N^*(0,0,0,1) \cap N^*(0,0,0,1) \cap N^*(0,0,0,1)
\]

\[
= \{ (0,0,0,0,r) \mid 1 \leq r \leq t \}.
\]

By Lemma 14 and (12), we can name

\[
\{(0,0,0,m_4,r),(0,0,0,m_4,r),(0,0,0,m_4,r)\}
\]

\[
\{ (n_1,0,m_3,0,0,0) \mid 1 \leq r \leq t \}.
\]

By Lemma 14 and (13), we can name

\[
\{(0,0,0,m_4,r),(0,0,0,m_4,r),(0,0,0,m_4,r)\}
\]

\[
\{ (n_1,0,m_3,0,0,0) \mid 1 \leq r \leq t \}.
\]

Similarly, we can name

\[
\{(0,0,0,0,0,0,0,0,0) \mid 1 \leq r \leq t \}.
\]

For the \( V(F_2) \cap V(F_1) \) point rename, by Lemma 14 and the uniqueness of the name get the name and in \( F_2 \) name is the same, then \( V(F_3) = \{ (x_1,x_2,x_3,x_4,x_5) \mid 0 \leq x_1 \leq n_i, x_i \neq 1, i = 1,2,3,4; 1 \leq x_4 \leq t \} \), and two vertices \( x = (x_1,x_2,x_3,x_4,x_5) \) and \( y = (y_1,y_2,y_3,y_4,y_5) \) are adjacent to each other in the \( F_3 \), if and only if \( x \neq y \) and, for any \( x_i = y_i \) for \( i = 1,2,3,4 \).

Case 4. Named \( G \) all unnamed point, this kind of composition is set \( N^*(0,0,0,0,0) \cap N^*(n_1,n_2,n_3,n_4,1) \cap N^*(1,1,1,1,1,1) \). Let \( F_4 = G - N^*(2,2,2,2,2,2) = \)}
Let $G$ be an $m$-HPK-graph, then $V(\bar{G}) \geq (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t$.

Proof. For $m = 1$, by Lemma 10, we are done. Suppose that $m \geq 1$, it is true. For $m + 1$, let $G$ be a connected $(m + 1)$-HPK-graph, then $V(\bar{G}) \geq (n_1 + m + 1)(n_2 + m + 1)(n_3 + m + 1)(n_4 + m + 1) - (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t$.

Let $G = G - N^*(w)$, then $x, y, z$ is pairwise nonadjacent in $G$, for any $w \in V(G)$, it is true.

With comprehensive discussed above, then $G = m$-HPK-graph. The proof is complete.

3. Connected $m$-HPK-Graphs

Lemma 15. Assume that graph $G$ is $m$-HPK-graph, then $V(\bar{G}) \geq (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t$.

Proof. By Definitions 4, 5, 6, and $n_1, n_2, n_3, n_4 \geq 4$, we can turn out vertex $u_1, u_2, \ldots, u_{m+2}, u_{m+3}, u_{m+1} \in G_1 = G_0 - N^*(u_i), i = 1, 2, \ldots, m + 2, m + 3$, let $G_0 = G$, $u_1 \in V(G_0)$ for convenience, $u_1 \in V(G_0)$ instead of $u \in G$, let $u_1, u_2, \ldots, u_{m+1}, u_{m+2}, u_{m+3}$ be pairwise nonadjacent. Hence $x, y, z, w$ is pairwise nonadjacent in $F$, by Lemma 9, then

\[N^*(x) \cap N^*(y) \cap N^*(z) \cap N^*(w) \geq N^*_F(x) \cap N^*_F(y) \cap N^*_F(z) \cap N^*_F(w) \geq 24t.
\]

Lemma 16. Assume that graph $G$ is $m$-HPK-graph, then $V(\bar{G}) \geq (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t$.

Proof. For $m = 1$, by Lemma 10, we are done. Suppose that $m \geq 1$, it is true. For $m + 1$, let $G$ be a connected $(m + 1)$-HPK-graph, then $V(\bar{G}) \geq (n_1 + m + 1)(n_2 + m + 1)(n_3 + m + 1)(n_4 + m + 1) - (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t$.

Lemma 17. Assume that graph $G$ is $m$-HPK-graph, then $V(\bar{G}) \geq (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t$.

Proof. For $m = 1$, by Lemma 10, we are done. Suppose that $m \geq 1$, it is true.

\[d(u) \geq (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t - (n_1 + m - 1)(n_2 + m - 1)(n_3 + m - 1)(n_4 + m - 1)t + 1.
\]
Proof. For $m = 1$, by Theorem 12, we are done. Suppose that $m \geq 1$, it is true. For $m + 1$, let $G$ be a connected $(m + 1)$-HPK$(n_1, n_2, n_3, n_4)[K_1]$-residual graph, for any $u \in G$. Let $H = G - N^*(u)$, $H$ is $m$-HPK$(n_1, n_2, n_3, n_4)[K_1]$-residual graph. By inductive hypothesis and by Lemma 17, then

\[ v(H) \geq (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t, \]

\[ |N^*(u)| \geq [(n_1 + m + 1)(n_2 + m + 1)](n_3 + m + 1) - (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t, \]

\[ v(G) \geq P(H)|N^*(u)| \geq (n_1 + m + 1)(n_2 + m + 1)(n_3 + m + 1)(n_4 + m + 1)t. \]

(25)

Theorem 19. Let $G$ is $m$-HPK$(n_1, n_2, n_3, n_4)[K_1]$-residual graph, $n_1, n_2, n_3, n_4 \geq 4$, $v(G) = (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t$, then $G \equiv$ HPK$(n_1 + m, n_2 + m, n_3 + m, n_4 + m)[K_1]$. Proof. For $m = 1$, by Theorem 12, we are done. Suppose that $m \geq 1$, it is true. For $m + 1$, let $G$ be a connected $(m + 1)$-HPK$(n_1, n_2, n_3, n_4)[K_1]$-residual graph, $n_1, n_2, n_3, n_4 \geq 4$, $v(G) = (n_1 + m + 1)(n_2 + m + 1)(n_3 + m + 1)(n_4 + m + 1)t$, for any $u \in G$. Let $H = G - N^*(u)$, $H$ is $m$-HPK$(n_1, n_2, n_3, n_4)[K_1]$-residual graph. By inductive hypothesis and by Lemma 17, then

\[ v(H) = v(G) - |N^*(u)| \leq (n_1 + m + 1)(n_2 + m + 1)(n_3 + m + 1)(n_4 + m + 1)t - [(n_1 + m + 1)(n_2 + m + 1)(n_3 + m + 1)(n_4 + m + 1) - (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t = (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t, \]

(26)

by Theorem 18, $v(H) = (n_1 + m)(n_2 + m)(n_3 + m)(n_4 + m)t$. By inductive hypothesis, $H \equiv$ HPK$(n_1 + m, n_2 + m, n_3 + m, n_4 + m)[K_1]$. By Definition 6, let $G$ be a connected HPK$(n_1 + m, n_2 + m, n_3 + m, n_4 + m)[K_1]$-residual graph, $n_1 + m \geq n_2 \geq 4$, with conditions and Theorem 12, then $G \equiv$ HPK$(n_1 + m + 1, n_2 + m + 1, n_3 + m + 1, n_4 + m + 1)[K_1]$. 

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References
