Research Article

k -Tuple Total Restrained Domination in Complementary Prisms

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In a graph G with $\delta(G) \geq k \geq 1$, a k -tuple total restrained dominating set S is a subset of $V(G)$ such that each vertex of $V(G)$ is adjacent to at least k vertices of S and also each vertex of $V(G) - S$ is adjacent to at least k vertices of $V(G) - S$. The minimum number of vertices of such sets in G is the k -tuple total restrained domination number of G . In [k -tuple total restrained domination/domatic in graphs, BIMS], the author initiated the study of the k -tuple total restrained domination number in graphs. In this paper, we continue it in the complementary prism of a graph.

1. Introduction

Let G be a simple graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. The order $|V|$ and size $|E|$ of G are denoted by $n = n(G)$ and $m = m(G)$, respectively. The open neighborhood and the closed neighborhood of a vertex $v \in V$ are $N_G(v) = \{u \in V \mid uv \in E\}$ and $N_G[v] = N_G(v) \cup \{v\}$, respectively. Also the open neighborhood and the closed neighborhood of a subset $X \subseteq V(G)$ are $N_G(X) = \cup_{v \in X} N_G(v)$ and $N_G[X] = N_G(X) \cup X$, respectively. The degree of a vertex $v \in V$ is $\deg(v) = |N(v)|$. The minimum and maximum degree of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. If every vertex of G has degree k , then G is called k -regular. We write K_n , C_n , and P_n for a complete graph, a cycle and a path of order n , respectively, while $K_{n_1, \dots, n_p}$ denotes a complete p -partite graph. The complement of a graph G is denoted by $\overline{G}$ and is a graph with the vertex set $V(G)$ and for every two vertices v and w , $vw \in E(\overline{G})$ if and only if $vw \notin E(G)$.

For each integer $k \geq 1$, the k -join $G \circ_k H$ of a graph G to a graph H of order at least k is the graph obtained from the disjoint union of G and H by joining each vertex of G to at least k vertices of H [1]. Also, $G \circ_{*k} H$ denotes the k -join $G \circ_k H$ such that each vertex of G is joined to exact k vertices of H .

The complementary prism $G\overline{G}$ of G is the graph formed from the disjoint union $G \cup \overline{G}$ of G and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices

(same label) of G and $\overline{G}$ [2]. For example, the graph $C_5\overline{C_5}$ is the Petersen graph. Also, if $G = K_n$, the graph $K_n\overline{K_n}$ is the corona $K_n \circ K_1$, where the corona $G \circ K_1$ of a graph G is the graph obtained from G by attaching a pendant edge to each vertex of G .

The research of domination in graphs is an evergreen area of graph theory. Its basic concept is the dominating set. The literature on this subject has been surveyed and detailed in the two books by Haynes et al. [3, 4]. And many variants of the dominating set were introduced and the corresponding numerical invariants were defined for them. For example, the k -tuple total dominating set is defined in [1] by Henning and Kazemi, which is an extension of the total dominating set (for more information see [5, 6]).

Definition 1 (see [1]). Let $k \geq 1$ be an integer and let G be a graph with $\delta(G) \geq k$. A subset $S \subseteq V(G)$ is called a k -tuple total dominating set, briefly kTDS, in G , if for each $x \in V(G)$, $|N(x) \cap S| \geq k$. The minimum number of vertices of a k -tuple total dominating set in a graph G is called the k -tuple total domination number of G and denoted by $\gamma_{\times k, t}(G)$.

A numerical invariant of a graph which is in a certain sense dual to it is the domatic number of a graph. The domatic number $d(G)$ and the total domatic number $d_t(G)$ of a graph were introduced in [7, 8], respectively. Sheikholeslami and Volkmann extended the last definition to the k -tuple total domatic number $d_{\times k, t}(G)$ in [9] and Kazemi extended it to the star k -tuple total domatic number $d_{\times k, t}^*(G)$ in [10].

Definition 2. The k -tuple total domatic partition, briefly kTDP, of G is a partition $\mathbb{D}$ of the vertex set of G such that all classes of $\mathbb{D}$ are k -tuple total dominating sets in G . The maximum number of classes of a k -tuple total domatic partition of G is called the k -tuple total domatic number $d_{\times k,t}(G)$ of G [9]. The star k -tuple total domatic number $d_{\times k,t}^*(G)$ of G is the maximum number of classes of a kTDP of G such that at least one of the k -tuple total dominating sets in it has cardinality $\gamma_{\times k,t}(G)$ [10].

The author in [10] initiated the study of the k -tuple total restrained domination number and the k -tuple total restrained domatic number of graphs.

Definition 3 (see [10]). The k -tuple total restrained domatic partition, briefly kTRDP, of G is a partition $\mathbb{D}$ of the vertex set of G such that all classes of $\mathbb{D}$ are k -tuple total restrained dominating sets in G . The maximum number of classes of a k -tuple total restrained domatic partition of G is the k -tuple total restrained domatic number $d_{\times k,t}^r(G)$ of G . Similarly, the star k -tuple total restrained domatic number $d_{\times k,t}^{r*}(G)$ of G is the maximum number of classes of a kTRDP of G such that at least one of the k -tuple total restrained dominating sets in it has cardinality $\gamma_{\times k,t}^r(G)$.

In this paper, we continue our studies, which is initiated in [10] and find some sharp bounds for the k -tuple total restrained domination number of the complementary prism of a graph. Also we will find the k -tuple total restrained domination number of a cycle, a path, and a complete multipartite graph.

Through this paper, k is a positive integer, and for simplicity, we assume that $V(G\overline{G})$ is the disjoint union $V(G) \cup V(\overline{G})$ with $V(\overline{G}) = \{\bar{v} \mid v \in V(G)\}$ and $E(G\overline{G}) = E(G) \cup E(\overline{G}) \cup \{v\bar{v} \mid v \in V(G)\}$ such that $E(\overline{G}) = \{\bar{u}\bar{v} \mid uv \notin E(G)\}$. The vertices v and $\bar{v}$ are called *corresponding vertices*. Also for a subset $X \subseteq V(G)$, we show its corresponding subset in $\overline{G}$ by $\overline{X}$. Also we assume that $V(C_n) = V(P_n) = \{i \mid 1 \leq i \leq n\}$, $V(\overline{C_n}) = V(\overline{P_n}) = \{\bar{i} \mid 1 \leq i \leq n\}$, $E(C_n) = E(P_n) \cup \{1n\} = \{ij \mid 1 \leq i \leq n-1, \text{ and } j = i+1\} \cup \{1n\}$.

The next known results are useful for our investigations.

Proposition 4 (see Henning and Kazemi [1] 2010). Let $p \geq 2$ be an integer and let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph with $n_1 \leq n_2 \leq \dots \leq n_p$.

- (i) If $k < p$, then $\gamma_{\times k,t}(G) = k + 1$.
- (ii) If $k = p$ and $\sum_{i=1}^{k-1} n_i \geq k$, then $\gamma_{\times k,t}(G) = k + 2$.
- (iii) If $2 \leq p < k$ and $\lceil k/(p-1) \rceil \leq n_1 \leq n_2 \leq \dots \leq n_p$, then $\gamma_{\times k,t}(G) = \lceil kp/(p-1) \rceil$.

Proposition 5 (see Kazemi [6] 2011). Let $n \geq 4$. Then

$$\gamma_t(P_n \overline{P_n}) = \begin{cases} 2 \left\lceil \frac{n-2}{4} \right\rceil + 1 & \text{if } n \equiv 3 \pmod{4}, \\ 2 \left\lceil \frac{n-2}{4} \right\rceil + 2 & \text{otherwise.} \end{cases} \quad (1)$$

Proposition 6 (see Kazemi [6] 2011). Let $n \geq 4$. Then

$$\gamma_t(C_n \overline{C_n}) = \begin{cases} 2 \left\lceil \frac{n}{4} \right\rceil + 2 & \text{if } n \equiv 0 \pmod{4}, \\ 2 \left\lceil \frac{n}{4} \right\rceil + 1 & \text{if } n \equiv 3 \pmod{4}, \\ 2 \left\lceil \frac{n}{4} \right\rceil & \text{otherwise.} \end{cases} \quad (2)$$

Proposition 7 (see Kazemi [6] 2011). If $n \geq 5$, then $\gamma_{\times 2,t}(C_n \overline{C_n}) = n + 2$.

Proposition 8 (see Kazemi [6] 2011). Let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph with $V(G\overline{G}) = \bigcup_{1 \leq i \leq p} (X_i \cup \overline{X_i})$, when for each i , X_i is isomorphic to $\overline{K_{n_i}}$. If S is a kTDS of $G\overline{G}$, then for each $1 \leq i \leq p$, $|S \cap \overline{X_i}| \geq k$. Furthermore, if $|S \cap \overline{X_i}| = k$ for some i , then $|S \cap X_i| \geq k$.

Proposition 9 (see Kazemi [6] 2011). Let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph with $1 \leq n_1 \leq n_2 \leq \dots \leq n_p$. Then

$$\gamma_t(G\overline{G}) = 2p - \alpha, \quad (3)$$

where $\alpha = |\{i \mid 1 \leq i \leq p, \text{ and } n_i = 1\}|$.

Proposition 10 (see Kazemi [6] 2011). Let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph with $3 \leq n_1 \leq n_2 \leq \dots \leq n_p$. Then $\gamma_{\times 2,t}(G\overline{G}) = 3p + 2$.

Proposition 11 (see Kazemi [6] 2011). Let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph with $5 \leq k+2 \leq n_1 \leq \dots \leq n_p$. Then

$$\gamma_{\times k,t}(G\overline{G}) = \begin{cases} (p+1)(k+1) & \text{if } p \geq k+1, \\ (p+1)(k+1) + 1 & \text{if } p = k+4, \\ 16 & \text{if } p = k+3, \\ p(k+1) + \min \left\{ 2k-2, \left\lceil \frac{kp}{p-1} \right\rceil \right\} & \text{if } p < k. \end{cases} \quad (4)$$

Proposition 12 (see Kazemi [10] 2011). Let G be a graph of order n in which $\delta(G) \geq k$. Then

- (i) every vertex of degree at most $2k-1$ of G and at least its k neighbors belong to every kTRDS;
- (ii) $d_{\times k,t}^r(G) = 1$ if $\delta(G) \leq 2k-1$;
- (iii) $\Delta(G) \geq 2k$ if $\gamma_{\times k,t}^r(G) < n$. Hence, $n \geq 2k+2$;
- (iv) $\gamma_{\times k,t}(G) \leq \gamma_{\times k,t}^r(G)$, and so $d_{\times k,t}^r(G) \leq d_{\times k,t}(G)$.

Proposition 13 (see Kazemi [10] 2011). Let G be a graph with minimum degree at least k . If $d_{\times k,t}^*(G) \geq 2$, then $\gamma_{\times k,t}^r(G) = \gamma_{\times k,t}(G)$.

Proposition 14 (see Kazemi [10] 2011). *Let $n \geq k + 3 \geq 4$. Then*

$$\gamma_{\times k,t}^r(\overline{C_n}) = \begin{cases} n & \text{if } n \leq 2k + 2, \\ k + 2 & \text{if } 2k + 3 \leq n \leq 3k + 2, \\ k + 1 & \text{if } n \geq 3k + 3. \end{cases} \quad (5)$$

Proposition 15 (see Kazemi [10] 2011). *Let $k < n$ be positive integers. Then*

$$\gamma_{\times k,t}^r(K_n) = \begin{cases} n & \text{if } n \leq 2k + 1, \\ k + 1 & \text{Otherwise.} \end{cases} \quad (6)$$

Proposition 16 (see Kazemi [10] 2011). *Let $n \geq 4$. Then*

$$\gamma_t^r(C_n) = \begin{cases} 2 \left\lfloor \frac{n}{4} \right\rfloor - 1 & \text{if } n \equiv 1 \pmod{4}, \\ 2 \left\lfloor \frac{n}{4} \right\rfloor + 1 & \text{if } n \equiv 3 \pmod{4}, \\ 2 \left\lfloor \frac{n}{4} \right\rfloor & \text{otherwise.} \end{cases} \quad (7)$$

2. Some Bounds

We first give a sharp lower bound for the k -tuple total restrained domination number of a regular graph.

Theorem 17. *Let k and ℓ be integers such that $1 \leq k - 1 \leq \ell \leq 2k - 2$. If G is a ℓ -regular graph of order n , then*

$$\gamma_{\times k,t}^r(G\overline{G}) \geq n + k, \quad (8)$$

with equality if and only if $n \geq \ell + 2k$ and $V(\overline{G})$ contains a k -subset T such that for each vertex $\bar{i} \in V(\overline{G})$, $|N(\bar{i}) \cap T| \geq k - 1$ and also if $\bar{i} \in V(\overline{G}) - T$, then $|N(\bar{i}) \cap (V(\overline{G}) - T)| \geq k$.

Proof. Let $V(G\overline{G}) = V(G) \cup V(\overline{G})$ such that $V(G) = \{i \mid 1 \leq i \leq n\}$ and $V(\overline{G}) = \{\bar{i} \mid 1 \leq i \leq n\}$. Let $n \geq 2k + \ell$, and let S be an arbitrary k TRDS of $G\overline{G}$. Then Proposition 12(i) and this fact that every vertex $i \in V(G)$ has degree $\ell + 1 \leq 2k - 1$ imply that $V(G) \subseteq S$. Let $\bar{i} \notin S$. Then $|N(\bar{i}) \cap V(\overline{G}) \cap S| \geq k - 1$. If $|N(\bar{i}) \cap V(\overline{G}) \cap S| \geq k$, then we have nothing to prove. Thus, let $N(\bar{i}) \cap V(\overline{G}) \cap S = \{\bar{j}_m \mid 1 \leq m \leq k - 1\}$. But, this implies that there exists a vertex $\bar{t} \in S - \{\bar{j}_m \mid 1 \leq m \leq k - 1\}$ such that its corresponding vertex t in $V(G)$ is adjacent to some vertex j_m , when $1 \leq m \leq k - 1$. Then $|S| \geq n + k$, and since S was arbitrary, we conclude that $\gamma_{\times k,t}^r(G\overline{G}) \geq n + k$.

Obviously, it can be seen that $\gamma_{\times k,t}^r(G\overline{G}) = n + k$ if and only if $n \geq \ell + 2k$ and $V(\overline{G})$ contains a k -subset T such that for each vertex $\bar{i} \in V(\overline{G})$, $|N(\bar{i}) \cap T| \geq k - 1$ and also if $\bar{i} \in V(\overline{G}) - T$, then $|N(\bar{i}) \cap (V(\overline{G}) - T)| \geq k$. $\square$

Theorem 17 and Proposition 12(i) imply the next corollary.

Corollary 18. *Let k and ℓ be integers such that $1 \leq k - 1 \leq \ell \leq 2k - 2$. If G is a ℓ -regular graph of order $n \leq \ell + 2k - 1$, then $\gamma_{\times k}^r(G\overline{G}) = 2n$.*

The next theorem gives lower and upper bounds for $\gamma_{\times k,t}^r(G\overline{G})$, when G is an arbitrary graph.

Theorem 19. *If G is a graph of order n with $k \leq \min\{\delta(G), \delta(\overline{G})\}$, then*

$$\begin{aligned} \gamma_{\times(k-1),t}^r(G) + \gamma_{\times(k-1),t}^r(\overline{G}) &\leq \gamma_{\times k,t}^r(G\overline{G}) \\ &\leq \gamma_{\times k,t}^r(G) + \gamma_{\times k,t}^r(\overline{G}), \end{aligned} \quad (9)$$

when $k \geq 1$ in the upper bound and $k \geq 2$ in the lower bound.

Proof. To prove $\gamma_{\times(k-1),t}^r(G) + \gamma_{\times(k-1),t}^r(\overline{G}) \leq \gamma_{\times k,t}^r(G\overline{G})$, let $k \geq 2$ and let D be a k TRDS of $G\overline{G}$. Since every vertex of $V(G)$ (resp., $V(\overline{G})$) is adjacent to only one vertex of $V(\overline{G})$ (resp., $V(G)$), we conclude that $D \cap V(G)$ is a $(k - 1)$ TRDS of G and $D \cap V(\overline{G})$ is a $(k - 1)$ TRDS of $\overline{G}$. Then

$$\begin{aligned} \gamma_{\times(k-1),t}^r(G) + \gamma_{\times(k-1),t}^r(\overline{G}) &\leq |D \cap V(G)| + |D \cap V(\overline{G})| \\ &= |D| = \gamma_{\times k,t}^r(G\overline{G}). \end{aligned} \quad (10)$$

We now prove that $\gamma_{\times k,t}^r(G\overline{G}) \leq \gamma_{\times k,t}^r(G) + \gamma_{\times k,t}^r(\overline{G})$, when $k \geq 1$. Since for every k TRDS S of G and every k TRDS S' of $\overline{G}$, the set $S \cup S'$ is a k TRDS of $G\overline{G}$, we have

$$\gamma_{\times k,t}^r(G\overline{G}) \leq \gamma_{\times k,t}^r(G) + \gamma_{\times k,t}^r(\overline{G}). \quad (11)$$

$\square$

In Section 4, we will show that the given bounds in Theorem 19 are sharp.

3. The Complementary Prism of Some Graphs

In this section, we will determine $\gamma_{\times k,t}^r(G\overline{G})$ when G is a cycle, a path, or a complete multipartite graph. First, let G be a cycle.

Proposition 20. *Let $n \geq 4$. Then*

$$\gamma_{\times 2,t}^r(C_n\overline{C_n}) = \begin{cases} 2n & \text{if } n = 4, 5, \\ n + 2 & \text{if } n \geq 6. \end{cases} \quad (12)$$

Proof. Corollary 18 implies that $\gamma_{\times 2,t}^r(C_n\overline{C_n}) = 2n$ if $n = 4, 5$. Now let $n \geq 6$. Proposition 7 with Proposition 12(iv) implies $\gamma_{\times 2,t}^r(C_n\overline{C_n}) \geq n + 2$. Since also $V(C_n) \cup \{\bar{1}, \bar{4}\}$ is a 2TRDS of $C_n\overline{C_n}$, we get $\gamma_{\times 2,t}^r(C_n\overline{C_n}) = n + 2$ when $n \geq 6$. $\square$

To calculate $\gamma_t(C_n\overline{C_n})$ we need to prove that $d_t^*(C_n\overline{C_n}) \geq 2$.

Proposition 21. *Let $n \geq 4$. Then $d_t^*(C_n\overline{C_n}) \geq 2$.*

Proof. We prove the proposition in the following four cases.

Case 1 ($n \equiv 0 \pmod{4}$). For $n = 4$, we set $S = \{1, \bar{1}, 2, \bar{2}\}$ and $S' = \{3, \bar{3}, 4, \bar{4}\}$. If $n > 4$, we set $S = \{1, \bar{1}, 2, \bar{2}\} \cup \{5 + 4i, 6 + 4i \mid 0 \leq i \leq \lfloor n/4 \rfloor - 2\}$ and $S' = \{3, \bar{3}, 4, \bar{4}\} \cup \{7 + 4i, 8 + 4i \mid 0 \leq i \leq \lfloor n/4 \rfloor - 2\}$.

Case 2 ($n \equiv 1 \pmod{4}$). For $n = 5$, we set $S = \{1, \bar{1}, 4, \bar{4}\}$ and $S' = \{2, \bar{2}, 5, \bar{5}\}$ and for $n = 9$, we set $S = \{1, \bar{1}, 4, \bar{4}, 7, \bar{7}\}$ and $S' = \{2, \bar{2}, 5, \bar{5}, 8, \bar{8}\}$. If $n > 9$, we set $S = \{1, \bar{1}, 4, \bar{4}, 7, \bar{7}\} \cup \{10 + 4i, 11 + 4i \mid 0 \leq i \leq \lceil n/4 \rceil - 4\}$ and $S' = \{3, \bar{3}, 6, \bar{6}, 9, \bar{9}\} \cup \{12 + 4i, 13 + 4i \mid 0 \leq i \leq \lceil n/4 \rceil - 4\}$.

Case 3 ($n \equiv 2 \pmod{4}$). For $n = 6$, we set $S = \{1, \bar{1}, 4, \bar{4}\}$ and $S' = \{2, \bar{2}, 5, \bar{5}\}$. If $n > 6$, we set $S = \{1, \bar{1}, 4, \bar{4}\} \cup \{7 + 4i, 8 + 4i \mid 0 \leq i \leq \lceil n/4 \rceil - 3\}$ and $S' = \{3, \bar{3}, 6, \bar{6}\} \cup \{9 + 4i, 10 + 4i \mid 0 \leq i \leq \lceil n/4 \rceil - 3\}$.

Case 4 ($n \equiv 3 \pmod{4}$). For $n = 7$, we set $S = \{1, \bar{1}, 4, \bar{4}, \bar{6}\}$ and $S' = \{2, \bar{2}, 5, \bar{5}, \bar{7}\}$. If $n > 7$, we set $S = \{1, \bar{1}, 4, \bar{4}, n - 1\} \cup \{7 + 4i, 8 + 4i \mid 0 \leq i \leq \lceil n/4 \rceil - 3\}$ and $S' = \{2, 3, \bar{3}, 6, \bar{6}\} \cup \{9 + 4i, 10 + 4i \mid 0 \leq i \leq \lceil n/4 \rceil - 3\}$.

Since, in all cases, S and S' are two disjoint $\gamma_t(C_n \overline{C_n})$ -sets, we have $d_t^*(C_n \overline{C_n}) \geq 2$. $\square$

By Propositions 6, 13, and 21 we obtain the next result.

Proposition 22. *Let $n \geq 4$. Then*

$$\gamma_t^r(C_n \overline{C_n}) = \begin{cases} 2 \left\lceil \frac{n}{4} \right\rceil + 2 & \text{if } n \equiv 0 \pmod{4}, \\ 2 \left\lceil \frac{n}{4} \right\rceil + 1 & \text{if } n \equiv 3 \pmod{4}, \\ 2 \left\lceil \frac{n}{4} \right\rceil & \text{otherwise.} \end{cases} \quad (13)$$

Now we continue our work when G is a path.

Proposition 23. *Let $n \geq 4$. Then*

$$\gamma_t^r(P_n \overline{P_n}) = \begin{cases} 2 \left\lceil \frac{n}{4} \right\rceil + 2 & \text{if } n \equiv 0 \pmod{4}, \\ 2 \left\lceil \frac{n}{4} \right\rceil + 1 & \text{if } n \equiv 2 \pmod{4}, \\ 2 \left\lceil \frac{n}{4} \right\rceil & \text{otherwise.} \end{cases} \quad (14)$$

Proof. Propositions 5 and 12(iv) imply that

$$\begin{aligned} \gamma_t^r(P_n \overline{P_n}) &\geq \gamma_t(P_n \overline{P_n}) \\ &= \begin{cases} 2 \left\lceil \frac{n-2}{4} \right\rceil + 1 & \text{if } n \equiv 0 \pmod{4}, \\ 2 \left\lceil \frac{n-2}{4} \right\rceil + 2 & \text{otherwise.} \end{cases} \end{aligned} \quad (15)$$

Let $n \equiv 0 \pmod{4}$. For $n = 8$, we set $S = \{\bar{1}, \bar{8}, 3, 4, 5, 6\}$ and for $n > 8$ we set

$$\begin{aligned} S &= \{\bar{1}, \overline{n-6}, \overline{n-5}, \bar{n}, n-3, n-2\} \\ &\cup \left\{ 3 + 4i, 4 + 4i \mid 0 \leq i \leq \left\lceil \frac{n}{4} \right\rceil - 3 \right\}. \end{aligned} \quad (16)$$

If $n \equiv 1, 2, 3 \pmod{4}$, then we set $S = \{\bar{1}, \overline{n-2}, \bar{n}, n-2\} \cup \{3 + 4i, 4 + 4i \mid 0 \leq i \leq \lceil n/4 \rceil - 2\}$, $S = \{\bar{1}, \bar{n}\} \cup \{3 + 4i, 4 + 4i \mid 0 \leq i \leq \lceil n/4 \rceil - 1\}$, and $S = \{\bar{1}, n-1, \bar{n}\} \cup \{3 + 4i, 4 + 4i \mid 0 \leq i \leq \lceil n/4 \rceil - 1\}$, respectively. Since, in each case, S is a TRDS of $P_n \overline{P_n}$ of cardinality $\gamma_t(P_n \overline{P_n})$, we have completed our proof. $\square$

In the next propositions, we calculate $\gamma_{\times k, t}^r(G\overline{G})$ when G is a complete multipartite graph.

Proposition 24. *If $G = K_{n_1, n_2, \dots, n_p}$ is a complete p -partite graph, then*

$$d_t(G\overline{G}) = \min \{n_i \mid 1 \leq i \leq p\}. \quad (17)$$

Proof. Let $V(G\overline{G}) = \bigcup_{1 \leq i \leq p} (X_i \cup \overline{X_i})$ and let $\ell = \min \{n_i \mid 1 \leq i \leq p\}$. Proposition 8 implies that for every TDS D of $G\overline{G}$, $|D \cap \overline{X_i}| \geq 1$ when $1 \leq i \leq p$. Hence, $d_t(G\overline{G}) \leq \ell$. Now let $D^1, D^2, \dots, D^\ell$ be ℓ disjoint $2p$ -sets of $V(G\overline{G})$ such that for every $j = 1, 2, \dots, \ell$ and every $i = 1, 2, \dots, p$, $|D^j \cap X_i| = |D^j \cap \overline{X_i}| = 1$ and $x \in D^j \cap X_i$ if and only if $\bar{x} \in D^j \cap \overline{X_i}$. Since $D^1, D^2, \dots, D^\ell$ are ℓ disjoint γ_t -sets of $G\overline{G}$, by Proposition 9, we get $d_t(G\overline{G}) \geq \ell$, and so $d_t(G\overline{G}) = \ell$. $\square$

Proposition 25. *If $G = K_{n_1, \dots, n_p}$ is a complete p -partite graph with $2 \leq n_1 \leq \dots \leq n_p$, then $\gamma_t^r(G\overline{G}) = 2p$.*

Proof. Since obviously $d_t(G\overline{G}) = d_t^*(G\overline{G})$, we obtain $\gamma_t^r(G\overline{G}) = 2p$, by Propositions 9, 13, and 24. $\square$

Proposition 26. *Let $G = K_{n_1, n_2}$ be a complete bipartite graph with $4 \leq n_1 \leq n_2$. Then $\gamma_{\times 2, t}^r(G\overline{G}) = 8$.*

Proof. Let S be a set of vertices such that for $i = 1, 2$, $|S \cap X_i| = |S \cap \overline{X_i}| = 2$ and $x \in S \cap X_i$ if and only if $\bar{x} \in S \cap \overline{X_i}$. Since S is a 2TRDS of $G\overline{G}$ of cardinality $\gamma_{\times 2, t}(G\overline{G})$, we get $\gamma_{\times 2, t}^r(G\overline{G}) = 8$, by Propositions 10 and 12(iv). $\square$

Proposition 27. *Let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph with $5 \leq n_1 \leq \dots \leq n_p$. Then $\gamma_{\times 2, t}^r(G\overline{G}) = 3p + 2$.*

Proof. Let S be a set of vertices such that for $i = 1, 2$, $|S \cap X_i| = |S \cap \overline{X_i}| = 2$ and $x \in S \cap X_i$ if and only if $\bar{x} \in S \cap \overline{X_i}$, and for $i = 3, \dots, p$, $|S \cap \overline{X_i}| = 3$. Obviously S is a $\gamma_{\times 2, t}(G\overline{G})$ -set. It can easily be verified that every vertex in out of S is adjacent to at least two vertices in out of S . Hence, S is a 2TRDS of $G\overline{G}$ of cardinality $\gamma_{\times 2, t}(G\overline{G})$, and so $\gamma_{\times 2, t}^r(G\overline{G}) = 3p + 2$, by Propositions 10 and 12(iv). $\square$

Proposition 28. *Let $G = K_{n_1, n_2, n_3}$ be a complete 3-partite graph with $6 \leq n_1 \leq n_2 \leq n_3$. Then $\gamma_{\times 3, t}^r(G\overline{G}) = 16$.*

Proof. Let S be a set of vertices such that $|S \cap \overline{X_3}| = 4$, and for $i = 1, 2$, $|S \cap X_i| = |S \cap \overline{X_i}| = 3$ such that $x \in S \cap X_i$ if and only if $\bar{x} \in S \cap \overline{X_i}$. Since S is a 3TRDS of $G\overline{G}$ of cardinality $\gamma_{\times 3,t}(G\overline{G})$, we get $\gamma_{\times 3,t}^r(G\overline{G}) = 16$, by Propositions 11 and 12(iv). $\square$

Proposition 29. Let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph with $7 \leq 2k + 1 \leq n_1 \leq \dots \leq n_p$. Then

$$\gamma_{\times k,t}^r(G\overline{G}) = \begin{cases} (p+1)(k+1) & \text{if } p \geq k+1, \\ (p+1)(k+1)+1 & \text{if } p = k \geq 4, \\ 16 & \text{if } p = k = 3, \\ p(k+1) + \min\left\{2k-2, \left\lceil \frac{kp}{p-1} \right\rceil\right\} & \text{if } p < k. \end{cases} \quad (18)$$

Proof. We prove the proposition in the following three cases.

Case 1 ($p \geq k+1$ or $p = k \geq 4$). Let S be a set of vertices such that $S \cap V(G)$ is a $\gamma_{\times k,t}(G)$ -set and for $i = 1, 2, \dots, p$, $|S \cap \overline{X_i}| = k+1$. Obviously, S is a $\gamma_{\times k,t}(G\overline{G})$ -set. It can easily be verified that every vertex in out of S is adjacent to at least k vertices in out of S . Hence, S is a kTRDS of $G\overline{G}$ of cardinality $\gamma_{\times k,t}(G\overline{G})$, and so $\gamma_{\times k,t}^r(G\overline{G})$ is $(p+1)(k+1)$ if $p \geq k+1$ and is $(p+1)(k+1)+1$ if $p = k \geq 4$, by Propositions 11 and 12(iv).

Case 2 ($p = k = 3$). This case is proved in Proposition 28 when $6 = 2k \leq n_1 \leq n_2 \leq n_3$.

Case 3 ($p < k$). If $\min\{2k-2, \lceil kp/(p-1) \rceil\} = 2k-2$, let S be a set of vertices such that for $i = 1, 2$, $|S \cap X_i| = |S \cap \overline{X_i}| = k$ and $x \in S \cap X_i$ if and only if $\bar{x} \in S \cap \overline{X_i}$, and for $i = 3, \dots, p$, $|S \cap \overline{X_i}| = k+1$. If $\min\{2k-2, \lceil kp/(p-1) \rceil\} = \lceil kp/(p-1) \rceil$, let S be a set of vertices such that $S \cap V(G)$ is a $\gamma_{\times k,t}(G)$ -set and for $i = 1, 2, \dots, p$, $|S \cap \overline{X_i}| = k+1$. In all cases, obviously S is a $\gamma_{\times k,t}(G\overline{G})$ -set, and it can easily be verified that every vertex in out of S is adjacent to at least k vertices in out of S . Hence, S is a kTRDS of $G\overline{G}$ of cardinality $\gamma_{\times k,t}(G\overline{G})$, and so $\gamma_{\times k,t}^r(G\overline{G}) = p(k+1) + \min\{2k-2, \lceil kp/(p-1) \rceil\}$, by Propositions 11 and 12(iv). $\square$

4. The Given Bounds in Theorem 19 Are Sharp

In this section, we show that the given bounds in Theorem 19 are sharp. For this aim, we first calculate the k -tuple total restrained domination number of a complete multipartite graph and its complement.

Proposition 30. Let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph of order n with $k+1 \leq n_1 \leq \dots \leq n_t \leq 2k+1 < n_{t+1} \leq \dots \leq n_p$. If $p \geq k+1$, then

$$\gamma_{\times k,t}^r(G\overline{G}) = n_1 + \dots + n_t + (p-t)(k+1). \quad (19)$$

Proof. Since $\overline{G}$ is the union of disjoint complete graphs $K_{n_1}, K_{n_2}, \dots, K_{n_p}$, Proposition 15 implies that

$$\begin{aligned} \gamma_{\times k,t}^r(G\overline{G}) &= \gamma_{\times k,t}^r(K_{n_1}) + \gamma_{\times k,t}^r(K_{n_2}) + \dots + \gamma_{\times k,t}^r(K_{n_p}) \\ &= n_1 + \dots + n_t + (p-t)(k+1). \end{aligned} \quad (20)$$

$\square$

Proposition 31. Let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph with $2 \leq n_1 \leq \dots \leq n_p$. If $p \geq k+1$, then $\gamma_{\times k,t}^r(G) = k+1$.

Proof. Let S be a set of vertices of cardinality $k+1$ such that for every index $i = 1, 2, \dots, k+1$, $|S \cap X_i| = 1$. Since S is a kTRDS of G of cardinality $\gamma_{\times k,t}(G)$, Propositions 12(iv) and 4(i) imply that $\gamma_{\times k,t}^r(G) = k+1$. $\square$

Proposition 32. Let $G = K_{n_1, n_2, \dots, n_k}$ be a complete k -partite graph of order n with $n_1 \leq n_2 \leq \dots \leq n_k$. If $n - n_{k-1} \geq n - n_k \geq 2k$ and $n - n_1 \geq \dots \geq n - n_{k-2} \geq 2k+1$, then $\gamma_{\times k,t}^r(G) = k+2$.

Proof. Let S be a set of vertices of cardinality $k+2$ such that $|S \cap X_i| = 2$ for $i = k-1, k$, and $|S \cap X_i| = 1$ for $i = 1, 2, \dots, k-2$. Since S is a kTRDS of G of cardinality $\gamma_{\times k,t}(G)$, Propositions 4(ii) and 12(iv) imply that $\gamma_{\times k,t}^r(G) = k+2$. $\square$

Proposition 33. Let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph of order n with $\lceil k/(p-1) \rceil \leq n_1 \leq \dots \leq n_p$, and let $p < k$. If $p-1 \mid k$ and, for each index $i = 1, 2, \dots, p$, we have $n - n_i \geq 2k$, then $\gamma_{\times k,t}^r(G) = kp/(p-1)$.

Proof. Let S be a set of vertices of cardinality $k p/(p-1)$ such that for each $i = 1, 2, \dots, p$, $|S \cap X_i| = k/(p-1)$. Since S is a kTRDS of G of cardinality $\gamma_{\times k,t}(G)$, Propositions 4(iii) and 12(iv) imply that $\gamma_{\times k,t}^r(G) = k p/(p-1)$. $\square$

Proposition 34. Let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph of order n with $\lceil k/(p-1) \rceil \leq n_1 \leq \dots \leq n_p$, and let $p < k$. Let also $k = (p-1) \cdot \lceil k/(p-1) \rceil + r$ for some integer $1 \leq r \leq p-2$. If $n - n_{p-r} \geq n - n_{p-r+1} \geq \dots \geq n - n_p \geq \lceil kp/(p-1) \rceil - \lceil k/(p-1) \rceil + k$ and $n - n_1 \geq \dots \geq n - n_{p-r-1} \geq \lceil kp/(p-1) \rceil - \lceil k/(p-1) \rceil + k + 1$, then $\gamma_{\times k,t}^r(G) = \lceil kp/(p-1) \rceil$.

Proof. Let S be a set of vertices of cardinality $1 \leq i \leq p-r-1$, $|S \cap X_i| = \lceil k/(p-1) \rceil$ and for $p-r \leq i \leq p$, $|S \cap X_i| = \lceil k/(p-1) \rceil = \lceil k/(p-1) \rceil + 1$. Since S is a kTRDS of G of cardinality $\gamma_{\times k,t}(G)$, Propositions 4(iii) and 12(iv) imply that $\gamma_{\times k,t}^r(G) = \lceil kp/(p-1) \rceil$. $\square$

Comparing Propositions 29, 30, 31, 32, 33, and 34 shows that the following two results, which state the given bounds in Theorem 19 are sharp, are proved.

Proposition 35. For every integer $k \geq 1$, let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph of order n with $2k+2 \leq n_1 \leq n_2 \leq \dots \leq n_p$. If either $p \geq k+1$ or $p = k \geq 4$ and $2k \leq n - n_k \leq n - n_{k-1}$, $2k+1 \leq n - n_{k-2} \leq \dots \leq n - n_1$, then

$$\gamma_{\times k,t}^r(G\overline{G}) = \gamma_{\times k,t}^r(G) + \gamma_{\times k,t}^r(G\overline{G}). \quad (21)$$

Proposition 36. Let $G = K_{n_1, n_2, \dots, n_p}$ be a complete p -partite graph, and let $k \geq 1$ be an integer. If $k \leq n_1 \leq \dots \leq n_t \leq 2k - 1 < n_{t+1} \leq \dots \leq n_p$ and $n_1 + n_2 + \dots + n_t = p + 1 + tk$ for some integer $t \geq 1$, then

$$\gamma_{\times k, t}^r(G\overline{G}) = \gamma_{\times(k-1), t}^r(G) + \gamma_{\times(k-1), t}^r(\overline{G}). \quad (22)$$

We note that there are some complete multipartite graphs that satisfy Proposition 36. For example, if $k \geq 4$, then the complete $(k + 1)$ -partite graph $G = K_{n_1, n_2, \dots, n_{k+1}}$ with the conditions $n_1 = k + 3$, $n_2 = 2k - 1$, and $2k \leq n_3 \leq \dots \leq n_{k+1}$ satisfies it.

Also Propositions 14, 16, and 20 by this fact that $\gamma_{\times 2, t}^r(C_n) = n$, which is obtained by Proposition 12(i), imply that

$$\gamma_{\times 2, t}^r(C_n \overline{C_n}) = \gamma_{\times 2, t}^r(C_n) + \gamma_{\times k, t}^r(\overline{C_n}) \quad (23)$$

if and only if $n = 5$. Also they imply that

$$\begin{aligned} \gamma_t^r(C_n) + \gamma_t^r(\overline{C_n}) &< \gamma_{\times 2, t}^r(C_n \overline{C_n}) \\ &< \gamma_{\times 2, t}^r(C_n) + \gamma_{\times k, t}^r(\overline{C_n}), \end{aligned} \quad (24)$$

when $n \geq 4$ in the lower bound and $n \geq 6$ in the upper bound.

5. Open Problem

Finally, we finish our paper with the following problems.

Problem 37. Characterize graphs G that satisfy $\gamma_{\times k, t}^r(G\overline{G}) = \gamma_{\times k, t}^r(G) + \gamma_{\times k, t}^r(\overline{G})$.

Problem 38. Characterize graphs G that satisfy $\gamma_{\times k, t}^r(G\overline{G}) = \gamma_{\times(k-1), t}^r(G) + \gamma_{\times(k-1), t}^r(\overline{G})$.

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