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*Research Article*

# Jensen Functionals on Time Scales for Several Variables

**Matloob Anwar,[1] Rabia Bibi,[1] Martin Bohner,[2] and Josip Pečarić[3]**

[1] *School of Natural Sciences, National University of Sciences and Technology, H-12, Islamabad 44000, Pakistan*
[2] *Department of Mathematics and Statistics, Missouri University of Science and Technology, Rolla, MO 65409-0020, USA*
[3] *Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia*

Correspondence should be addressed to Rabia Bibi; emaorr@gmail.com

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We define Jensen functionals and concerned generalized means for several variables on time scales. We derive properties of Jensen functionals and apply them to generalized means. In this setting, we obtain generalizations, refinements, and conversions of many remarkable inequalities.

## 1. Introduction

Jensen's inequality is well known in analysis and many other areas of mathematics. Most of the classical inequalities can be obtained by using the Jensen inequality. For time scale theory, Jensen's inequality for one variable is obtained by Agarwal et al. [1], and now there are various extensions and generalizations of it given by many researchers (see [2–8]). In [3], it is shown that the Jensen inequality for one variable holds for time scale integrals including the Cauchy delta, Cauchy nabla, diamond-$\alpha$, Riemann, Lebesgue, multiple Riemann, and multiple Lebesgue integrals. Further, in [4], we give properties and applications of Jensen functionals on time scales for one variable.

In this paper, we obtain the Jensen inequality for several variables and deduce Jensen functionals. We discuss several properties and applications of Jensen functionals. In the sequel, we give all the results for Lebesgue delta integrals. For other time scale integrals, as mentioned above, all those results can be obtained in a similar way. These results generalize the results given in [4] for one variable. Now, we give a brief introduction of time scale integrals; for a detailed introduction we refer to [1, 9–12]. A time scale $\mathbb{T}$ is an arbitrary closed subset of $\mathbb{R}$, and time scale calculus provides unification and extension of classical results. For example, when $\mathbb{T} = \mathbb{R}$, the time scale integral is an ordinary integral, and when $\mathbb{T} = \mathbb{Z}$, the time scale integral becomes a sum. In [10, Chapter 5], the Lebesgue integral is introduced: let $[a,b) \subseteq \mathbb{T}$ be a time scale interval defined by

$$[a,b) = \{t \in \mathbb{T} : a \le t < b\}, \tag{1}$$

where $a, b \in \mathbb{T}$ with $a \le b$. Let $\mu_\Delta$ be the Lebesgue $\Delta$-measure on $[a,b)$. Suppose $f : [a,b) \to \mathbb{R}$ is a $\mu_\Delta$-measurable function. Then the Lebesgue $\Delta$-integral of $f$ on $[a,b)$ is denoted by

$$\int_{[a,b)} f d\mu_\Delta, \qquad \int_{[a,b)} f(t)\, d\mu_\Delta(t), \quad \text{or} \quad \int_{[a,b)} f(t)\, \Delta t. \tag{2}$$

All theorems of the general Lebesgue integration theory, including the Lebesgue dominated convergence theorem, hold also for Lebesgue $\Delta$-integrals on $\mathbb{T}$. Now, we give some properties of Lebesgue $\Delta$-integrals and state Jensen's inequality and Hölder's inequality for Lebesgue $\Delta$-integrals. Throughout this paper, $[a,b)$ denotes a time scale interval otherwise is specified.

**Theorem 1** (see [3, Theorem 3.2]). *If $f$ and $g$ are $\Delta$-integrable functions on $[a,b)$, then*

$$\int_{[a,b)} (\alpha f + \beta g)\, d\mu_\Delta = \alpha \int_{[a,b)} f d\mu_\Delta + \beta \int_{[a,b)} g d\mu_\Delta \quad \forall \alpha, \beta \in \mathbb{R},$$
$$f(t) \geq 0 \quad \forall t \in [a,b) \text{ implies } \int_{[a,b)} f d\mu_\Delta \geq 0. \tag{3}$$

**Theorem 2** (see [3, Theorem 4.2]). *Assume $\Phi \in C(I,\mathbb{R})$ is convex, where $I \subseteq \mathbb{R}$ is an interval. Suppose $f : [a,b) \to I$ is $\Delta$-integrable. Moreover, let $p : [a,b) \to \mathbb{R}$ be nonnegative and $\Delta$-integrable such that $\int_{[a,b)} p d\mu_\Delta > 0$. Then*

$$\Phi\left(\frac{\int_{[a,b)} p f d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right) \leq \frac{\int_{[a,b)} p(\Phi \circ f)\, d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}. \tag{4}$$

**Theorem 3** (see [3, Theorem 6.2]). *For $p \neq 1$, define $q = p/(p-1)$. Let $w$, $f$, $g$ be nonnegative functions such that $wf^p$, $wg^q$, $wfg$ are $\Delta$-integrable on $[a,b)$. If $p > 1$, then*

$$\int_{[a,b)} wfg d\mu_\Delta \leq \left(\int_{[a,b)} wf^p d\mu_\Delta\right)^{1/p} \left(\int_{[a,b)} wg^q d\mu_\Delta\right)^{1/q}. \tag{5}$$

*If $0 < p < 1$ and $\int_{[a,b)} wg^q d\mu_\Delta > 0$, or if $p < 0$ and $\int_{[a,b)} wf^p d\mu_\Delta > 0$, then (5) is reversed.*

*Remark 4.* Theorem 1 recalls that the Lebesgue $\Delta$-integral is an isotonic linear functional (see [13]). So we can also use the approach of isotonic linear functionals whenever results are known for isotonic linear functionals.

In the next section, we give Jensen inequality on time scales for several variables and define Jensen functionals. In Section 3, we investigate properties of Jensen functionals and some of its consequences regarding superadditivity and monotonicity. In Section 4, we apply these results to weighted general means, defined on time scales, and give many applications. Finally in Section 5, we give applications to Hölder's inequality on time scales.

## 2. Jensen Inequality and Jensen Functionals

Let $\mathbf{f}(t) = (f_1(t), \ldots, f_n(t))$ be an $n$-tuple of functions such that $f_1, \ldots, f_n$ are $\Delta$-integrable on $[a,b)$. Then $\int_{[a,b)} \mathbf{f} d\mu_\Delta$ denotes the $n$-tuple:

$$\left(\int_{[a,b)} f_1 d\mu_\Delta, \ldots, \int_{[a,b)} f_n d\mu_\Delta\right). \tag{6}$$

That is, $\Delta$-integral acts on each component of $\mathbf{f}$.

**Theorem 5** (Jensen inequality). *Assume $\Phi \in C(K,\mathbb{R})$ is convex, where $K \subseteq \mathbb{R}^n$ is closed and convex. Suppose $f_i$, $i = 1, 2, \ldots, n$, are $\Delta$-integrable on $[a,b)$ such that $\mathbf{f}(t) = (f_1(t), \ldots, f_n(t)) \in K$ for all $t \in [a,b)$. Moreover, let $p : [a,b) \to \mathbb{R}$ be nonnegative and $\Delta$-integrable such that $\int_{[a,b)} p d\mu_\Delta > 0$. Then*

$$\Phi\left(\frac{\int_{[a,b)} p\mathbf{f} d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right) \leq \frac{\int_{[a,b)} p\Phi(\mathbf{f})\, d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}. \tag{7}$$

*Proof.* Suppose $\Phi$ is convex on $K \subset \mathbb{R}^n$. Therefore, for every point $\mathbf{x}_0 \in K$, there exists a point $\boldsymbol{\lambda} \in \mathbb{R}^n$ (see [13, Theorem 1.31]) such that

$$\Phi(\mathbf{x}) - \Phi(\mathbf{x}_0) \geq \langle \boldsymbol{\lambda}, \mathbf{x} - \mathbf{x}_0 \rangle. \tag{8}$$

Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. By (8), we get

$$\begin{aligned}
&\frac{\int_{[a,b)} p\Phi(\mathbf{f})\, d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} - \Phi\left(\frac{\int_{[a,b)} p\mathbf{f} d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right) \\
&= \frac{\int_{[a,b)} p\left\{\Phi(\mathbf{f}) - \Phi\left(\int_{[a,b)} p\mathbf{f} d\mu_\Delta / \int_{[a,b)} p d\mu_\Delta\right)\right\} d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \\
&\geq \frac{\int_{[a,b)} p\left\langle \lambda, \mathbf{f} - \left(\int_{[a,b)} p\mathbf{f} d\mu_\Delta / \int_{[a,b)} p d\mu_\Delta\right)\right\rangle d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \\
&= \frac{\int_{[a,b)} p \sum_{i=1}^n \lambda_i \left(f_i - \left(\int_{[a,b)} p f_i d\mu_\Delta / \int_{[a,b)} p d\mu_\Delta\right)\right) d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \\
&= 0,
\end{aligned} \tag{9}$$

and hence the proof is completed. □

*Remark 6.* By using the fact that the time scale integral is an isotonic linear functional, Theorem 5 can also be obtained by using Theorem 1 and [13, Theorem 2.6].

*Definition 7.* Assume $\Phi \in C(K,\mathbb{R})$, where $K \subseteq \mathbb{R}^n$ is closed and convex. Suppose $f_i$, $i = 1, 2, \ldots, n$, are $\Delta$-integrable on $[a,b)$ such that $\mathbf{f}(t) = (f_1(t), \ldots, f_n(t)) \in K$ for all $t \in [a,b)$. Moreover, let $p : [a,b) \to \mathbb{R}$ be nonnegative and $\Delta$-integrable such that $\int_{[a,b)} p d\mu_\Delta > 0$. Then one defines the Jensen functional on time scales for several variables by

$$\begin{aligned}
\mathbf{J}_\Delta(\Phi, \mathbf{f}, p) = &\int_{[a,b)} p\Phi(\mathbf{f})\, d\mu_\Delta \\
&- \int_{[a,b)} p d\mu_\Delta \Phi\left(\frac{\int_{[a,b)} p\mathbf{f} d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)
\end{aligned} \tag{10}$$

*Remark 8.* By Theorem 5, the following statements are obvious. If $\Phi$ is convex, then

$$\mathbf{J}_\Delta(\Phi, \mathbf{f}, p) \geq 0 \tag{11}$$

while if $\Phi$ is concave, then

$$\mathbf{J}_\Delta(\Phi, \mathbf{f}, p) \leq 0. \tag{12}$$

*Example 9.* Let $[a,b) = \{1,2,\ldots,n\}$, $f_1(i) = x_{1_i},\ldots,f_n(i) = x_{n_i}$, and $p(i) = p_i$, $i = 1,2,\ldots,n$ in (10). Then the Jensen functional (10) becomes

$$\mathbf{J}_n(\Phi,\mathbf{X},\mathbf{p}) = \sum_{i=1}^{n} p_i\Phi(\mathbf{x}_i) - P_n\Phi\left(\frac{\sum_{i=1}^{n} p_i\mathbf{x}_i}{P_n}\right), \tag{13}$$

where $\mathbf{X} = (\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_n)$ with $\mathbf{x}_i = (x_{i_1},x_{i_2},\ldots,x_{i_n})$, $\mathbf{p} = (p_1,\ldots,p_n)$, and $P_n = \sum_{i=1}^{n} p_i > 0$. Some properties of the Jensen functional $\mathbf{J}_n$ are investigated in [14, 15].

*Example 10.* If $[a,b)$ is a real interval, then Jensen's functional (10) becomes

$$\begin{aligned}
&\int_{[a,b)} p(t)\,\Phi(f_1(t),f_2(t),\ldots,f_n(t))\,d\mu(t)\\
&\quad-\int_{[a,b)} p(t)\,d\mu(t)\\
&\quad\times\Phi\left(\frac{\int_{[a,b)} p(t)\,f_1(t)\,d\mu(t)}{\int_{[a,b)} p(t)\,d\mu(t)},\frac{\int_{[a,b)} p(t)\,f_2(t)\,d\mu(t)}{\int_{[a,b)} p(t)\,d\mu(t)},\ldots,\right.\\
&\qquad\left.\frac{\int_{[a,b)} p(t)\,f_n(t)\,d\mu(t)}{\int_{[a,b)} p(t)\,d\mu(t)}\right).
\end{aligned} \tag{14}$$

## 3. Properties of Jensen Functionals

In the following theorem, we give our main result concerning the properties of the Jensen functional (10).

**Theorem 11.** *Assume $\Phi \in C(K,\mathbb{R})$, where $K \subseteq \mathbb{R}^n$ is closed and convex. Suppose $f_i$, $i = 1,2,\ldots,n$, are $\Delta$-integrable on $[a,b)$ such that $\mathbf{f}(t) = (f_1(t),\ldots,f_n(t)) \in K$ for all $t \in [a,b)$. Moreover, let $p,q : [a,b) \to \mathbb{R}$ be nonnegative and $\Delta$-integrable such that $\int_{[a,b)} pd\mu_\Delta > 0$ and $\int_{[a,b)} qd\mu_\Delta > 0$. If $\Phi$ is convex, then $\mathbf{J}_\Delta(\Phi,\mathbf{f},\cdot)$ is superadditive; that is,*

$$\mathbf{J}_\Delta(\Phi,\mathbf{f},p+q) \geq \mathbf{J}_\Delta(\Phi,\mathbf{f},p) + \mathbf{J}_\Delta(\Phi,\mathbf{f},q), \tag{15}$$

*and $\mathbf{J}_\Delta(\Phi,\mathbf{f},\cdot)$ is increasing; that is, $p \geq q$ with $\int_{[a,b)} pd\mu_\Delta > \int_{[a,b)} qd\mu_\Delta$ implies*

$$\mathbf{J}_\Delta(\Phi,\mathbf{f},p) \geq \mathbf{J}_\Delta(\Phi,\mathbf{f},q). \tag{16}$$

*Moreover, if $\Phi$ is concave, then $\mathbf{J}_\Delta(\Phi,\mathbf{f},\cdot)$ is subadditive and decreasing; that is,* (15) *and* (16) *hold in reverse order.*

*Proof.* Let $\Phi$ be convex. Because the time scales integral is linear (see Theorem 1), it follows from Definition 7 that

$$\begin{aligned}
&\mathbf{J}_\Delta(\Phi,\mathbf{f},p+q)\\
&\quad=\int_{[a,b)}(p+q)\,\Phi(\mathbf{f})\,d\mu_\Delta\\
&\qquad-\int_{[a,b)}(p+q)\,d\mu_\Delta\Phi\left(\frac{\int_{[a,b)}(p+q)\,\mathbf{f}d\mu_\Delta}{\int_{[a,b)}(p+q)\,d\mu_\Delta}\right)\\
&\quad=\int_{[a,b)} p\Phi(\mathbf{f})\,d\mu_\Delta+\int_{[a,b)} q\Phi(\mathbf{f})\,d\mu_\Delta\\
&\qquad-\left(\int_{[a,b)} pd\mu_\Delta+\int_{[a,b)} qd\mu_\Delta\right)\\
&\qquad\times\Phi\left(\frac{\int_{[a,b)} p\mathbf{f}d\mu_\Delta+\int_{[a,b)} q\mathbf{f}d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta+\int_{[a,b)} qd\mu_\Delta}\right)\\
&\quad\geq\int_{[a,b)} p\Phi(\mathbf{f})\,d\mu_\Delta+\int_{[a,b)} q\Phi(\mathbf{f})\,d\mu_\Delta\\
&\qquad-\int_{[a,b)} pd\mu_\Delta\Phi\left(\frac{\int_{[a,b)} p\mathbf{f}d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}\right)\\
&\qquad-\int_{[a,b)} qd\mu_\Delta\Phi\left(\frac{\int_{[a,b)} q\mathbf{f}d\mu_\Delta}{\int_{[a,b)} qd\mu_\Delta}\right)\\
&\quad=\mathbf{J}_\Delta(\Phi,\mathbf{f},p)+\mathbf{J}_\Delta(\Phi,\mathbf{f},q).
\end{aligned} \tag{17}$$

If $p \geq q$, we have $p - q \geq 0$. Now, because Jensen's functional is superadditive and nonnegative, we have

$$\begin{aligned}
\mathbf{J}_\Delta(\Phi,\mathbf{f},p) &= \mathbf{J}_\Delta(\Phi,\mathbf{f},p-q+q)\\
&\geq \mathbf{J}_\Delta(\Phi,\mathbf{f},p-q)+\mathbf{J}_\Delta(\Phi,\mathbf{f},q)\\
&\geq \mathbf{J}_\Delta(\Phi,\mathbf{f},q).
\end{aligned} \tag{18}$$

On the other hand, if $\Phi$ is concave, then the reversed inequalities of (15) and (16) can be obtained in a similar way. □

**Corollary 12.** *Let $\Phi$, $\mathbf{f}$, $p$, $q$ satisfy the hypotheses of Theorem 11. Further, suppose there exist nonnegative constants $m$ and $M$ such that*

$$\begin{aligned}
&Mq(t) \geq p(t) \geq mq(t) \quad \forall t \in [a,b),\\
&M\int_{[a,b)} qd\mu_\Delta > \int_{[a,b)} pd\mu_\Delta > m\int_{[a,b)} qd\mu_\Delta.
\end{aligned} \tag{19}$$

*If $\Phi$ is convex, then*

$$M\mathbf{J}_\Delta(\Phi,\mathbf{f},q) \geq \mathbf{J}_\Delta(\Phi,\mathbf{f},p) \geq m\mathbf{J}_\Delta(\Phi,\mathbf{f},q), \tag{20}$$

*while if $\Phi$ is concave, then the inequalities in* (20) *hold in reverse order.*

*Proof.* By using (10), we have

$$\begin{aligned} \mathbf{J}_{\Delta}(\Phi, \mathbf{f}, mq) &= m\mathbf{J}_{\Delta}(\Phi, \mathbf{f}, q), \\ \mathbf{J}_{\Delta}(\Phi, \mathbf{f}, Mq) &= M\mathbf{J}_{\Delta}(\Phi, \mathbf{f}, q). \end{aligned} \tag{21}$$

Now the result follows from the second property of Theorem 11. □

**Corollary 13.** *Let $\Phi, \mathbf{f}, p$ satisfy the hypotheses of Theorem 11. Further, assume that $p$ attains its minimum value and its maximum value on its domain. If $\Phi$ is convex, then*

$$\begin{aligned} \left[\max_{t\in[a,b)} p(t)\right] \mathfrak{J}_{\Delta}(\Phi, \mathbf{f}) &\geq \mathbf{J}_{\Delta}(\Phi, \mathbf{f}, p) \\ &\geq \left[\min_{t\in[a,b)} p(t)\right] \mathfrak{J}_{\Delta}(\Phi, \mathbf{f}), \end{aligned} \tag{22}$$

*where*

$$\mathfrak{J}_{\Delta}(\Phi, \mathbf{f}) = \int_{[a,b)} \Phi(\mathbf{f})\, d\mu_{\Delta} - (b-a)\,\Phi\left(\frac{\int_{[a,b)} \mathbf{f} d\mu_{\Delta}}{b-a}\right). \tag{23}$$

*Moreover, if $\Phi$ is concave, then the inequalities in (22) hold in reverse order.*

*Proof.* Let $p$ attain its minimum and maximum values on its domain $[a,b)$. Then

$$\max_{t\in[a,b)} p(t) \geq p(t) \geq \min_{t\in[a,b)} p(t). \tag{24}$$

Let

$$\overline{p}(t) = \max_{t\in[a,b)} p(t), \qquad \underline{p}(t) = \min_{t\in[a,b)} p(t). \tag{25}$$

By using (10), we have

$$\begin{aligned} \mathbf{J}_{\Delta}(\Phi, \mathbf{f}, \overline{p}) &= \left[\max_{t\in[a,b)} p(t)\right] \mathfrak{J}_{\Delta}(\Phi, \mathbf{f}), \\ \mathbf{J}_{\Delta}(\Phi, \mathbf{f}, \underline{p}) &= \left[\min_{t\in[a,b)} p(t)\right] \mathfrak{J}_{\Delta}(\Phi, \mathbf{f}). \end{aligned} \tag{26}$$

Now the result follows from the second property of Theorem 11. □

*Example 14.* Let the functional $\mathbf{J}_n(\Phi, \mathbf{X}, \mathbf{p})$ be defined as in Example 9. Let $\mathbf{q} = (q_1, \ldots, q_n)$ with $q_i \geq 0$ and $\sum_{i=1}^{n} q_i = Q_n > 0$. If $\Phi$ is convex, then Theorem 11 implies $\mathbf{J}_n(\Phi, \mathbf{X}, \cdot)$ is superadditive; that is,

$$\mathbf{J}_n(\Phi, \mathbf{X}, \mathbf{p}+\mathbf{q}) \geq \mathbf{J}_n(\Phi, \mathbf{X}, \mathbf{p}) + \mathbf{J}_n(\Phi, \mathbf{X}, \mathbf{q}), \tag{27}$$

and $\mathbf{J}_n(\Phi, \mathbf{X}, \cdot)$ is increasing; that is, if $\mathbf{p} \geq \mathbf{q}$ such that $P_n > Q_n$, then

$$\mathbf{J}_n(\Phi, \mathbf{X}, \mathbf{p}) \geq \mathbf{J}_n(\Phi, \mathbf{X}, \mathbf{q}). \tag{28}$$

Moreover, if $\Phi$ is concave, then the inequalities in (27) and (28) hold in reverse order. If $\mathbf{p}$ attains its minimum and maximum values on its domain, then Corollary 13 yields

$$\max_{1\leq i\leq n}\{p_i\}\, \mathfrak{J}_n(\Phi, \mathbf{X}) \geq \mathbf{J}_n(\Phi, \mathbf{X}, \mathbf{p}) \geq \min_{1\leq i\leq n}\{p_i\}\, \mathfrak{J}_n(\Phi, \mathbf{X}), \tag{29}$$

where

$$\mathfrak{J}_n(\Phi, \mathbf{X}) = \sum_{i=1}^{n} \Phi(\mathbf{x}_i) - n\Phi\left(\frac{\sum_{i=1}^{n} \mathbf{x}_i}{n}\right), \tag{30}$$

if $\Phi$ is convex. Further, the inequalities in (29) hold in reverse order if $\Phi$ is concave.

## 4. Applications to Weighted Generalized Means

In the sequel, $I \subset \mathbb{R}$ is an interval and $K \subset \mathbb{R}^n$ is closed and convex.

*Definition 15.* Assume $\chi \in C(I, \mathbb{R})$ is strictly monotone and $\varphi : K \to I$ is a function of $n$ variables. Suppose $f_i$, $i = 1, 2, \ldots, n$, are $\Delta$-integrable on $[a,b)$ such that $\mathbf{f}(t) = (f_1(t), \ldots, f_n(t)) \in K$ for all $t \in [a,b)$. Moreover, let $p : [a,b) \to \mathbb{R}$ be nonnegative and $\Delta$-integrable such that $p\chi(\varphi(\mathbf{f}))$ is $\Delta$-integrable and $\int_{[a,b)} p d\mu_{\Delta} > 0$. Then one defines the weighted generalized mean on time scales by

$$\mathbf{M}_{\Delta}(\chi, \varphi(\mathbf{f}), p) = \chi^{-1}\left(\frac{\int_{[a,b)} p\chi(\varphi(\mathbf{f}))\, d\mu_{\Delta}}{\int_{[a,b)} p d\mu_{\Delta}}\right). \tag{31}$$

**Theorem 16.** *Assume $\chi, \psi_i \in C(I, \mathbb{R})$, $i = 1, 2, \ldots, n$, are strictly monotone and $\varphi : K \to I \subseteq \mathbb{R}$ is a function of $n$ variables. Suppose $f_i : [a,b) \to I$, $i = 1, 2, \ldots, n$, are $\Delta$-integrable such that $\mathbf{f}(t) = (f_1(t), \ldots, f_n(t)) \in K$ for all $t \in [a,b)$. Moreover, let $p, q : [a,b) \to \mathbb{R}$ be nonnegative and $\Delta$-integrable such that $p\chi(\varphi(\mathbf{f}))$, $q\chi(\varphi(\mathbf{f}))$, $p\psi_i(f_i))$, $q\psi_i(f_i))$, $i = 1, 2, \ldots, n$, are $\Delta$-integrable and $\int_{[a,b)} p d\mu_{\Delta} > 0$, $\int_{[a,b)} q d\mu_{\Delta} > 0$. If $H$ defined by*

$$H(s_1, \ldots, s_n) = \chi \circ \varphi\left(\psi_1^{-1}(s_1), \ldots, \psi_n^{-1}(s_n)\right) \tag{32}$$

*is convex, then the functional*

$$\int_{[a,b)} p d\mu_{\Delta} \left[\chi\left(\mathbf{M}_{\Delta}(\chi, \varphi(\mathbf{f}), p)\right) - \chi \circ \varphi\left(\mathbf{M}_{\Delta}(\psi_1, f_1, p), \ldots, \mathbf{M}_{\Delta}(\psi_n, f_n, p)\right)\right] \tag{33}$$

*is superadditive, that is,*

$$\begin{aligned}
&\int_{[a,b)} (p+q)\, d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta\left(\chi, \varphi(\mathbf{f}), p+q\right)\right)\right. \\
&\qquad - \chi\circ\varphi\left(\mathbf{M}_\Delta\left(\psi_1, f_1, p+q\right), \ldots,\right. \\
&\qquad\qquad \left.\left.\mathbf{M}_\Delta\left(\psi_n, f_n, p+q\right)\right)\right] \\
&\geq \int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta\left(\chi, \varphi(\mathbf{f}), p\right)\right)\right. \\
&\qquad - \chi\circ\varphi\left(\mathbf{M}_\Delta\left(\psi_1, f_1, p\right), \ldots,\right. \\
&\qquad\qquad \left.\left.\mathbf{M}_\Delta\left(\psi_n, f_n, p\right)\right)\right] \\
&\quad + \int_{[a,b)} q d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta\left(\chi, \varphi(\mathbf{f}), q\right)\right)\right. \\
&\qquad - \chi\circ\varphi\left(\mathbf{M}_\Delta\left(\psi_1, f_1, q\right), \ldots,\right. \\
&\qquad\qquad \left.\left.\mathbf{M}_\Delta\left(\psi_n, f_n, q\right)\right)\right],
\end{aligned} \tag{34}$$

*and increasing; that is,* $p \geq q$ *with* $\int_{[a,b)} p d\mu_\Delta > \int_{[a,b)} q d\mu_\Delta$ *implies*

$$\begin{aligned}
&\int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta\left(\chi, \varphi(\mathbf{f}), p\right)\right)\right. \\
&\qquad - \chi\circ\varphi\left(\mathbf{M}_\Delta\left(\psi_1, f_1, p\right), \ldots,\right. \\
&\qquad\qquad \left.\left.\mathbf{M}_\Delta\left(\psi_n, f_n, p\right)\right)\right] \\
&\geq \int_{[a,b)} q d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta\left(\chi, \varphi(\mathbf{f}), q\right)\right)\right. \\
&\qquad - \chi\circ\varphi\left(\mathbf{M}_\Delta\left(\psi_1, f_1, q\right), \ldots,\right. \\
&\qquad\qquad \left.\left.\mathbf{M}_\Delta\left(\psi_n, f_n, q\right)\right)\right].
\end{aligned} \tag{35}$$

*Moreover, if $H$ is continuous and concave, then* (33) *is subadditive and decreasing; that is,* (34) *and* (35) *hold in reverse order.*

*Proof.* The functional defined in (33) is obtained by replacing $\Phi$ with $H$ and $f_i$ with $\psi_i(f_i)$, $i = 1, 2, \ldots, n$, in the Jensen functional (10) and letting $\Psi(\mathbf{f}) = (\psi_1(f_1), \ldots, \psi_n(f_n))$; that is,

$$\begin{aligned}
&\mathbf{J}_\Delta\left(H, \Psi(\mathbf{f}), p\right) \\
&\quad = \int_{[a,b)} p\chi\circ\varphi\left(f_1, \ldots, f_n\right) d\mu_\Delta - \int_{[a,b)} p d\mu_\Delta \\
&\qquad \times H\left(\frac{\int_{[a,b)} p\psi_1\left(f_1\right) d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}, \ldots, \frac{\int_{[a,b)} p\psi_n\left(f_n\right) d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right) \\
&\quad = \int_{[a,b)} p d\mu_\Delta \chi\left(\mathbf{M}_\Delta\left(\chi, \varphi(\mathbf{f}), p\right)\right) \\
&\qquad - \int_{[a,b)} p d\mu_\Delta \chi\circ\varphi\left(\mathbf{M}_\Delta\left(\psi_1, f_1, p\right), \ldots, \mathbf{M}_\Delta\left(\psi_n, f_n, p\right)\right) \\
&\quad = \int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta\left(\chi, \varphi(\mathbf{f}), p\right)\right)\right. \\
&\qquad - \chi\circ\varphi\left(\mathbf{M}_\Delta\left(\psi_1, f_1, p\right), \ldots,\right. \\
&\qquad\qquad \left.\left.\mathbf{M}_\Delta\left(\psi_n, f_n, p\right)\right)\right].
\end{aligned} \tag{36}$$

Now, all claims follow immediately from Theorem 11. □

**Corollary 17.** *Let $H$, $\varphi$, $\mathbf{f}$, $p$, $\chi$, $f_i$, and $\psi_i$, $i = 1, \ldots, n$, satisfy the hypothesis of Theorem 16. Further, assume that $p$ attains its minimum value and its maximum value on its domain. If $H$ is convex, then*

$$\begin{aligned}
&\left[\max_{t\in[a,b)} p(t)\right](b-a) \\
&\quad \times \left[\chi\left(\mathfrak{M}_\Delta\left(\chi, \varphi(\mathbf{f})\right)\right) - \chi\circ\varphi\left(\mathfrak{M}_\Delta\left(\psi_1, f_1\right), \ldots,\right.\right. \\
&\qquad\qquad \left.\left.\mathfrak{M}_\Delta\left(\psi_n, f_n\right)\right)\right] \\
&\geq \int_{[a,b)} p d\mu_\Delta \left[\chi\left(\mathbf{M}_\Delta\left(\chi, \varphi(\mathbf{f}), p\right)\right)\right. \\
&\qquad \left.-\chi\circ\varphi\left(\mathbf{M}_\Delta\left(\psi_1, f_1, p\right), \ldots, \mathbf{M}_\Delta\left(\psi_n, f_n, p\right)\right)\right] \\
&\geq \left[\min_{t\in[a,b)} p(t)\right](b-a) \\
&\quad \times \left[\chi\left(\mathfrak{M}_\Delta\left(\chi, \varphi(\mathbf{f})\right)\right) - \chi\circ\varphi\left(\mathfrak{M}_\Delta\left(\psi_1, f_1\right), \ldots,\right.\right. \\
&\qquad\qquad \left.\left.\mathfrak{M}_\Delta\left(\psi_n, f_n\right)\right)\right],
\end{aligned} \tag{37}$$

*where*

$$\mathfrak{M}_\Delta\left(\chi, \varphi(\mathbf{f})\right) = \chi^{-1}\left(\frac{\int_{[a,b)} \chi\left(\varphi(\mathbf{f})\right) d\mu_\Delta}{b-a}\right). \tag{38}$$

*Moreover, if $H$ is concave, then the inequalities in* (37) *hold in reverse order.*

*Proof.* The proof is omitted as it is similar to the proof of Corollary 13. □

*Remark 18.* If we take the discrete form of the weighted generalized mean (31) with $\int_{[a,b)} p d\mu_\Delta = 1$, then we obtain the quasiarithmetic mean. Namely, let $\psi : I \subseteq \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be continuous and strictly monotone, $\mathbf{a} = (a_1, \ldots, a_n)$ with $a_k \in I$, $k = 1, \ldots, n$, and $\mathbf{w} = (w_1, \ldots, w_n)$ with $w_k \geq 0$ and $\sum_{k=1}^n w_k = 1$. Then the quasiarithmetic mean of $\mathbf{a}$ with weight $\mathbf{w}$ is defined by

$$\mathbf{M}_n = \psi^{-1}\left(\sum_{k=1}^n w_k \psi\left(a_k\right)\right). \tag{39}$$

Now the following examples connect the quasiarithmetic mean (39) and the properties of Jensen functionals.

*Example 19* (see [16, Corollary 3]). Let $\mathbf{w}$ and $\psi$ be defined as in Remark 18 and let $\psi$ be strictly increasing and strictly

convex with continuous derivatives of second order such that $\psi'/\psi''$ is concave. Further, let $\mathbf{X}$, $\mathbf{p}$, $\mathbf{x}_i$, $i = 1, \ldots, n$, be defined as in Example 9, and $\mathbf{q} = (q_1, \ldots, q_n)$ with $q_i \geq 0$, $i = 1, \ldots, n$, and $\sum_{i=1}^{n} q_i = Q_n > 0$. Then, $\Phi_{\mathbf{M}_n}(\mathbf{x}_i) = \psi^{-1}(\sum_{k=1}^{n} w_k \psi(x_{i_k}))$ is a convex function (see [17, Theorem 1, page 197]). Hence by Theorem 11, the functional

$$\mathbf{J}_n\left(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{p}\right) = \sum_{i=1}^{n} p_i \Phi_{\mathbf{M}_n}(\mathbf{x}_i) - P_n \Phi_{\mathbf{M}_n}\left(\frac{\sum_{i=1}^{n} p_i \mathbf{x}_i}{P_n}\right) \quad (40)$$

is superadditive, that is,

$$\mathbf{J}_n\left(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{p}+\mathbf{q}\right) \geq \mathbf{J}_n\left(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{p}\right) + \mathbf{J}_n\left(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{q}\right), \quad (41)$$

and increasing; that is, if $\mathbf{p} \geq \mathbf{q}$ such that $P_n > Q_n$, then

$$\mathbf{J}_n\left(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{p}\right) \geq \mathbf{J}_n\left(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{q}\right). \quad (42)$$

Also, by Corollary 12, we have

$$\begin{aligned} \max_{1\leq i\leq n}\{p_i\}\, \mathfrak{J}_n\left(\Phi_{\mathbf{M}_n}, \mathbf{X}\right) &\geq \mathbf{J}_n\left(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{p}\right) \\ &\geq \min_{1\leq i\leq n}\{p_i\}\, \mathfrak{J}_n\left(\Phi_{\mathbf{M}_n}, \mathbf{X}\right), \end{aligned} \quad (43)$$

where

$$\mathfrak{J}_n\left(\Phi_{\mathbf{M}_n}, \mathbf{X}\right) = \sum_{i=1}^{n} \Phi_{\mathbf{M}_n}(\mathbf{x}_i) - n\Phi_{\mathbf{M}_n}\left(\frac{\sum_{i=1}^{n} \mathbf{x}_i}{n}\right). \quad (44)$$

*Example 20* (see [16, Corollary 4]). Consider (39), but with different conditions on $\psi$ and $\mathbf{w}$. Namely, if

(i) $w_i \geq 1$ for $i = 1, \ldots, n$;

(ii) $\psi : \mathbb{R}^+ \to \mathbb{R}^+$;

(iii) $\lim_{x\to 0}\psi(x) = \infty$ or $\lim_{x\to\infty}\psi(x) = \infty$,

then we define

$$\widetilde{\mathbf{M}}_n = \psi^{-1}\left(\sum_{k=1}^{n} w_k \psi(a_k)\right). \quad (45)$$

Let $\mathbf{X}$, $\mathbf{p}$, $\mathbf{x}_i$, $i = 1, \ldots, n$, be defined as in Example 9 and $\mathbf{q} = (q_1, \ldots, q_n)$ with $q_i \geq 0$ and $\sum_{i=1}^{n} q_i = Q_n > 0$. Let $\psi$ be strictly increasing and strictly convex with continuous derivatives of second order such that $\psi/\psi'$ is convex. Then $\Phi_{\widetilde{\mathbf{M}}_n}(\mathbf{x}_i) = \psi^{-1}(\sum_{k=1}^{n} w_k \psi(x_{i_k}))$ is a convex function (see [17, Theorem 2, page 197]). Hence, by Theorem 11, the functional

$$\mathbf{J}_n\left(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{p}\right) = \sum_{i=1}^{n} p_i \Phi_{\widetilde{\mathbf{M}}_n}\left(x_{i_k}\right) - P_n \Phi_{\widetilde{\mathbf{M}}_n}\left(\frac{\sum_{i=1}^{n} p_i x_{i_k}}{P_n}\right) \quad (46)$$

is superadditive, that is,

$$\mathbf{J}_n\left(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{p}+\mathbf{q}\right) \geq \mathbf{J}_n\left(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{p}\right) + \mathbf{J}_n\left(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{q}\right), \quad (47)$$

and increasing; that is, if $p \geq q$, then

$$\mathbf{J}_n\left(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{p}\right) \geq \mathbf{J}_n\left(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}, \mathbf{q}\right). \quad (48)$$

Also, by Corollary 12, we have

$$\begin{aligned} \max_{1\leq i\leq n}\{p_i\}\, \mathfrak{J}_n\left(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}\right) &\geq \mathbf{J}_n\left(\Phi_{\mathbf{M}_n}, \mathbf{X}, \mathbf{p}\right) \\ &\geq \min_{1\leq i\leq n}\{p_i\}\, \mathfrak{J}_n\left(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}\right), \end{aligned} \quad (49)$$

where

$$\mathfrak{J}_n\left(\Phi_{\widetilde{\mathbf{M}}_n}, \mathbf{X}\right) = \sum_{i=1}^{n} \Phi_{\widetilde{\mathbf{M}}_n}\left(x_{i_k}\right) - n\Phi_{\widetilde{\mathbf{M}}_n}\left(\frac{\sum_{i=1}^{n} x_{i_k}}{n}\right). \quad (50)$$

*Example 21* (see [16, Corollary 5]). For a real-valued function $f$ defined on interval $[a, b]$, an $n$th order divided difference of $f$ at distinct points $x_0, \ldots, x_n \in [a, b]$ is defined recursively by

$$\begin{aligned} [x_i]\, f &= f(x_i), \quad i = 0, \ldots, n, \\ [x_0, \ldots, x_n]\, f &= \frac{[x_1, \ldots, x_n]\, f - [x_0, \ldots, x_{n-1}]\, f}{x_n - x_0}. \end{aligned} \quad (51)$$

Further, $f$ is $n$-convex on $[a, b]$, $n \geq 0$, if and only if, for all choices of $n+1$ distinct points in $[a, b]$,

$$[x_0, \ldots, x_n]\, f \geq 0. \quad (52)$$

Let $\mathbf{X}$, $\mathbf{p}$, $\mathbf{x}_i$, $i = 1, \ldots, n$, be defined as in Example 9 and $\mathbf{q} = (q_1, \ldots, q_n)$, with $q_i \geq 0$ and $\sum_{i=1}^{n} q_i = Q_n > 0$. Let $f : I \to \mathbb{R}$ be $(n+1)$-convex, where $I \subset \mathbb{R}$ is a closed and bounded interval. Then by Theorem 11, for $\Phi_G(\mathbf{x}_i) = [x_{i_1}, \ldots, x_{i_n}]\, f$, the functional

$$\mathbf{J}_n\left(\Phi_G, \mathbf{X}, \mathbf{p}\right) = \sum_{i=1}^{n} p_i \Phi_G(\mathbf{x}_i) - P_n \Phi_G\left(\frac{\sum_{i=1}^{n} p_i \mathbf{x}_i}{P_n}\right) \quad (53)$$

is superadditive, that is,

$$\mathbf{J}_n\left(\Phi_G, \mathbf{X}, \mathbf{p}+\mathbf{q}\right) \geq \mathbf{J}_n\left(\Phi_G, \mathbf{X}, \mathbf{p}\right) + \mathbf{J}_n\left(\Phi_G, \mathbf{X}, \mathbf{q}\right), \quad (54)$$

and increasing; that is, if $\mathbf{p} \geq \mathbf{q}$ such that $P_n > Q_n$, then

$$\mathbf{J}_n\left(\Phi_G, \mathbf{X}, \mathbf{p}\right) \geq \mathbf{J}_n\left(\Phi_G, \mathbf{X}, \mathbf{q}\right). \quad (55)$$

Also, by Corollary 12, we have

$$\begin{aligned} \max_{1\leq i\leq n}\{p_i\}\, \mathfrak{J}_n\left(\Phi_G, \mathbf{X}\right) &\geq \mathbf{J}_n\left(\Phi_G, \mathbf{X}, \mathbf{p}\right) \\ &\geq \min_{1\leq i\leq n}\{p_i\}\, \mathfrak{J}_n\left(\Phi_G, \mathbf{X}\right), \end{aligned} \quad (56)$$

where

$$\mathfrak{J}_n\left(\Phi_G, \mathbf{X}\right) = \sum_{i=1}^{n} \Phi_G(\mathbf{x}_i) - n\Phi_G\left(\frac{\sum_{i=1}^{n} \mathbf{x}_i}{n}\right). \quad (57)$$

**Corollary 22.** *Assume $\chi$, $\psi_1$, and $\psi_2 \in C^2(I, \mathbb{R})$ are strictly monotone. Suppose $f_1, f_2 : [a, b) \to I$ are $\Delta$-integrable such that $f_1(t) + f_2(t) \in I$ for all $t \in [a, b)$ and $p, q : [a, b) \to \mathbb{R}$ are nonnegative and $\Delta$-integrable such that $p\chi(f_1 + f_2)$,*

*$q\chi(f_1+f_2)$, $p\psi_i(f_i)$, and $q\psi_i(f_i)$, $i=1,2$, are $\Delta$-integrable and $\int_{[a,b)} pd\mu_\Delta > 0$, $\int_{[a,b)} qd\mu_\Delta > 0$. Further, let*

$$E=\frac{\psi_1'}{\psi_1''},\qquad F=\frac{\psi_2'}{\psi_2''},\qquad G=\frac{\chi'}{\chi''}. \tag{58}$$

*If $\psi_1'$, $\psi_2'$, and $\chi'$ are positive and $\psi_1''$, $\psi_2''$, and $\chi''$ are negative, then the functional*

$$\begin{aligned}\int_{[a,b)} pd\mu_\Delta\,[&\chi(\mathbf{M}_\Delta(\chi,f_1+f_2,p))\\ &-\chi(\mathbf{M}_\Delta(\psi_1,f_1,p)+\mathbf{M}_\Delta(\psi_2,f_2,p))]\end{aligned} \tag{59}$$

*is superadditive, that is,*

$$\begin{aligned}\int_{[a,b)}(p+q)\,d\mu_\Delta\,[&\chi(\mathbf{M}_\Delta(\chi,f_1+f_2,p+q))\\ &-\chi(\mathbf{M}_\Delta(\psi_1,f_1,p+q)\\ &\quad+\mathbf{M}_\Delta(\psi_2,f_2,p+q))]\\ \geq\int_{[a,b)} pd\mu_\Delta\,[&\chi(\mathbf{M}_\Delta(\chi,f_1+f_2,p))\\ &-\chi(\mathbf{M}_\Delta(\psi_1,f_1,p)+\mathbf{M}_\Delta(\psi_2,f_2,p))]\\ +\int_{[a,b)} qd\mu_\Delta\,[&\chi(\mathbf{M}_\Delta(\chi,f_1+f_2,q))\\ &-\chi(\mathbf{M}_\Delta(\psi_1,f_1,q)+\mathbf{M}_\Delta(\psi_2,f_2,q))],\end{aligned} \tag{60}$$

*and increasing; that is, if $p\geq q$ such that $\int_{[a,b)} pd\mu_\Delta > \int_{[a,b)} qd\mu_\Delta$, then*

$$\begin{aligned}\int_{[a,b)} pd\mu_\Delta\,[&\chi(\mathbf{M}_\Delta(\chi,f_1+f_2,p))\\ &-\chi(\mathbf{M}_\Delta(\psi_1,f_1,p)+\mathbf{M}_\Delta(\psi_2,f_2,p))]\\ \geq\int_{[a,b)} qd\mu_\Delta\,[&\chi(\mathbf{M}_\Delta(\chi,f_1+f_2,q))\\ &-\chi(\mathbf{M}_\Delta(\psi_1,f_1,q)+\mathbf{M}_\Delta(\psi_2,f_2,q))],\end{aligned} \tag{61}$$

*if and only if $G(x+y)\leq E(x)+F(y)$. If $p$ attains its minimum and maximum values on its domain $[a,b)$, then (61) yields*

$$\begin{aligned}\left[\max_{t\in[a,b)} p(t)\right](b-a)\,[&\chi(\mathfrak{M}_\Delta(\chi,f_1+f_2))\\ &-\chi(\mathfrak{M}_\Delta(\psi_1,f_1)+\mathfrak{M}_\Delta(\psi_2,f_2))]\\ \geq\int_{[a,b)} pd\mu_\Delta\,[&\chi(\mathbf{M}_\Delta(\chi,f_1+f_2,p))\\ &-\chi(\mathbf{M}_\Delta(\psi_1,f_1,p)+\mathbf{M}_\Delta(\psi_2,f_2,p))]\\ \geq\left[\min_{t\in[a,b)} p(t)\right](b-a)\,[&\chi(\mathfrak{M}_\Delta(\chi,f_1+f_2))\\ &-\chi(\mathfrak{M}_\Delta(\psi_1,f_1)+\mathfrak{M}_\Delta(\psi_2,f_2))].\end{aligned} \tag{62}$$

*Moreover, if $\psi_1'$, $\psi_2'$, $\chi'$, $\psi_1''$, $\psi_2''$, and $\chi''$ are all positive, then the inequalities in (60), (61), and (62) are reversed if and only if $G(x+y)\geq E(x)+F(y)$.*

*Proof.* Let $n=2$ in Theorem 16. By setting $\varphi(x,y)=x+y$, we have

$$H(s_1,s_2)=\chi\left(\psi_1^{-1}(s_1)+\psi_2^{-1}(s_2)\right). \tag{63}$$

If $\psi_1'$, $\psi_2'$, and $\chi'$ are positive and $\psi_1''$, $\psi_2''$, and $\chi''$ are negative, then $H$ is convex if and only if $G(x+y)\leq E(x)+F(y)$ (see [18]). If $\psi_1'$, $\psi_2'$, $\chi'$, $\psi_1''$, $\psi_2''$, and $\chi''$ are all positive, then $H$ is concave if and only if $G(x+y)\geq E(x)+F(y)$ (see [18]). Now, all claims follow immediately from Theorem 16. □

**Corollary 23.** *Assume $\chi$, $\psi_1$, and $\psi_2\in C^2(I,\mathbb{R})$ are strictly monotone. Suppose $f_1,f_2:[a,b)\to I$ are $\Delta$-integrable such that $f_1(t)f_2(t)\in I$ for all $t\in[a,b)$ and $p,q:[a,b)\to\mathbb{R}$ are nonnegative and $\Delta$-integrable such that $p\chi(f_1\cdot f_2)$, $q\chi(f_1\cdot f_2)$, $p\psi_i(f_i)$, and $q\psi_i(f_i)$, $i=1,2$, are $\Delta$-integrable and $\int_{[a,b)} pd\mu_\Delta>0$, $\int_{[a,b)} qd\mu_\Delta>0$. Further, let*

$$\begin{aligned}A(t)&=\frac{\psi_1'(t)}{\psi_1'(t)+t\psi_1''(t)},\qquad B(t)=\frac{\psi_2'(t)}{\psi_2'(t)+t\psi_2''(t)},\\ C(t)&=\frac{\chi'(t)}{\chi'(t)+t\chi''(t)}.\end{aligned} \tag{64}$$

*If $\psi_1'$, $\psi_2'$, and $\chi'$ are positive and $A$, $B$, and $C$ are negative, then the functional*

$$\begin{aligned}\int_{[a,b)} pd\mu_\Delta\,[&\chi(\mathbf{M}_\Delta(\chi,f_1\cdot f_2,p))\\ &-\chi(\mathbf{M}_\Delta(\psi_1,f_1,p)\cdot\mathbf{M}_\Delta(\psi_2,f_2,p))]\end{aligned} \tag{65}$$

*is superadditive, that is,*

$$\begin{aligned}\int_{[a,b)}(p+q)\,d\mu_\Delta\,[&\chi(\mathbf{M}_\Delta(\chi,f_1\cdot f_2,p+q))\\ &-\chi(\mathbf{M}_\Delta(\psi_1,f_1,p+q)\\ &\quad\cdot\mathbf{M}_\Delta(\psi_2,f_2,p+q))]\\ \geq\int_{[a,b)} pd\mu_\Delta\,[&\chi(\mathbf{M}_\Delta(\chi,f_1\cdot f_2,p))\\ &-\chi(\mathbf{M}_\Delta(\psi_1,f_1,p)\cdot\mathbf{M}_\Delta(\psi_2,f_2,p))]\\ +\int_{[a,b)} qd\mu_\Delta\,[&\chi(\mathbf{M}_\Delta(\chi,f_1\cdot f_2,q))\\ &-\chi(\mathbf{M}_\Delta(\psi_1,f_1,q)\cdot\mathbf{M}_\Delta(\psi_2,f_2,q))],\end{aligned} \tag{66}$$

*and increasing; that is, if* $p \geq q$ *such that* $\int_{[a,b)} p d\mu_\Delta > \int_{[a,b)} q d\mu_\Delta$, *then*

$$\begin{aligned}\int_{[a,b)} p d\mu_\Delta &[\chi(\mathbf{M}_\Delta(\chi, f_1 \cdot f_2, p)) \\ &\quad -\chi(\mathbf{M}_\Delta(\psi_1, f_1, p) \cdot \mathbf{M}_\Delta(\psi_2, f_2, p))] \\ \geq \int_{[a,b)} q d\mu_\Delta &[\chi(\mathbf{M}_\Delta(\chi, f_1 \cdot f_2, q)) \\ &\quad -\chi(\mathbf{M}_\Delta(\psi_1, f_1, q) \cdot \mathbf{M}_\Delta(\psi_2, f_2, q))],\end{aligned} \tag{67}$$

*if and only if* $C(x \cdot y) \leq A(x) + B(y)$. *If* $p$ *attains its minimum and maximum values on its domain* $[a,b)$, *then* (67) *yields*

$$\begin{aligned}\left[\max_{t\in[a,b)} p(t)\right](b-a) &[\chi(\mathfrak{M}_\Delta(\chi, f_1 \cdot f_2)) \\ &\quad -\chi(\mathfrak{M}_\Delta(\psi_1, f_1) \cdot \mathfrak{M}_\Delta(\psi_2, f_2))] \\ \geq \int_{[a,b)} p d\mu_\Delta &[\chi(\mathbf{M}_\Delta(\chi, f_1 \cdot f_2, p)) \\ &\quad -\chi(\mathbf{M}_\Delta(\psi_1, f_1, p) \cdot \mathbf{M}_\Delta(\psi_2, f_2, p))] \\ \geq \left[\min_{t\in[a,b)} p(t)\right](b-a) &[\chi(\mathfrak{M}_\Delta(\chi, f_1 \cdot f_2)) \\ &\quad -\chi(\mathfrak{M}_\Delta(\psi_1, f_1) \cdot \mathfrak{M}_\Delta(\psi_2, f_2))].\end{aligned} \tag{68}$$

*If* $\psi_1', \psi_2', \chi', A, B,$ *and* $C$ *are all positive, then the inequalities in* (66), (67), *and* (68) *are reversed if and only if* $C(x \cdot y) \geq A(x) + B(y)$.

*Proof.* Let $n = 2$ in Theorem 16. By setting $\varphi(x, y) = x \cdot y$, we have

$$H(s_1, s_2) = \chi\left(\psi_1^{-1}(s_1) \cdot \psi_2^{-1}(s_2)\right). \tag{69}$$

If $\psi_1'$, $\psi_2'$, and $\chi'$ are positive and $A$, $B$, and $C$ are negative, then $H$ is convex if and only if $C(x \cdot y) \leq A(x) + B(y)$. If $\psi_1'$, $\psi_2'$, $\chi'$, $A$, $B$, and $C$ are all positive, then $H$ is concave if and only if $C(x \cdot y) \geq A(x) + B(y)$ (see [18]). Now, all claims follow immediately from Theorem 16. □

**Corollary 24.** *Let* $\lambda, \omega, \nu \in \mathbb{R}$ *be such that*

(a) $\lambda < 0 < \omega, \nu$, *or* $\omega, \nu < 0 < \lambda$;

(b) $\lambda < \omega, \nu < 0$, *or* $\nu < 0 < \omega < \lambda$, *or* $\omega < 0 < \nu < \lambda$, *for* $1/\lambda \leq 1/\omega + 1/\nu$;

(c) $\lambda < \omega < 0 < \nu$, *or* $\lambda < \nu < 0 < \omega$, *for* $1/\lambda \geq 1/\omega + 1/\nu$.

*Suppose* $f_1, f_2 : [a,b) \to \mathbb{R}$ *are* $\Delta$*-integrable and* $p, q : [a,b) \to \mathbb{R}$ *are nonnegative and* $\Delta$*-integrable such that* $p \cdot f_1^\lambda \cdot f_2^\lambda$, $q \cdot f_1^\lambda \cdot f_2^\lambda$, $pf_1^\omega$, $qf_1^\omega$, $pf_2^\nu$, *and* $qf_2^\nu$ *are* $\Delta$*-integrable and* $\int_{[a,b)} p d\mu_\Delta > 0$, $\int_{[a,b)} q d\mu_\Delta > 0$. *Then the functional*

$$\begin{aligned}&\int_{[a,b)} p \cdot f_1^\lambda \cdot f_2^\lambda d\mu_\Delta \\ &\quad - \int_{[a,b)} p d\mu_\Delta \left[\left(\frac{\int_{[a,b)} pf_1^\omega d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)^{1/\omega} \left(\frac{\int_{[a,b)} pf_2^\nu d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)^{1/\nu}\right]^\lambda\end{aligned} \tag{70}$$

*is superadditive, that is,*

$$\begin{aligned}&\int_{[a,b)} (p+q) \cdot f_1^\lambda \cdot f_2^\lambda d\mu_\Delta \\ &\quad - \int_{[a,b)} (p+q) d\mu_\Delta \left[\left(\frac{\int_{[a,b)} (p+q) f_1^\omega d\mu_\Delta}{\int_{[a,b)} (p+q) d\mu_\Delta}\right)^{1/\omega} \right. \\ &\qquad \left. \times \left(\frac{\int_{[a,b)} (p+q) f_2^\nu d\mu_\Delta}{\int_{[a,b)} (p+q) d\mu_\Delta}\right)^{1/\nu}\right]^\lambda \\ &\geq \int_{[a,b)} p \cdot f_1^\lambda \cdot f_2^\lambda d\mu_\Delta \\ &\quad - \int_{[a,b)} p d\mu_\Delta \left[\left(\frac{\int_{[a,b)} pf_1^\omega d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)^{1/\omega} \right. \\ &\qquad \left. \times \left(\frac{\int_{[a,b)} pf_2^\nu d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)^{1/\nu}\right]^\lambda \\ &\quad + \int_{[a,b)} q \cdot f_1^\lambda \cdot f_2^\lambda d\mu_\Delta \\ &\quad - \int_{[a,b)} q d\mu_\Delta \left[\left(\frac{\int_{[a,b)} qf_1^\omega d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta}\right)^{1/\omega} \right. \\ &\qquad \left. \times \left(\frac{\int_{[a,b)} qf_2^\nu d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta}\right)^{1/\nu}\right]^\lambda,\end{aligned} \tag{71}$$

*and increasing; that is, if* $p \geq q$ *such that* $\int_{[a,b)} p d\mu_\Delta > \int_{[a,b)} q d\mu_\Delta$, *then*

$$\begin{aligned}&\int_{[a,b)} p \cdot f_1^\lambda \cdot f_2^\lambda d\mu_\Delta \\ &\quad - \int_{[a,b)} p d\mu_\Delta \left[\left(\frac{\int_{[a,b)} pf_1^\omega d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)^{1/\omega} \left(\frac{\int_{[a,b)} pf_2^\nu d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)^{1/\nu}\right]^\lambda\end{aligned}$$

$$\geq \int_{[a,b)} q \cdot f_1^{\lambda} \cdot f_2^{\lambda} d\mu_{\Delta} - \int_{[a,b)} q d\mu_{\Delta} \left[ \left( \frac{\int_{[a,b)} q f_1^{\omega} d\mu_{\Delta}}{\int_{[a,b)} q d\mu_{\Delta}} \right)^{1/\omega} \left( \frac{\int_{[a,b)} q f_2^{\nu} d\mu_{\Delta}}{\int_{[a,b)} q d\mu_{\Delta}} \right)^{1/\nu} \right]^{\lambda}. \tag{72}$$

*If $p$ attains its minimum and maximum values on its domain, then*

$$\begin{aligned}
&\max_{t\in[a,b)} p(t) \left[ \int_{[a,b)} f_1^{\lambda} \cdot f_2^{\lambda} d\mu_{\Delta} - (b-a) \left[ \left( \frac{\int_{[a,b)} f_1^{\omega} d\mu_{\Delta}}{b-a} \right)^{1/\omega} \times \left( \frac{\int_{[a,b)} f_2^{\nu} d\mu_{\Delta}}{b-a} \right)^{1/\nu} \right]^{\lambda} \right] \\
&\geq \int_{[a,b)} p \cdot f_1^{\lambda} \cdot f_2^{\lambda} d\mu_{\Delta} - \int_{[a,b)} p d\mu_{\Delta} \left[ \left( \frac{\int_{[a,b)} p f_1^{\omega} d\mu_{\Delta}}{\int_{[a,b)} p d\mu_{\Delta}} \right)^{1/\omega} \times \left( \frac{\int_{[a,b)} p f_2^{\nu} d\mu_{\Delta}}{\int_{[a,b)} p d\mu_{\Delta}} \right)^{1/\nu} \right]^{\lambda} \\
&\geq \min_{t\in[a,b)} p(t) \left[ \int_{[a,b)} f_1^{\lambda} \cdot f_2^{\lambda} d\mu_{\Delta} - (b-a) \left[ \left( \frac{\int_{[a,b)} f_1^{\omega} d\mu_{\Delta}}{b-a} \right)^{1/\omega} \times \left( \frac{\int_{[a,b)} f_2^{\nu} d\mu_{\Delta}}{b-a} \right)^{1/\nu} \right]^{\lambda} \right].
\end{aligned} \tag{73}$$

*Moreover, the inequalities in* (71), (72), *and* (73) *are reversed provided that*

(a′) $\omega, \nu > \lambda > 0$, *for* $1/\lambda \geq 1/\omega + 1/\nu$;

(b′) $\omega, \nu < \lambda < 0$, *for* $1/\lambda \leq 1/\omega + 1/\nu$.

*Proof.* Let $n = 2$ in Theorem 16. By setting $\varphi(x, y) = x \cdot y$, $\chi(t) = t^{\lambda}$, $\psi_1(t) = t^{\omega}$, and $\psi_2(t) = t^{\nu}$, we have

$$H(s_1, s_2) = \chi\left(\psi_1^{-1}(s_1) \cdot \psi_2^{-1}(s_2)\right) = \left(s_1^{1/\omega} s_2^{1/\nu}\right)^{\lambda}. \tag{74}$$

Now, $H$ is convex if and only if $d^2 H \geq 0$, which implies

$$\frac{\lambda}{\omega}\left(\frac{\lambda}{\omega} - 1\right) \geq 0, \qquad \frac{\lambda}{\nu}\left(\frac{\lambda}{\nu} - 1\right) \geq 0, \qquad \frac{\lambda^3}{\omega\nu}\left(\frac{1}{\lambda} - \frac{1}{\omega} - \frac{1}{\nu}\right) \geq 0, \tag{75}$$

and these are satisfied if $\lambda$, $\omega$, and $\nu$ satisfy conditions (a), (b), and (c). $H$ is concave if and only if $d^2 H \leq 0$, and this implies

$$\frac{\lambda}{\omega}\left(\frac{\lambda}{\omega} - 1\right) \leq 0, \qquad \frac{\lambda}{\nu}\left(\frac{\lambda}{\nu} - 1\right) \leq 0, \qquad \frac{\lambda^3}{\omega\nu}\left(\frac{1}{\lambda} - \frac{1}{\omega} - \frac{1}{\nu}\right) \geq 0. \tag{76}$$

These are satisfied if $\lambda$, $\omega$, and $\nu$ satisfy conditions (a′) and (b′). Now, all claims follow immediately from Theorem 16. □

**Corollary 25.** *Let $\lambda, \omega, \nu \in \mathbb{R}$ be such that $\lambda, \omega, \nu > 0$, $\lambda, \omega, \nu \neq 1$ and*

(a) $\lambda < 1 < \omega, \nu$, *or* $\omega, \nu < 1 < \lambda$;

(b) $\lambda < \omega, \nu < 1$, *or* $\nu < 1 < \omega < \lambda$, *or* $\omega < 1 < \nu < \lambda$, *for* $1/\log\lambda \leq 1/\log\omega + 1/\log\nu$;

(c) $\lambda < \omega < 1 < \nu$, *or* $\lambda < \nu < 1 < \omega$, *for* $1/\log\lambda \geq 1/\log\omega + 1/\log\nu$.

*Suppose $f_1, f_2 : [a,b) \to \mathbb{R}$ are $\Delta$-integrable and $p, q : [a,b) \to \mathbb{R}$ are nonnegative and $\Delta$-integrable such that $p\lambda^{f_1+f_2}$, $q\lambda^{f_1+f_2}$, $p\omega^{f_1}$, $q\omega^{f_1}$, $p\nu^{f_2}$, and $q\nu^{f_2}$ are $\Delta$-integrable and $\int_{[a,b)} p d\mu_{\Delta} > 0$, $\int_{[a,b)} q d\mu_{\Delta} > 0$. Then the functional*

$$\int_{[a,b)} p\lambda^{f_1+f_2} d\mu_{\Delta} - \int_{[a,b)} p d\mu_{\Delta} \times \lambda^{\log_{\omega}\left(\int_{[a,b)} p\cdot\omega^{f_1} d\mu_{\Delta} / \int_{[a,b)} p d\mu_{\Delta}\right) + \log_{\nu}\left(\int_{[a,b)} p\cdot\nu^{f_2} d\mu_{\Delta} / \int_{[a,b)} p d\mu_{\Delta}\right)} \tag{77}$$

*is superadditive, that is,*

$$\begin{aligned}
&\int_{[a,b)} (p+q)\lambda^{f_1+f_2} d\mu_{\Delta} - \int_{[a,b)} (p+q) d\mu_{\Delta} \left(\lambda^{\log_{\omega}\left(\int_{[a,b)} (p+q)\cdot\omega^{f_1} d\mu_{\Delta} / \int_{[a,b)} (p+q) d\mu_{\Delta}\right)} \times \lambda^{\log_{\nu}\left(\int_{[a,b)} (p+q)\cdot\nu^{f_2} d\mu_{\Delta} / \int_{[a,b)} (p+q) d\mu_{\Delta}\right)}\right) \\
&\geq \int_{[a,b)} p\lambda^{f_1+f_2} d\mu_{\Delta} - \int_{[a,b)} p d\mu_{\Delta} \times \lambda^{\log_{\omega}\left(\int_{[a,b)} p\cdot\omega^{f_1} d\mu_{\Delta} / \int_{[a,b)} p d\mu_{\Delta}\right) + \log_{\nu}\left(\int_{[a,b)} p\cdot\nu^{f_2} d\mu_{\Delta} / \int_{[a,b)} p d\mu_{\Delta}\right)} \\
&\quad + \int_{[a,b)} q\lambda^{f_1+f_2} d\mu_{\Delta} - \int_{[a,b)} q d\mu_{\Delta} \times \lambda^{\log_{\omega}\left(\int_{[a,b)} q\cdot\omega^{f_1} d\mu_{\Delta} / \int_{[a,b)} q d\mu_{\Delta}\right) + \log_{\nu}\left(\int_{[a,b)} q\cdot\nu^{f_2} d\mu_{\Delta} / \int_{[a,b)} q d\mu_{\Delta}\right)},
\end{aligned} \tag{78}$$

*and increasing; that is, if* $p \geq q$ *such that* $\int_{[a,b)} p d\mu_\Delta > \int_{[a,b)} q d\mu_\Delta$, *then*

$$\begin{aligned}
&\int_{[a,b)} p\lambda^{f_1+f_2} d\mu_\Delta - \int_{[a,b)} p d\mu_\Delta \\
&\quad \times \lambda^{\log_\omega(\int_{[a,b)} p\cdot\omega^{f_1} d\mu_\Delta / \int_{[a,b)} p d\mu_\Delta) + \log_\nu(\int_{[a,b)} p\cdot\nu^{f_2} d\mu_\Delta / \int_{[a,b)} p d\mu_\Delta)} \\
&\geq \int_{[a,b)} q\lambda^{f_1+f_2} d\mu_\Delta - \int_{[a,b)} q d\mu_\Delta \\
&\quad \times \lambda^{\log_\omega(\int_{[a,b)} q\cdot\omega^{f_1} d\mu_\Delta / \int_{[a,b)} q d\mu_\Delta) + \log_\nu(\int_{[a,b)} q\cdot\nu^{f_2} d\mu_\Delta / \int_{[a,b)} q d\mu_\Delta)}.
\end{aligned} \tag{79}$$

*If* $p$ *attains its minimum and maximum values on its domain, then*

$$\begin{aligned}
&\max_{t\in[a,b)} p(t) \left[ \int_{[a,b)} \lambda^{f_1+f_2} d\mu_\Delta - (b-a) \right. \\
&\qquad \left. \times \lambda^{\log_\omega(\int_{[a,b)} \omega^{f_1} d\mu_\Delta/(b-a)) + \log_\nu(\int_{[a,b)} \nu^{f_2} d\mu_\Delta/(b-a))} \right] \\
&\geq \int_{[a,b)} p\lambda^{f_1+f_2} d\mu_\Delta - \int_{[a,b)} p d\mu_\Delta \\
&\quad \times \lambda^{\log_\omega(\int_{[a,b)} p\cdot\omega^{f_1} d\mu_\Delta / \int_{[a,b)} p d\mu_\Delta) + \log_\nu(\int_{[a,b)} p\cdot\nu^{f_2} d\mu_\Delta / \int_{[a,b)} p d\mu_\Delta)} \\
&\geq \min_{t\in[a,b)} p(t) \left[ \int_{[a,b)} \lambda^{f_1+f_2} d\mu_\Delta - (b-a) \right. \\
&\qquad \left. \times \lambda^{\log_\omega(\int_{[a,b)} \omega^{f_1} d\mu_\Delta/(b-a)) + \log_\nu(\int_{[a,b)} \nu^{f_2} d\mu_\Delta/(b-a))} \right].
\end{aligned} \tag{80}$$

*Moreover, the inequalities in* (78), (79), *and* (80) *are reversed provided that*

(a′) $\omega, \nu > \lambda > 1$, *for* $1/\log\lambda \geq 1/\log\omega + 1/\log\nu$;

(b′) $\omega, \nu < \lambda < 0$, *for* $1/\log\lambda \leq 1/\log\omega + 1/\log\nu$.

*Proof.* Let $n = 2$ in Theorem 16. By setting $\varphi(x, y) = x + y$, $\chi(t) = \lambda^t$, $\psi_1(t) = \omega^t$, and $\psi_2(t) = \nu^t$, we have

$$H(s_1, s_2) = \left(s_1^{1/\log\omega} \cdot s_2^{1/\log\nu}\right)^{\log\lambda}. \tag{81}$$

Now, the proof is similar to the proof of Corollary 24. □

**Corollary 26.** *Let* $\lambda, \omega, \nu \in \mathbb{R}$ *be such that*

(a) $0 < \omega, \nu \leq \lambda < 1$, *for all* $f_1, f_2 > 0$;

(b) $0 < \nu \leq \lambda \leq \omega < 1$, *for* $f_2 \geq (((\omega-\lambda)(1-\nu))/((\lambda-\nu)(1-\omega)))f_1 \geq 0$;

(c) $0 < \omega \leq \lambda \leq \nu < 1$, *for* $(((\lambda-\omega)(1-\nu))/((\nu-\lambda)(1-\omega)))f_1 \geq f_2 \geq 0$.

*Suppose* $f_1, f_2 : [a,b) \to \mathbb{R}$ *are* $\Delta$*-integrable and* $p, q : [a,b) \to \mathbb{R}$ *are nonnegative and* $\Delta$*-integrable such that* $p(f_1+f_2)^\lambda$, $q(f_1+f_2)^\lambda$, $pf_1^\omega$, $qf_1^\omega$, $pf_2^\nu$, *and* $qf_2^\nu$ *are* $\Delta$*-integrable and* $\int_{[a,b)} p d\mu_\Delta > 0, \int_{[a,b)} q d\mu_\Delta > 0$. *Then the functional*

$$\begin{aligned}
&\int_{[a,b)} p\cdot(f_1+f_2)^\lambda d\mu_\Delta - \int_{[a,b)} p d\mu_\Delta \\
&\quad \times \left[ \left( \frac{\int_{[a,b)} pf_1^\omega d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \right)^{1/\omega} + \left( \frac{\int_{[a,b)} pf_2^\nu d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \right)^{1/\nu} \right]^\lambda
\end{aligned} \tag{82}$$

*is superadditive, that is,*

$$\begin{aligned}
&\int_{[a,b)} (p+q)\cdot(f_1+f_2)^\lambda d\mu_\Delta \\
&\quad - \int_{[a,b)} (p+q) d\mu_\Delta \left[ \left( \frac{\int_{[a,b)} (p+q) f_1^\omega d\mu_\Delta}{\int_{[a,b)} (p+q) d\mu_\Delta} \right)^{1/\omega} \right. \\
&\qquad \left. + \left( \frac{\int_{[a,b)} (p+q) f_2^\nu d\mu_\Delta}{\int_{[a,b)} (p+q) d\mu_\Delta} \right)^{1/\nu} \right]^\lambda \\
&\geq \int_{[a,b)} p\cdot(f_1+f_2)^\lambda d\mu_\Delta - \int_{[a,b)} p d\mu_\Delta \\
&\quad \times \left[ \left( \frac{\int_{[a,b)} pf_1^\omega d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \right)^{1/\omega} + \left( \frac{\int_{[a,b)} pf_2^\nu d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \right)^{1/\nu} \right]^\lambda \\
&\quad + \int_{[a,b)} q\cdot(f_1+f_2)^\lambda d\mu_\Delta - \int_{[a,b)} q d\mu_\Delta \\
&\quad \times \left[ \left( \frac{\int_{[a,b)} qf_1^\omega d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta} \right)^{1/\omega} + \left( \frac{\int_{[a,b)} qf_2^\nu d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta} \right)^{1/\nu} \right]^\lambda,
\end{aligned} \tag{83}$$

*and increasing; that is, if* $p \geq q$ *such that* $\int_{[a,b)} p d\mu_\Delta > \int_{[a,b)} q d\mu_\Delta$, *then*

$$\begin{aligned}
&\int_{[a,b)} p\cdot(f_1+f_2)^\lambda d\mu_\Delta - \int_{[a,b)} p d\mu_\Delta \\
&\quad \times \left[ \left( \frac{\int_{[a,b)} pf_1^\omega d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \right)^{1/\omega} + \left( \frac{\int_{[a,b)} pf_2^\nu d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \right)^{1/\nu} \right]^\lambda \\
&\geq \int_{[a,b)} q\cdot(f_1+f_2)^\lambda d\mu_\Delta - \int_{[a,b)} q d\mu_\Delta \\
&\quad \times \left[ \left( \frac{\int_{[a,b)} qf_1^\omega d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta} \right)^{1/\omega} + \left( \frac{\int_{[a,b)} qf_2^\nu d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta} \right)^{1/\nu} \right]^\lambda.
\end{aligned} \tag{84}$$

*If $p$ attains its minimum and maximum values on its domain, then*

$$\begin{aligned}
&\max_{t\in[a,b)} p(t)\left[\int_{[a,b)}(f_1+f_2)^{\lambda}d\mu_\Delta\right.\\
&\qquad -(b-a)\left[\left(\frac{\int_{[a,b)} pf_1^{\omega}d\mu_\Delta}{b-a}\right)^{1/\omega}\right.\\
&\qquad\left.\left.+\left(\frac{\int_{[a,b)} pf_2^{\nu}d\mu_\Delta}{b-a}\right)^{1/\nu}\right]^{\lambda}\right]\\
&\geq \int_{[a,b)} p\cdot(f_1+f_2)^{\lambda}d\mu_\Delta-\int_{[a,b)} pd\mu_\Delta\\
&\quad\times\left[\left(\frac{\int_{[a,b)} pf_1^{\omega}d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}\right)^{1/\omega}+\left(\frac{\int_{[a,b)} pf_2^{\nu}d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}\right)^{1/\nu}\right]^{\lambda}\\
&\geq \min_{t\in[a,b)} p(t)\left[\int_{[a,b)}(f_1+f_2)^{\lambda}d\mu_\Delta\right.\\
&\qquad -(b-a)\left[\left(\frac{\int_{[a,b)} pf_1^{\omega}d\mu_\Delta}{b-a}\right)^{1/\omega}\right.\\
&\qquad\left.\left.+\left(\frac{\int_{[a,b)} pf_2^{\nu}d\mu_\Delta}{b-a}\right)^{1/\nu}\right]^{\lambda}\right].
\end{aligned}\tag{85}$$

*Moreover, the inequalities in* (83), (84), *and* (85) *are reversed provided that*

(a′) $1<\lambda\leq\omega,\nu$, *for all* $f_1,f_2>0$;

(b′) $1<\nu\leq\lambda\leq\omega$, *for* $0\leq f_2\leq(((\omega-\lambda)(\nu-1))/((\lambda-\nu)(\omega-1)))f_1$;

(c′) $1<\omega\leq\lambda\leq\nu$, *for* $f_2\geq(((\lambda-\omega)(\nu-1))/((\nu-\lambda)(\omega-1)))f_1\geq 0$.

*Proof.* Let $n=2$ in Theorem 16. By setting $\varphi(x,y)=x+y$, $\chi(t)=t^{\lambda}$, $\psi_1(t)=t^{\omega}$, and $\psi_2(t)=t^{\nu}$, we have

$$H(s_1,s_2)=\left(s_1^{1/\omega}+s_2^{1/\nu}\right)^{\lambda}.\tag{86}$$

Now, the proof is similar to the proof of Corollary 22, with some extra considerations of the definitions of $E$, $F$, and $G$. □

**Corollary 27.** *Suppose $f_1,f_2:[a,b)\to[0,\pi/4]$ are $\Delta$-integrable. Moreover, let $p,q:[a,b)\to\mathbb{R}$ be nonnegative and $\Delta$-integrable such that $p\cos(f_1+f_2)$, $q\cos(f_1+f_2)$, $p\cos(f_i)$, and $q\cos(f_i)$, $i=1,2$, are $\Delta$-integrable and $\int_{[a,b)} pd\mu_\Delta>0$, $\int_{[a,b)} qd\mu_\Delta>0$. Then the functional*

$$\begin{aligned}
&\int_{[a,b)} pd\mu_\Delta\cdot\cos\left[\arccos\left(\frac{\int_{[a,b)} p\cdot\cos(f_1)\,d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}\right)\right.\\
&\qquad\left.+\arccos\left(\frac{\int_{[a,b)} p\cdot\cos(f_2)\,d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}\right)\right]\\
&\quad-\int_{[a,b)} p\cos(f_1+f_2)\,d\mu_\Delta
\end{aligned}\tag{87}$$

*is subadditive, that is,*

$$\begin{aligned}
&\int_{[a,b)}(p+q)\,d\mu_\Delta\\
&\quad\cdot\cos\left[\arccos\left(\frac{\int_{[a,b)}(p+q)\cdot\cos(f_1)\,d\mu_\Delta}{\int_{[a,b)}(p+q)\,d\mu_\Delta}\right)\right.\\
&\qquad\left.+\arccos\left(\frac{\int_{[a,b)}(p+q)\cdot\cos(f_2)\,d\mu_\Delta}{\int_{[a,b)}(p+q)\,d\mu_\Delta}\right)\right]\\
&\quad-\int_{[a,b)}(p+q)\cdot\cos(f_1+f_2)\,d\mu_\Delta\\
&\leq\int_{[a,b)} pd\mu_\Delta\cdot\cos\left[\arccos\left(\frac{\int_{[a,b)} p\cdot\cos(f_1)\,d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}\right)\right.\\
&\qquad\left.+\arccos\left(\frac{\int_{[a,b)} p\cdot\cos(f_2)\,d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}\right)\right]\\
&\quad-\int_{[a,b)} p\cdot\cos(f_1+f_2)\,d\mu_\Delta\\
&\quad+\int_{[a,b)} qd\mu_\Delta\cdot\cos\left[\arccos\left(\frac{\int_{[a,b)} q\cdot\cos(f_1)\,d\mu_\Delta}{\int_{[a,b)} qd\mu_\Delta}\right)\right.\\
&\qquad\left.+\arccos\left(\frac{\int_{[a,b)} q\cdot\cos(f_2)\,d\mu_\Delta}{\int_{[a,b)} qd\mu_\Delta}\right)\right]\\
&\quad-\int_{[a,b)} q\cdot\cos(f_1+f_2)\,d\mu_\Delta,
\end{aligned}\tag{88}$$

*and decreasing; that is, if $p\geq q$ such that $\int_{[a,b)} pd\mu_\Delta>\int_{[a,b)} qd\mu_\Delta$, then*

$$\int_{[a,b)} pd\mu_\Delta\cdot\cos\left[\arccos\left(\frac{\int_{[a,b)} p\cdot\cos(f_1)\,d\mu_\Delta}{\int_{[a,b)} pd\mu_\Delta}\right)\right.$$

$$+ \arccos\left(\frac{\int_{[a,b)} p\cdot\cos(f_2)\,d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)\Bigg]$$
$$-\int_{[a,b)} p\cdot\cos(f_1+f_2)\,d\mu_\Delta$$
$$\le \int_{[a,b)} q d\mu_\Delta\cdot\cos\Bigg[\arccos\left(\frac{\int_{[a,b)} q\cdot\cos(f_1)\,d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta}\right)$$
$$+ \arccos\left(\frac{\int_{[a,b)} q\cdot\cos(f_2)\,d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta}\right)\Bigg]$$
$$-\int_{[a,b)} q\cdot\cos(f_1+f_2)\,d\mu_\Delta. \tag{89}$$

*If $p$ attains its minimum and maximum values on its domain, then*

$$\left[\max_{t\in[a,b)} p(t)\right](b-a)\cos\Bigg[\arccos\left(\frac{\int_{[a,b)}\cos(f_1)\,d\mu_\Delta}{b-a}\right)$$
$$+ \arccos\left(\frac{\int_{[a,b)}\cos(f_2)\,d\mu_\Delta}{b-a}\right)\Bigg]$$
$$-\int_{[a,b)}\cos(f_1+f_2)\,d\mu_\Delta$$
$$\le \int_{[a,b)} p d\mu_\Delta\cdot\cos\Bigg[\arccos\left(\frac{\int_{[a,b)} p\cdot\cos(f_1)\,d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)$$
$$+ \arccos\left(\frac{\int_{[a,b)} p\cdot\cos(f_2)\,d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta}\right)\Bigg]$$
$$-\int_{[a,b)} p\cdot\cos(f_1+f_2)\,d\mu_\Delta$$
$$\le \left[\min_{t\in[a,b)} p(t)\right](b-a)\cos\Bigg[\arccos\left(\frac{\int_{[a,b)}\cos(f_1)\,d\mu_\Delta}{b-a}\right)$$
$$+ \arccos\left(\frac{\int_{[a,b)}\cos(f_2)\,d\mu_\Delta}{b-a}\right)\Bigg]$$
$$-\int_{[a,b)}\cos(f_1+f_2)\,d\mu_\Delta. \tag{90}$$

*Proof.* Let $n = 2$ in Theorem 16. By setting $\varphi(x,y) = x+y$ and $\chi(t) = \psi_1(t) = \psi_2(t) = -\cos(t)$, we have

$$H(s_1,s_2) = -\cos(\arccos(-s_1)+\arccos(-s_2)). \tag{91}$$

Now, the proof is similar to the proof of Corollary 22. □

## 5. Applications to Hölder's Inequality

Suppose $f_i$, $i = 1,2,\ldots,n$, are nonnegative $\Delta$-integrable functions on $[a,b)$ such that $\prod_{i=1}^n f_i^{\alpha_i}$ is $\Delta$-integrable, where $\alpha_i \ge 0$, $i = 1,\ldots,n$, are such that $\sum_{i=1}^n \alpha_i = 1$. Then, by using Theorem 3 (Hölder's inequality on time scales), we have

$$\int_{[a,b)}\prod_{i=1}^n f_i^{\alpha_i} d\mu_\Delta \le \prod_{i=1}^n\left(\int_{[a,b)} f_i d\mu_\Delta\right)^{\alpha_i}. \tag{92}$$

If $\sum_{i=1}^n \alpha_i = A_n > 0$, then (92) implies

$$\int_{[a,b)}\prod_{i=1}^n f_i^{\alpha_i/A_n} d\mu_\Delta \le \prod_{i=1}^n\left(\int_{[a,b)} f_i d\mu_\Delta\right)^{\alpha_i/A_n} \tag{93}$$

or

$$\left(\int_{[a,b)}\prod_{i=1}^n f_i^{\alpha_i/A_n} d\mu_\Delta\right)^{A_n} \le \prod_{i=1}^n\left(\int_{[a,b)} f_i d\mu_\Delta\right)^{\alpha_i}. \tag{94}$$

In this section, we discuss properties of the functional, deduced from the Hölder inequality (93), defined in the following way.

*Definition 28.* Suppose $\mathbf{f} = (f_1,\ldots,f_n)$ is such that $f_i$, $i = 1,\ldots,n$, are nonnegative $\Delta$-integrable functions on $[a,b)$. Let $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$ be such that $\alpha_i \ge 0$ and $\sum_{i=1}^n \alpha_i = A_n > 0$. Then one defines the functional $\mathbf{H}_\Delta$ by

$$\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\alpha}) = \frac{\prod_{i=1}^n\left(\int_{[a,b)} f_i d\mu_\Delta\right)^{\alpha_i}}{\left(\int_{[a,b)}\prod_{i=1}^n f_i^{\alpha_i/A_n} d\mu_\Delta\right)^{A_n}}. \tag{95}$$

**Theorem 29.** *Let $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_n)$ and $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_n)$ be real $n$-tuples with $\alpha_i \ge 0$, $\beta_i \ge 0$ and $\sum_{i=1}^n \alpha_i = A_n > 0$, $\sum_{i=1}^n \beta_i = B_n > 0$. Suppose $f_i$, $i = 1,\ldots,n$, are nonnegative $\Delta$-integrable on $[a,b)$ such that $\prod_{i=1}^n f_i^{\alpha_i/A_n}$ and $\prod_{i=1}^n f_i^{\beta_i/B_n}$ are $\Delta$-integrable. Then*

$$\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\alpha}+\boldsymbol{\beta}) \ge \mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\alpha})\cdot\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\beta}), \tag{96}$$

*and $\mathbf{H}_\Delta(\mathbf{f},\cdot,\mu_\Delta)$ is increasing; that is, if $\boldsymbol{\alpha} \ge \boldsymbol{\beta}$ such that $A_n > B_n$, then*

$$\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\alpha}) \ge \mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\beta}). \tag{97}$$

*Proof.* By Definition 28, we have

$$\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\alpha}+\boldsymbol{\beta}) = \frac{\prod_{i=1}^n\left(\int_{[a,b)} f_i d\mu_\Delta\right)^{\alpha_i+\beta_i}}{\left(\int_{[a,b)}\prod_{i=1}^n f_i^{(\alpha_i+\beta_i)/(A_n+B_n)} d\mu_\Delta\right)^{A_n+B_n}}, \tag{98}$$

where

$$\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{(\alpha_i+\beta_i)/(A_n+B_n)}d\mu_\Delta\right)^{A_n+B_n}$$
$$=\left[\int_{[a,b)}\left(\prod_{i=1}^{n} f_i^{\alpha_i/A_n}\right)^{A_n/(A_n+B_n)}\times\left(\prod_{i=1}^{n} f_i^{\beta_i/B_n}\right)^{B_n/(A_n+B_n)}d\mu_\Delta\right]^{A_n+B_n}$$
$$\leq\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\alpha_i/A_n}d\mu_\Delta\right)^{A_n}\left(\int_{[a,b)}\prod_{i=1}^{n} f_i^{\beta_i/B_n}d\mu_\Delta\right)^{B_n}. \tag{99}$$

Now, by combining (98) and (99), we have

$$\begin{aligned}\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\alpha}+\boldsymbol{\beta})&\geq\frac{\prod_{i=1}^{n}\left(\int_{[a,b)}f_i d\mu_\Delta\right)^{\alpha_i}\prod_{i=1}^{n}\left(\int_{[a,b)}f_i d\mu_\Delta\right)^{\beta_i}}{\left(\int_{[a,b)}\prod_{i=1}^{n}f_i^{\alpha_i/A_n}d\mu_\Delta\right)^{A_n}\left(\int_{[a,b)}\prod_{i=1}^{n}f_i^{\beta_i/B_n}d\mu_\Delta\right)^{B_n}}\\&=\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\alpha})\cdot\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\beta}).\end{aligned} \tag{100}$$

If $\boldsymbol{\alpha}\geq\boldsymbol{\beta}$, then $\boldsymbol{\alpha}-\boldsymbol{\beta}\geq 0$, and therefore

$$\begin{aligned}\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\alpha})&=\mathbf{H}_\Delta(\mathbf{f},(\boldsymbol{\alpha}-\boldsymbol{\beta})+\boldsymbol{\beta})\\&\geq\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\alpha}-\boldsymbol{\beta})\cdot\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\beta})\\&\geq\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\beta}).\end{aligned} \tag{101}$$

This completes the proof. □

**Corollary 30.** *Let* $\mathbf{f}$ *and* $\boldsymbol{\alpha}$ *satisfy the hypothesis of Theorem 29. Then*

$$\left[\frac{\prod_{i=1}^{n}\int_{[a,b)}f_i d\mu_\Delta}{\left(\int_{[a,b)}\prod_{i=1}^{n}f_i^{1/n}d\mu_\Delta\right)^n}\right]^{\max_{1\leq i\leq n}\{\alpha_i\}}\geq\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\alpha})\geq\left[\frac{\prod_{i=1}^{n}\int_{[a,b)}f_i d\mu_\Delta}{\left(\int_{[a,b)}\prod_{i=1}^{n}f_i^{1/n}d\mu_\Delta\right)^n}\right]^{\min_{1\leq i\leq n}\{\alpha_i\}}. \tag{102}$$

*Proof.* Let

$$\boldsymbol{\alpha}_{\max}=\left(\max_{1\leq i\leq n}\{\alpha_i\},\ldots,\max_{1\leq i\leq n}\{\alpha_i\}\right),\qquad\boldsymbol{\alpha}_{\min}=\left(\min_{1\leq i\leq n}\{\alpha_i\},\ldots,\min_{1\leq i\leq n}\{\alpha_i\}\right). \tag{103}$$

By Definition 28, we have

$$\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\alpha}_{\max})=\left[\frac{\prod_{i=1}^{n}\int_{[a,b)}f_i d\mu_\Delta}{\left(\int_{[a,b)}\prod_{i=1}^{n}f_i^{1/n}d\mu_\Delta\right)^n}\right]^{\max_{1\leq i\leq n}\{\alpha_i\}},\qquad\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\alpha}_{\min})=\left[\frac{\prod_{i=1}^{n}\int_{[a,b)}f_i d\mu_\Delta}{\left(\int_{[a,b)}\prod_{i=1}^{n}f_i^{1/n}d\mu_\Delta\right)^n}\right]^{\min_{1\leq i\leq n}\{\alpha_i\}}. \tag{104}$$

Since $\boldsymbol{\alpha}_{\max}\geq\boldsymbol{\alpha}\geq\boldsymbol{\alpha}_{\min}$, the result follows from the second property of Theorem 29. □

**Corollary 31.** *Let* $\mathbf{f}$, $\boldsymbol{\alpha}$, *and* $\boldsymbol{\beta}$ *satisfy the hypothesis of Theorem 29 with* $A_n=B_n=1$. *If there exist constants* $M>1>m$ *such that* $M\boldsymbol{\beta}\geq\boldsymbol{\alpha}\geq m\boldsymbol{\beta}$, *then*

$$\mathbf{H}_\Delta(\mathbf{f},M\boldsymbol{\beta})\geq\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\alpha})\geq\mathbf{H}_\Delta(\mathbf{f},m\boldsymbol{\beta}). \tag{105}$$

*Proof.* By Definition 28, we have

$$\mathbf{H}_\Delta(\mathbf{f},M\boldsymbol{\beta})=M\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\beta}),\qquad\mathbf{H}_\Delta(\mathbf{f},m\boldsymbol{\beta})=m\mathbf{H}_\Delta(\mathbf{f},\boldsymbol{\beta}). \tag{106}$$

Now the result follows from the second property of Theorem 29. □

*Remark 32.* Some results for isotonic linear functionals related to the results given in this paper can be found in [16].

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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