Jensen Functionals on Time Scales for Several Variables

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1. Introduction

Jensen’s inequality is well known in analysis and many other areas of mathematics. Most of the classical inequalities can be obtained by using the Jensen inequality. For time scale theory, Jensen’s inequality for one variable is obtained by Agarwal et al. [1], and now there are various extensions and generalizations of it given by many researchers (see [2–8]). In [3], it is shown that the Jensen inequality for one variable holds for time scale integrals including the Cauchy delta, Cauchy nabla, diamond-\(\alpha\), Riemann, Lebesgue, multiple Riemann, and multiple Lebesgue integrals. Further, in [4], we give properties and applications of Jensen functionals on time scales for one variable.

In this paper, we obtain the Jensen inequality for several variables and deduce Jensen functionals. We discuss several properties and applications of Jensen functionals. In the sequel, we give all the results for Lebesgue delta integrals. For other time scale integrals, as mentioned above, all those results can be obtained in a similar way. These results generalize the results given in [4] for one variable. Now, we give a brief introduction of time scale integrals; for a detailed introduction we refer to [1, 9–12]. A time scale \(\mathbb{T}\) is an arbitrary closed subset of \(\mathbb{R}\), and time scale calculus provides unification and extension of classical results. For example, when \(\mathbb{T} = \mathbb{R}\), the time scale integral is an ordinary integral, and when \(\mathbb{T} = \mathbb{Z}\), the time scale integral becomes a sum. In [10, Chapter 5], the Lebesgue integral is introduced: let \([a,b) \subseteq \mathbb{T}\) be a time scale interval defined by

\[ [a,b) = \{ t \in \mathbb{T} : a \leq t < b \}, \]

where \(a, b \in \mathbb{T}\) with \(a \leq b\). Let \(\mu_\Delta\) be the Lebesgue \(\Delta\)-measure on \([a,b)\). Suppose \(f : [a,b) \to \mathbb{R}\) is a \(\mu_\Delta\)-measurable function. Then the Lebesgue \(\Delta\)-integral of \(f\) on \([a,b)\) is denoted by

\[
\int_{[a,b)} f d\mu_\Delta, \quad \int_{[a,b)} f(t) d\mu_\Delta(t), \quad \text{or} \quad \int_{(a,b)} f(t)\,\Delta t.
\]

All theorems of the general Lebesgue integration theory, including the Lebesgue dominated convergence theorem, hold also for Lebesgue \(\Delta\)-integrals on \(\mathbb{T}\). Now, we give some properties of Lebesgue \(\Delta\)-integrals and state Jensen’s inequality and Hölder’s inequality for Lebesgue \(\Delta\)-integrals. Throughout this paper, \([a,b)\) denotes a time scale interval otherwise is specified.
Theorem 1 (see [3, Theorem 3.2]). If \( f \) and \( g \) are \( \Delta \)-integrable functions on \([a, b)\), then
\[
\int_{(a,b)} (af + bg) \, d\mu_\Delta = \alpha \int_{(a,b)} f \, d\mu_\Delta + \beta \int_{(a,b)} g \, d\mu_\Delta
\]
\[\forall \alpha, \beta \in \mathbb{R},\]
\[
f(t) \geq 0 \ \forall t \in [a, b) \implies \int_{(a,b)} f \, d\mu_\Delta \geq 0.
\]
(3)

Theorem 2 (see [3, Theorem 4.2]). Assume \( \Phi \in \mathcal{C}(I, \mathbb{R}) \) is convex, where \( I \subseteq \mathbb{R} \) is an interval. Suppose \( f : [a, b) \to \Delta \)-integrable. Moreover, let \( p : [a, b) \to \mathbb{R} \) be nonnegative and \( \Delta \)-integrable such that \( \int_{[a,b]} p \, d\mu_\Delta > 0 \). Then
\[
\Phi \left( \frac{\int_{[a,b]} p f \, d\mu_\Delta}{\int_{[a,b]} p \, d\mu_\Delta} \right) \leq \frac{\int_{[a,b]} p (\Phi \circ f) \, d\mu_\Delta}{\int_{[a,b]} p \, d\mu_\Delta}.
\]
(4)

Theorem 3 (see [3, Theorem 6.2]). For \( p \neq 1 \), define \( q = p/(p-1) \). Let \( f, g \) be nonnegative functions such that \( wfg \) and \( wfg^q \) are \( \Delta \)-integrable on \([a, b)\). If \( p > 1 \), then
\[
\int_{[a,b]} wfg \, d\mu_\Delta \leq \left( \int_{[a,b]} wfg^p \, d\mu_\Delta \right)^{1/p} \left( \int_{[a,b]} w^q \, d\mu_\Delta \right)^{1/q}.
\]
(5)

If \( 0 < p < 1 \) and \( \int_{[a,b]} wfg \, d\mu_\Delta > 0 \), or if \( p < 0 \) and \( \int_{[a,b]} wfg \, d\mu_\Delta > 0 \), then (5) is reversed.

Remark 4. Theorem 1 recalls that the Lebesgue \( \Delta \)-integral is an isotonic linear functional (see [13]). So we can also use the approach of isotonic linear functionals whenever results are known for isotonic linear functionals.

In the next section, we give Jensen inequality on time scales for several variables and define Jensen functionals. In Section 3, we investigate properties of Jensen functionals and some of its consequences regarding superadditivity and monotonicity. In Section 4, we apply these results to weighted general means, defined on time scales, and give many applications. Finally in Section 5, we give applications to Hölder’s inequality on time scales.

2. Jensen Inequality and Jensen Functionals

Let \( f(t) = (f_1(t), \ldots, f_n(t)) \) be an \( n \)-tuple of functions such that \( f_1, \ldots, f_n \) are \( \Delta \)-integrable on \([a, b)\). Then \( \int_{[a,b]} f \, d\mu_\Delta \) denotes the \( n \)-tuple:
\[
\left( \int_{[a,b]} f_1 \, d\mu_\Delta, \ldots, \int_{[a,b]} f_n \, d\mu_\Delta \right).
\]
(6)

That is, \( \Delta \)-integral acts on each component of \( f \).

Theorem 5 (Jensen inequality). Assume \( \Phi \in \mathcal{C}(K, \mathbb{R}) \) is convex, where \( K \subseteq \mathbb{R}^n \) is closed and convex. Suppose \( f_i, i = 1, 2, \ldots, n, \) are \( \Delta \)-integrable on \([a, b)\) such that \( f(t) = (f_1(t), \ldots, f_n(t)) \in K \) for all \( t \in [a, b) \). Moreover, let \( p : [a, b) \to \mathbb{R} \) be nonnegative and \( \Delta \)-integrable such that \( \int_{[a,b]} p \, d\mu_\Delta > 0 \). Then
\[
\Phi \left( \frac{\int_{[a,b]} p f_i \, d\mu_\Delta}{\int_{[a,b]} p \, d\mu_\Delta} \right) \leq \frac{\int_{[a,b]} p \Phi(f) \, d\mu_\Delta}{\int_{[a,b]} p \, d\mu_\Delta}.
\]
(7)

Proof. Suppose \( \Phi \) is convex on \( K \subseteq \mathbb{R}^n \). Therefore, for every point \( x \in K \), there exists a point \( \lambda \in \mathbb{R}^n \) (see [13, Theorem 1.31]) such that
\[
\Phi(x) - \Phi(x_0) \geq \lambda(x - x_0).
\]
(8)

Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \). By (8), we get
\[
\frac{\int_{[a,b]} p \Phi(f) \, d\mu_\Delta}{\int_{[a,b]} p \, d\mu_\Delta} - \Phi \left( \frac{\int_{[a,b]} p f_i \, d\mu_\Delta}{\int_{[a,b]} p \, d\mu_\Delta} \right)
\]
\[
= \int_{[a,b]} p \left( \Phi(f) - \Phi \left( \frac{\int_{[a,b]} p f_i \, d\mu_\Delta}{\int_{[a,b]} p \, d\mu_\Delta} \right) \right) \, d\mu_\Delta
\]
\[
\geq \int_{[a,b]} p \left( \lambda_i f_i - \left( \int_{[a,b]} p f_i \, d\mu_\Delta / \int_{[a,b]} p \, d\mu_\Delta \right) \right) \, d\mu_\Delta
\]
\[
= \int_{[a,b]} p \sum_{i=1}^n \lambda_i \left( f_i - \left( \int_{[a,b]} p f_i \, d\mu_\Delta / \int_{[a,b]} p \, d\mu_\Delta \right) \right) \, d\mu_\Delta
\]
\[
= 0,
\]
(9)
and hence the proof is completed.

Remark 6. By using the fact that the time scale integral is an isotonic linear functional, Theorem 5 can also be obtained by using Theorem I and [13, Theorem 2.6].

Definition 7. Assume \( \Phi \in \mathcal{C}(K, \mathbb{R}) \), where \( K \subseteq \mathbb{R}^n \) is closed and convex. Suppose \( f_i, i = 1, 2, \ldots, n, \) are \( \Delta \)-integrable on \([a, b)\) such that \( f(t) = (f_1(t), \ldots, f_n(t)) \in K \) for all \( t \in [a, b) \). Moreover, let \( p : [a, b) \to \mathbb{R} \) be nonnegative and \( \Delta \)-integrable such that \( \int_{[a,b]} p \, d\mu_\Delta > 0 \). Then one defines the Jensen functional on time scales for several variables by
\[
J_\Delta(\Phi, f, p) = \int_{[a,b]} p \Phi(f) \, d\mu_\Delta
\]
(10)

Remark 8. By Theorem 5, the following statements are obvious. If \( \Phi \) is convex, then
\[
J_\Delta(\Phi, f, p) \geq 0
\]
(11)
while if \( \Phi \) is concave, then
\[
J_\Delta(\Phi, f, p) \leq 0.
\]
(12)
Example 9. Let \([a, b) = \{1, 2, \ldots, n\}\), \(f_1(i) = x_1i, \ldots, f_n(i) = x_ni\), and \(p(i) = p_i, i = 1, 2, \ldots, n\) in (10). Then the Jensen functional (10) becomes

\[
J_n(\Phi, X, p) = \sum_{i=1}^{n} p_i \Phi(x_i) - P_n \Phi\left(\frac{\sum_{i=1}^{n} p_i x_i}{P_n}\right),
\]

where \(X = (x_1, x_2, \ldots, x_n)\) with \(x_i = (x_1i, x_12, \ldots, x_1n)\), \(p = (p_1, p_2, \ldots, p_n)\), and \(P_n = \sum_{i=1}^{n} p_i > 0\). Some properties of the Jensen functional \(J_n\) are investigated in [14, 15].

Moreover, let \(J_n\) be convex. Because the time scales integral is linear (see Theorem 1), it follows from Definition 7 that

\[
J_n(\Phi, f, p + q) \geq J_n(\Phi, f, p) + J_n(\Phi, f, q).
\]

If \(p \geq q\), we have \(p - q \geq 0\). Now, because Jensen's functional is superadditive and nonnegative, we have

\[
J_n(\Phi, f, p) \geq J_n(\Phi, f, p - q + q) \geq J_n(\Phi, f, p - q) + J_n(\Phi, f, q)
\]

(18)

On the other hand, if \(\Phi\) is concave, then the reversed inequalities of (15) and (16) can be obtained in a similar way. \(\square\)

Corollary 12. Let \(\Phi, f, p, q\) satisfy the hypotheses of Theorem 11. Further, suppose there exist nonnegative constants \(m\) and \(M\) such that

\[
Mq(t) \geq p(t) \geq mq(t) \quad \forall t \in [a, b],
\]

(19)

If \(\Phi\) is convex, then

\[
MJ_n(\Phi, f, q) \geq J_n(\Phi, f, p) \geq mJ_n(\Phi, f, q), \quad (20)
\]

while if \(\Phi\) is concave, then the inequalities in (20) hold in reverse order.
Proof. By using (10), we have
\[ J_\Delta (\Phi, f, m q) = m J_\Delta (\Phi, f, q), \]
\[ J_\Delta (\Phi, f, M q) = M J_\Delta (\Phi, f, q). \] 
(21)

Now the result follows from the second property of Theorem II. \( \square \)

**Corollary 13.** Let \( \Phi, f, p \) satisfy the hypotheses of Theorem II. Further, assume that \( p \) attains its minimum value and its maximum value on its domain. If \( \Phi \) is convex, then
\[ J_\Delta (\Phi, f, p) \geq J_\Delta (\Phi, f, p), \] 
(22)

where
\[ J_\Delta (\Phi, f) = \int _{[a,b]} \Phi (f) d \mu_\Delta - (b - a) \Phi \left( \frac{\int _{[a,b]} f d \mu_\Delta}{b - a} \right). \] 
(23)

Moreover, if \( \Phi \) is concave, then the inequalities in (22) hold in reverse order.

**Proof.** Let \( p \) attain its minimum and maximum values on its domain \([a, b]\). Then
\[ \max_{t \in [a,b]} p(t) \geq p(t) \geq \min_{t \in [a,b]} p(t). \] 
(24)

Let
\[ \overline{p}(t) = \max_{t \in [a,b]} p(t), \quad \underline{p}(t) = \min_{t \in [a,b]} p(t). \] 
(25)

By using (10), we have
\[ J_\Delta (\Phi, f, \overline{p}) = \left[ \max_{t \in [a,b]} p(t) \right] J_\Delta (\Phi, f, p), \]
\[ J_\Delta (\Phi, f, \underline{p}) = \left[ \min_{t \in [a,b]} p(t) \right] J_\Delta (\Phi, f, p). \] 
(26)

Now the result follows from the second property of Theorem II. \( \square \)

**Example 9.** Let the functional \( J_n(\Phi, X, p) \) be defined as in Example 9. Let \( q = (q_1, \ldots, q_n) \) with \( q_i \geq 0 \) and \( \sum_{i=1}^n q_i = Q_n > 0 \). If \( \Phi \) is convex, then Theorem II implies \( J_n(\Phi, X, \cdot) \) is superadditive; that is,
\[ J_n(\Phi, X, p + q) \geq J_n(\Phi, X, p) + J_n(\Phi, X, q), \] 
(27)

and \( J_n(\Phi, X, \cdot) \) is increasing; that is, if \( p \geq q \) such that \( P_n > Q_n \), then
\[ J_n(\Phi, X, p) \geq J_n(\Phi, X, q). \] 
(28)

Moreover, if \( \Phi \) is concave, then the inequalities in (27) and (28) hold in reverse order. If \( p \) attains its minimum and maximum values on its domain, then Corollary 13 yields
\[ \max_{1 \leq i \leq n} \mathfrak{T}_n (\Phi, X) \geq J_n (\Phi, X, p) \geq \min_{1 \leq i \leq n} \mathfrak{T}_n (\Phi, X), \] 
(29)

where
\[ \mathfrak{T}_n (\Phi, X) = \sum_{i=1}^n \Phi (x_i) - n \Phi \left( \frac{\sum_{i=1}^n x_i}{n} \right). \] 
(30)

if \( \Phi \) is convex. Further, the inequalities in (29) hold in reverse order if \( \Phi \) is concave.

**4. Applications to Weighted Generalized Means**

In the sequel, \( I \subset \mathbb{R} \) is an interval and \( K \subset \mathbb{R}^n \) is closed and convex.

**Definition 15.** Assume \( \chi \in C(I, \mathbb{R}) \) is strictly monotone and \( \varphi : K \rightarrow I \) is a function of \( n \) variables. Suppose \( f_i, i = 1, 2, \ldots, n \), are \( \Delta \)-integrable on \([a,b]\) such that \( f(t) = (f_1(t), \ldots, f_n(t)) \in K \) for all \( t \in [a,b] \). Moreover, let \( p : [a,b] \rightarrow \mathbb{R} \) be nonnegative and \( \Delta \)-integrable such that \( p \chi (\varphi (f)) \) is \( \Delta \)-integrable and \( \int _{[a,b]} p d \mu_\Delta > 0 \). Then one defines the weighted generalized mean on time scales by
\[ M_\Delta (\chi, \varphi (f), p) = \chi^{-1} \left( \int _{[a,b]} p \chi (\varphi (f)) d \mu_\Delta \right). \] 
(31)

**Theorem 16.** Assume \( \chi, \psi_i \in C(I, \mathbb{R}), i = 1, 2, \ldots, n \), are strictly monotone and \( \varphi : K \rightarrow I \subset \mathbb{R} \) is a function of \( n \) variables. Suppose \( f_i : [a,b] \rightarrow [1,2, \ldots, n] \), are \( \Delta \)-integrable such that \( f(t) = (f_1(t), \ldots, f_n(t)) \in K \) for all \( t \in [a,b] \). Moreover, let \( p, q : [a,b] \rightarrow \mathbb{R} \) be nonnegative and \( \Delta \)-integrable such that \( p \chi (\varphi (f)), q \chi (\varphi (f)), \psi_i (f_i), i = 1, 2, \ldots, n \), are \( \Delta \)-integrable and \( \int _{[a,b]} p d \mu_\Delta > 0, \int _{[a,b]} q d \mu_\Delta > 0 \). If \( H \) defined by
\[ H(s_1, \ldots, s_n) = \chi \circ \varphi \left( \psi_1^{-1} (s_1), \ldots, \psi_n^{-1} (s_n) \right) \] 
(32)

is convex, then the functional
\[ \int _{[a,b]} p d \mu_\Delta \left[ \chi (M_\Delta (\chi, \varphi (f), p)) - \chi \circ \varphi (M_\Delta (\psi_1, f_1, p), \ldots, \right. \]
\[ \left. M_\Delta (\psi_n, f_n, p) \right] \] 
(33)
is superadditive, that is,
\[
\int_{(a,b)} (p + q) \, d\mu_{\Delta} \left[ \chi \left( M_{\Delta} (\chi, \varphi (f), p + q) \right) \right. \\
\left. - \chi \circ \varphi \left( M_{\Delta} (\psi_1, f_1, p + q), \ldots, M_{\Delta} (\psi_n, f_n, p + q) \right) \right] \\
\geq \int_{(a,b)} p \, d\mu_{\Delta} \left[ \chi \left( M_{\Delta} (\chi, \varphi (f), p) \right) \right. \\
\left. - \chi \circ \varphi \left( M_{\Delta} (\psi_1, f_1, p), \ldots, M_{\Delta} (\psi_n, f_n, p) \right) \right] \\
+ \int_{(a,b)} q \, d\mu_{\Delta} \left[ \chi \left( M_{\Delta} (\chi, \varphi (f), q) \right) \right. \\
\left. - \chi \circ \varphi \left( M_{\Delta} (\psi_1, f_1, q), \ldots, M_{\Delta} (\psi_n, f_n, q) \right) \right],
\]
and increasing; that is, \( p \geq q \) with \( \int_{(a,b)} p \, d\mu_{\Delta} > \int_{(a,b)} q \, d\mu_{\Delta} \) implies
\[
\int_{(a,b)} p \, d\mu_{\Delta} \left[ \chi \left( M_{\Delta} (\chi, \varphi (f), p) \right) \right. \\
\left. - \chi \circ \varphi \left( M_{\Delta} (\psi_1, f_1, p), \ldots, M_{\Delta} (\psi_n, f_n, p) \right) \right] \\
\geq \int_{(a,b)} q \, d\mu_{\Delta} \left[ \chi \left( M_{\Delta} (\chi, \varphi (f), q) \right) \right. \\
\left. - \chi \circ \varphi \left( M_{\Delta} (\psi_1, f_1, q), \ldots, M_{\Delta} (\psi_n, f_n, q) \right) \right].
\]
Moreover, if \( H \) is continuous and concave, then (33) is subadditive and decreasing; that is, (34) and (35) hold in reverse order.

Proof. The functional defined in (33) is obtained by replacing \( \Phi \) with \( H \) and \( f_i \) with \( \psi_i (f_i), i = 1, 2, \ldots, n \), in the Jensen functional (10) and letting \( \Psi (f) = (\psi_1 (f_1), \ldots, \psi_n (f_n)) \); that is,
\[
\begin{align*}
\mathcal{J}_{\Delta} (H, \Psi (f), p) & = \int_{(a,b)} p \chi \circ \varphi \left( f_1, \ldots, f_n \right) \, d\mu_{\Delta} - \int_{(a,b)} p \, d\mu_{\Delta} \\
& \quad \times H \left( \frac{\int_{(a,b)} p \psi_1 (f_1) \, d\mu_{\Delta}}{\int_{(a,b)} p \, d\mu_{\Delta}}, \ldots, \frac{\int_{(a,b)} p \psi_n (f_n) \, d\mu_{\Delta}}{\int_{(a,b)} p \, d\mu_{\Delta}} \right) \\
& = \int_{(a,b)} p \, d\mu_{\Delta} \left[ \chi (M_{\Delta} (\chi, \varphi (f), p)) \\ - \chi \circ \varphi (M_{\Delta} (\psi_1, f_1, p), \ldots, M_{\Delta} (\psi_n, f_n, p)) \right]
\end{align*}
\]
Now, all claims follow immediately from Theorem 11. \( \square \)

**Corollary 17.** Let \( H, \varphi, f, p, \chi, f_i \), and \( \psi_i, i = 1, \ldots, n \), satisfy the hypothesis of Theorem 16. Further, assume that \( p \) attains its minimum value and its maximum value on its domain. If \( H \) is convex, then
\[
\begin{align*}
\max_{t \in (a,b)} p (t) & \left( b - a \right) \\
\times \left[ \chi (M_{\Delta} (\chi, \varphi (f))) - \chi \circ \varphi (M_{\Delta} (\psi_1, f_1), \ldots, M_{\Delta} (\psi_n, f_n)) \right] \\
& \geq \int_{(a,b)} p \, d\mu_{\Delta} \left[ \chi (M_{\Delta} (\chi, \varphi (f), p)) \\
- \chi \circ \varphi (M_{\Delta} (\psi_1, f_1, p), \ldots, M_{\Delta} (\psi_n, f_n, p)) \right]
\end{align*}
\]
where
\[
\begin{align*}
\psi (f) & = \chi^{-1} \left( \frac{\int_{(a,b)} \chi (\varphi (f)) \, d\mu_{\Delta}}{b - a} \right).
\end{align*}
\]
Moreover, if \( H \) is concave, then the inequalities in (37) hold in reverse order.

Proof. The proof is omitted as it is similar to the proof of Corollary 13. \( \square \)

**Remark 18.** If we take the discrete form of the weighted generalized mean (31) with \( \int_{(a,b)} p \, d\mu_{\Delta} = 1 \), then we obtain the quasiarithmetic mean. Namely, let \( \psi : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be continuous and strictly monotone, \( a = (a_1, \ldots, a_n) \) with \( a_k \in I, k = 1, \ldots, n \), and \( w = (w_1, \ldots, w_n) \) with \( w_k \geq 0 \) and \( \sum_{k=1}^{n} w_k = 1 \). Then the quasiarithmetic mean of \( a \) with weight \( w \) is defined by
\[
M_n = \psi^{-1} \left( \sum_{k=1}^{n} w_k \psi (a_k) \right).
\]
Now the following examples connect the quasiarithmetic mean (39) and the properties of Jensen functionals.

**Example 19** (see [16, Corollary 3]). Let \( w \) and \( \psi \) be defined as in Remark 18 and let \( \psi \) be strictly increasing and strictly
convex with continuous derivatives of second order such that $\psi'/\psi''$ is concave. Further, let $X, p, x_i, i = 1, \ldots, n$, be defined as in Example 9, and $q = (q_1, \ldots, q_n)$ with $q_i \geq 0$, $i = 1, \ldots, n$, and $\sum_{i=1}^n q_i = Q_n > 0$. Then, $\Phi_M(x_i) = \psi^{-1}(\sum_{k=1}^n w_k \psi(x_{k}))$ is a convex function (see [17, Theorem 1, page 197]). Hence by Theorem II, the functional

$$J_n(\Phi_M, X, p) = \sum_{i=1}^n p_i \Phi_M(x_i) - P_n \Phi_M\left(\frac{\sum_{i=1}^n p_i x_i}{P_n}\right)$$

is superadditive, that is,

$$J_n(\Phi_M, X, p + q) \geq J_n(\Phi_M, X, p) + J_n(\Phi_M, X, q),$$

and increasing; that is, if $p \geq q$, then

$$J_n(\Phi_M, X, p) \geq J_n(\Phi_M, X, q).$$

Also, by Corollary 12, we have

$$\max_{1 \leq i \leq n} \left\{ p_i \right\} \mathcal{J}_n(\Phi_M, X) \geq J_n(\Phi_M, X, p) \geq \min_{1 \leq i \leq n} \left\{ p_i \right\} \mathcal{J}_n(\Phi_M, X),$$

where

$$\mathcal{J}_n(\Phi_M, X) = \sum_{i=1}^n \Phi_M(x_i) - n\Phi_M\left(\frac{\sum_{i=1}^n x_i}{n}\right).$$

Example 20 (see [16, Corollary 4]). Consider (39), but with different conditions on $\psi$ and $w$. Namely, if

(i) $w_i \geq 0$ for $i = 1, \ldots, n$;

(ii) $\psi: \mathbb{R}^+ \to \mathbb{R}^+$;

(iii) $\lim_{x \to -\infty} \psi(x) = \infty$ or $\lim_{x \to -\infty} \psi(x) = \infty$,

then we define

$$\mathcal{M}_n = \psi^{-1}\left(\sum_{k=1}^n w_k \psi(q_k)\right).$$

Let $X, p, x_i, i = 1, \ldots, n$, be defined as in Example 9 and $q = (q_1, \ldots, q_n)$ with $q_i \geq 0$ and $\sum_{i=1}^n q_i = Q_n > 0$. Let $\psi$ be strictly increasing and strictly convex with continuous derivatives of second order such that $\psi'/\psi''$ is convex. Then $\Phi_M(x_i) = \psi^{-1}(\sum_{k=1}^n w_k \psi(x_{k}))$ is a convex function (see [17, Theorem 2, page 197]). Hence, by Theorem II, the functional

$$J_n(\Phi_M, X, p) = \sum_{i=1}^n p_i \Phi_M(x_i) - P_n \Phi_M\left(\frac{\sum_{i=1}^n p_i x_i}{P_n}\right)$$

is superadditive, that is,

$$J_n(\Phi_M, X, p + q) \geq J_n(\Phi_M, X, p) + J_n(\Phi_M, X, q),$$

and increasing; that is, if $p \geq q$, then

$$J_n(\Phi_M, X, p) \geq J_n(\Phi_M, X, q).$$

Also, by Corollary 12, we have

$$\max_{1 \leq i \leq n} \left\{ p_i \right\} \mathcal{J}_n(\Phi_M, X) \geq J_n(\Phi_M, X, p) \geq \min_{1 \leq i \leq n} \left\{ p_i \right\} \mathcal{J}_n(\Phi_M, X),$$

where

$$\mathcal{J}_n(\Phi_M, X) = \sum_{i=1}^n \Phi_M(x_i) - n\Phi_M\left(\frac{\sum_{i=1}^n x_i}{n}\right).$$

Corollary 22. Assume $\chi, \psi_1$, and $\psi_2 \in C^2([1, \infty])$ are strictly monotone. Suppose $f_1, f_2 : [a, b] \to \mathbb{R}$ are $\Delta$-integrable such that $f_1(t) + f_2(t) \in I$ for all $t \in [a, b]$ and $p, q : [a, b] \to \mathbb{R}$ are nonnegative and $\Delta$-integrable such that $p\chi(f_1 + f_2)$,
\[ q\chi(f_1 + f_2), p\psi_i(f_i), \text{ and } q\varphi_i(f_i), \ i = 1, 2, \text{ are } \Delta\text{-integrable and } \int_{[a,b]} p d\mu_\Delta > 0, \int_{[a,b]} q d\mu_\Delta > 0. \text{ Further, let} \]
\[
E = \frac{\psi'_1}{\psi'_1}, \quad F = \frac{\psi'_2}{\psi'_2}, \quad G = \frac{\chi'}{\chi''},
\]
(58)

If \( \psi'_1, \psi'_2, \text{ and } \chi' \) are positive and \( \psi''_1, \psi''_2, \text{ and } \chi'' \) are negative, then the functional
\[
\int_{[a,b]} p d\mu_\Delta \left[ \chi(M_\Delta(\chi, f_1 + f_2, p)) - \chi(M_\Delta(\psi_1, f_1, p) + M_\Delta(\psi_2, f_2, p)) \right]
\]
(59)
is superadditive, that is,
\[
\int_{[a,b]} (p + q) d\mu_\Delta \left[ \chi(M_\Delta(\chi, f_1 + f_2, p + q)) - \chi(M_\Delta(\psi_1, f_1, p + q) + M_\Delta(\psi_2, f_2, p + q)) \right]
\]
\[
\geq \int_{[a,b]} p d\mu_\Delta \left[ \chi(M_\Delta(\chi, f_1 + f_2, p)) - \chi(M_\Delta(\psi_1, f_1, p) + M_\Delta(\psi_2, f_2, p)) \right] + \int_{[a,b]} q d\mu_\Delta \left[ \chi(M_\Delta(\chi, f_1 + f_2, q)) - \chi(M_\Delta(\psi_1, f_1, q) + M_\Delta(\psi_2, f_2, q)) \right],
\]
(60)

and increasing; that is, if \( p \geq q \) such that \( \int_{[a,b]} p d\mu_\Delta > \int_{[a,b]} q d\mu_\Delta \), then
\[
\int_{[a,b]} p d\mu_\Delta \left[ \chi(M_\Delta(\chi, f_1 + f_2, p)) - \chi(M_\Delta(\psi_1, f_1, p) + M_\Delta(\psi_2, f_2, p)) \right]
\]
\[
\geq \int_{[a,b]} q d\mu_\Delta \left[ \chi(M_\Delta(\chi, f_1 + f_2, q)) - \chi(M_\Delta(\psi_1, f_1, q) + M_\Delta(\psi_2, f_2, q)) \right],
\]
(61)

if and only if \( G(x + y) \leq E(x) + F(y) \). If \( p \) attains its minimum and maximum values on its domain \([a, b]\), then (61) yields
\[
\left[ \max_{t \in [a,b]} p(t) \right] (b - a) \left[ \chi(\mathcal{M}_\Delta(\chi, f_1 + f_2)) - \chi(\mathcal{M}_\Delta(\psi_1, f_1) + \mathcal{M}_\Delta(\psi_2, f_2)) \right]
\]
\[
\geq \int_{[a,b]} p d\mu_\Delta \left[ \chi(M_\Delta(\chi, f_1 + f_2, p)) - \chi(M_\Delta(\psi_1, f_1, p) + M_\Delta(\psi_2, f_2, p)) \right]
\]
(62)

Moreover, if \( \psi'_1, \psi'_2, \chi'_1, \chi'_2, \psi''_1, \psi''_2, \text{ and } \chi'' \) are all positive, then the inequalities in (60), (61), and (62) are reversed if and only if \( G(x + y) \geq E(x) + F(y) \).

Proof. Let \( n = 2 \) in Theorem 16. By setting \( \varphi(x, y) = x + y \), we have
\[
H(s_1, s_2) = \chi(\psi^{-1}_1(s_1) + \psi^{-1}_2(s_2)).
\]
(63)

If \( \psi'_1, \psi'_2, \text{ and } \chi' \) are positive and \( \psi''_1, \psi''_2, \text{ and } \chi'' \) are negative, then \( H \) is convex if and only if \( G(x + y) \leq E(x) + F(y) \) (see [18]). If \( \psi'_1, \psi'_2, \chi'_1, \chi'_2, \psi''_1, \psi''_2, \text{ and } \chi'' \) are all positive, then \( H \) is concave if and only if \( G(x + y) \geq E(x) + F(y) \) (see [18]). Now, all claims follow immediately from Theorem 16.

**Corollary 23.** Assume \( \chi, \psi_1, \text{ and } \psi_2 \in C^2([1, \mathbb{R}) \) are strictly monotone. Suppose \( f_1, f_2 : [a, b] \rightarrow \mathbb{R} \) are nonnegative and \( \Delta\)-integrable such that \( p_f(f_1, f_2(t)) \in I \) for all \( t \in [a, b] \) and \( p, q : [a, b] \rightarrow \mathbb{R} \) are nonnegative and \( \Delta\)-integrable such that \( \psi_1(f_1, f_2), q\chi(f_1 + f_2), p\psi_i(f_i), \text{ and } q\varphi_i(f_i), i = 1, 2, \text{ are } \Delta\text{-integrable and } \int_{[a,b]} p d\mu_\Delta > 0, \int_{[a,b]} q d\mu_\Delta > 0. \text{ Further, let} \)
\[
A(t) = \frac{\psi'_1(t)}{\psi'_1(t) + t\psi''_1(t)}, \quad B(t) = \frac{\psi'_2(t)}{\psi'_2(t) + t\psi''_2(t)},
\]
\[
C(t) = \frac{\chi'(t)}{\chi'(t) + t\chi''(t)}.
\]
(64)

If \( \psi'_1, \psi'_2, \text{ and } \chi' \) are positive and \( A, B, \text{ and } C \) are negative, then the functional
\[
\int_{[a,b]} p d\mu_\Delta \left[ \chi(M_\Delta(\chi, f_1 + f_2, p)) - \chi(M_\Delta(\psi_1, f_1, p) \cdot M_\Delta(\psi_2, f_2, p)) \right]
\]
(65)
is superadditive, that is,
\[
\int_{[a,b]} (p + q) d\mu_\Delta \left[ \chi(M_\Delta(\chi, f_1 + f_2, p + q)) - \chi(M_\Delta(\psi_1, f_1, p + q) \cdot M_\Delta(\psi_2, f_2, p + q)) \right]
\]
\[
\geq \int_{[a,b]} p d\mu_\Delta \left[ \chi(M_\Delta(\chi, f_1 + f_2, p)) - \chi(M_\Delta(\psi_1, f_1, p) \cdot M_\Delta(\psi_2, f_2, p)) \right] + \int_{[a,b]} q d\mu_\Delta \left[ \chi(M_\Delta(\chi, f_1 + f_2, q)) - \chi(M_\Delta(\psi_1, f_1, q) \cdot M_\Delta(\psi_2, f_2, q)) \right],
\]
(66)
and increasing; that is, if \( p \geq q \) such that \( \int_{(a,b)} p \, d\mu_\Delta > \int_{(a,b)} q \, d\mu_\Delta \), then

\[
\int_{(a,b)} p \, d\mu_\Delta \left[ \chi (M_\Delta (x, f_1 \cdot f_2, p)) - \chi (M_\Delta (p_1, f_1, p) \cdot M_\Delta (p_2, f_2, p)) \right] \\
\geq \int_{(a,b)} q \, d\mu_\Delta \left[ \chi (M_\Delta (x, f_1 \cdot f_2, q)) - \chi (M_\Delta (p_1, f_1, q) \cdot M_\Delta (p_2, f_2, q)) \right],
\]

(67)

if and only if \( C(x \cdot y) \leq A(x) + B(y) \). If \( p \) attains its minimum and maximum values on its domain \([a, b]\), then (67) yields

\[
\left[ \max_{t \in [a,b]} p (t) \right] (b-a) \left[ \chi (\mathcal{M}_{\Delta} (x, f_1 \cdot f_2)) - \chi (\mathcal{M}_{\Delta} (p_1, f_1) \cdot \mathcal{M}_{\Delta} (p_2, f_2)) \right] \\
\geq \int_{(a,b)} p \, d\mu_\Delta \left[ \chi (M_\Delta (x, f_1 \cdot f_2, p)) - \chi (M_\Delta (p_1, f_1, p) \cdot M_\Delta (p_2, f_2, p)) \right] \\
\geq \left[ \min_{t \in [a,b]} p (t) \right] (b-a) \left[ \chi (\mathcal{M}_{\Delta} (x, f_1 \cdot f_2)) - \chi (\mathcal{M}_{\Delta} (p_1, f_1) \cdot \mathcal{M}_{\Delta} (p_2, f_2)) \right].
\]

(68)

If \( \psi_1', \psi_2', \chi', A, B, \) and \( C \) are all positive, then the inequalities in (66), (67), and (68) are reversed if and only if \( C(x \cdot y) \geq A(x) + B(y) \).

**Proof.** Let \( n = 2 \) in Theorem 16. By setting \( \varphi(x, y) = x \cdot y \), we have

\[ H (s_1, s_2) = \chi (\psi_1^{-1} (s_1) \cdot \psi_2^{-1} (s_2)). \]

(69)

If \( \psi_1', \psi_2', \) and \( \chi' \) are positive and \( A, B, \) and \( C \) are negative, then \( H \) is convex if and only if \( C(x \cdot y) \leq A(x) + B(y) \). If \( \psi_1', \psi_2', \chi', A, B, \) and \( C \) are all positive, then \( H \) is concave if and only if \( C(x \cdot y) \geq A(x) + B(y) \) (see [8]). Now, all claims follow immediately from Theorem 16.

**Corollary 24.** Let \( \lambda, \omega, \nu \in \mathbb{R} \) be such that

(a) \( \lambda < 0 \), \( \omega, \nu \), or \( \omega, \nu < 0 \), \( \lambda < 0 \);

(b) \( \lambda < \omega, \nu < 0 \), or \( \nu < 0 < \omega < \lambda \), or \( \omega < 0 < \nu < \lambda \), for \( 1/\lambda \leq 1/\omega + 1/\nu \);

(c) \( \lambda < 0 < \nu \), or \( \lambda < \nu < 0 \), for \( 1/\lambda \geq 1/\omega + 1/\nu \).

Suppose \( f_1, f_2 : [a, b] \rightarrow \mathbb{R} \) are \( \Delta \)-integrable and \( p, q : [a, b] \rightarrow \mathbb{R} \) are nonnegative and \( \Delta \)-integrable such that \( p \cdot f_1^3 \cdot f_2^3, q \cdot f_1^3 \cdot f_2^3, p f_1^\omega, q f_1^\omega, p f_2^\nu, \) and \( q f_2^\nu \) are \( \Delta \)-integrable and \( \int_{(a,b)} p \, d\mu_\Delta > 0, \int_{(a,b)} q \, d\mu_\Delta > 0 \). Then the functional

\[
\int_{(a,b)} p \cdot f_1^3 \cdot f_2^3 \, d\mu_\Delta \\
- \int_{(a,b)} p \, d\mu_\Delta \left[ \left( \frac{\int_{(a,b)} p f_1^\omega \, d\mu_\Delta}{\int_{(a,b)} p \, d\mu_\Delta} \right)^{1/\omega} \left( \frac{\int_{(a,b)} p f_2^\nu \, d\mu_\Delta}{\int_{(a,b)} p \, d\mu_\Delta} \right)^{1/\nu} \right]^\lambda
\]

is superadditive, that is,

\[
\int_{(a,b)} (p + q) \cdot f_1^3 \cdot f_2^3 \, d\mu_\Delta \\
- \int_{(a,b)} (p + q) \, d\mu_\Delta \left[ \left( \frac{\int_{(a,b)} (p + q) f_1^\omega \, d\mu_\Delta}{\int_{(a,b)} (p + q) \, d\mu_\Delta} \right)^{1/\omega} \left( \frac{\int_{(a,b)} (p + q) f_2^\nu \, d\mu_\Delta}{\int_{(a,b)} (p + q) \, d\mu_\Delta} \right)^{1/\nu} \right]^\lambda
\]

\[
\geq \int_{(a,b)} p \cdot f_1^3 \cdot f_2^3 \, d\mu_\Delta \\
- \int_{(a,b)} p \, d\mu_\Delta \left[ \left( \frac{\int_{(a,b)} p f_1^\omega \, d\mu_\Delta}{\int_{(a,b)} p \, d\mu_\Delta} \right)^{1/\omega} \left( \frac{\int_{(a,b)} p f_2^\nu \, d\mu_\Delta}{\int_{(a,b)} p \, d\mu_\Delta} \right)^{1/\nu} \right]^\lambda
\]

\[
+ \int_{(a,b)} q \cdot f_1^3 \cdot f_2^3 \, d\mu_\Delta \\
- \int_{(a,b)} q \, d\mu_\Delta \left[ \left( \frac{\int_{(a,b)} q f_1^\omega \, d\mu_\Delta}{\int_{(a,b)} q \, d\mu_\Delta} \right)^{1/\omega} \left( \frac{\int_{(a,b)} q f_2^\nu \, d\mu_\Delta}{\int_{(a,b)} q \, d\mu_\Delta} \right)^{1/\nu} \right]^\lambda
\]

(71)

and increasing; that is, if \( p \geq q \) such that \( \int_{(a,b)} p \, d\mu_\Delta > \int_{(a,b)} q \, d\mu_\Delta \), then

\[
\int_{(a,b)} p \cdot f_1^3 \cdot f_2^3 \, d\mu_\Delta \\
- \int_{(a,b)} p \, d\mu_\Delta \left[ \left( \frac{\int_{(a,b)} p f_1^\omega \, d\mu_\Delta}{\int_{(a,b)} p \, d\mu_\Delta} \right)^{1/\omega} \left( \frac{\int_{(a,b)} p f_2^\nu \, d\mu_\Delta}{\int_{(a,b)} p \, d\mu_\Delta} \right)^{1/\nu} \right]^\lambda
\]
\[
\int_{[a,b)} q \cdot f_1^3 \cdot f_2^3 \, d\mu_\Delta
\]
\[- \int_{[a,b)} q \, d\mu_\Delta \left[ \left( \frac{\int_{[a,b)} q f_1^w \, d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta} \right)^{1/w} \left( \frac{\int_{[a,b)} q f_2^w \, d\mu_\Delta}{\int_{[a,b)} q d\mu_\Delta} \right)^{1/y} \right]^h. 
\]

If \( p \) attains its minimum and maximum values on its domain, then
\[
\max_{t \in [a,b)} \left[ \int_{[a,b)} f_1^3 \cdot f_2^3 \, d\mu_\Delta \right.
\[- (b - a) \left[ \left( \frac{\int_{[a,b)} p f_1^w \, d\mu_\Delta}{b - a} \right)^{1/w} \right. 
\times \left( \frac{\int_{[a,b)} p f_2^w \, d\mu_\Delta}{b - a} \right)^{1/y} \right]^h
\]
\[
\int_{[a,b)} p \cdot f_1^3 \cdot f_2^3 \, d\mu_\Delta
\[- \int_{[a,b)} p d\mu_\Delta \left[ \left( \frac{\int_{[a,b)} p f_1^w \, d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \right)^{1/w} \right. 
\times \left( \frac{\int_{[a,b)} p f_2^w \, d\mu_\Delta}{\int_{[a,b)} p d\mu_\Delta} \right)^{1/y} \right]^h
\]
\[
\geq \min_{t \in [a,b)} \left[ \int_{[a,b)} f_1^3 \cdot f_2^3 \, d\mu_\Delta \right.
\[- (b - a) \left[ \left( \frac{\int_{[a,b)} f_1^w \, d\mu_\Delta}{b - a} \right)^{1/w} \right. 
\times \left( \frac{\int_{[a,b)} f_2^w \, d\mu_\Delta}{b - a} \right)^{1/y} \right]^h.
\]

Moreover, the inequalities in (71), (72), and (73) are reversed provided that
\[
(a') \omega, \nu > \lambda > 0, \text{ for } 1/\lambda \geq 1/\omega + 1/\nu;
\]
\[
(b') \omega, \nu < \lambda < 0, \text{ for } 1/\lambda \leq 1/\omega + 1/\nu.
\]

Proof. Let \( n = 2 \) in Theorem 16. By setting \( \varphi(x, y) = x \cdot y \), \( \chi(t) = t^4 \), \( \psi_1(t) = t^w \), and \( \psi_2(t) = t^y \), we have
\[
H(s_1, s_2) = \chi(\psi_1^{-1}(s_1) \cdot \psi_2^{-1}(s_2)) = \left( s_1^{1/w} s_2^{1/y} \right)^4.
\]

Now, \( H \) is convex if and only if \( d^2 H \geq 0 \), which implies
\[
\frac{\lambda}{\omega} \left( \frac{\lambda}{\omega} - 1 \right) \geq 0, \quad \frac{\lambda}{\nu} \left( \frac{\lambda}{\nu} - 1 \right) \geq 0,
\]
and these are satisfied if \( \lambda, \omega, \) and \( \nu \) satisfy conditions (a), (b), and (c). \( H \) is concave if and only if \( d^2 H \leq 0 \), and this implies
\[
\frac{\lambda}{\omega} \left( \frac{\lambda}{\omega} - 1 \right) \leq 0, \quad \frac{\lambda}{\nu} \left( \frac{\lambda}{\nu} - 1 \right) \leq 0,
\]
and these are satisfied if \( \lambda, \omega, \) and \( \nu \) satisfy conditions (a') and (b'). Now, all claims follow immediately from Theorem 16.

\[
\text{Corollary 25. Let } \lambda, \omega, \nu \in \mathbb{R} \text{ be such that } \lambda, \omega, \nu > 0, \lambda, \omega, \nu \neq 1 \text{ and }
\]
\[
(a) \lambda < 1 < \omega, \nu, \text{ or } \omega, \nu < 1 < \lambda;
\]
\[
(b) \lambda < \omega, \nu < 1, \text{ or } \omega < 1 < \lambda, \text{ or } \omega < 1 < \nu, \text{ for } 1/\log \lambda \leq 1/\log \omega + 1/\log \nu;
\]
\[
(c) \lambda < \omega < 1 < \nu, \text{ or } \lambda < \nu < 1 < \omega, \text{ for } 1/\log \lambda \geq 1/\log \omega + 1/\log \nu.
\]

Suppose \( f_1, f_2 : [a,b) \to \mathbb{R} \) are \( \Delta \)-integrable and \( p, q : [a,b) \to \mathbb{R} \) are nonnegative and \( \Delta \)-integrable such that \( p \lambda^{f_1 + f_2}, q \lambda^{f_1 + f_2}, p \omega^{f_1}, q \omega^{f_1}, p \nu^{f_2}, q \nu^{f_2} \) and \( q \lambda^{f_2} \) are \( \Delta \)-integrable and \( \int_{[a,b)} p d\mu_\Delta > 0, \int_{[a,b)} q d\mu_\Delta > 0 \). Then the functional
\[
\int_{[a,b)} p \lambda^{f_1 + f_2} d\mu_\Delta - \int_{[a,b)} p d\mu_\Delta 
\times \lambda^\log(\int_{[a,b)} p \omega^{f_1} d\mu_\Delta) \int_{[a,b)} p d\mu_\Delta + \log(\int_{[a,b)} p \nu^{f_2} d\mu_\Delta) \int_{[a,b)} p d\mu_\Delta
\]
\[
\geq \int_{[a,b)} p \lambda^{f_1 + f_2} d\mu_\Delta - \int_{[a,b)} p d\mu_\Delta 
\times \lambda^\log(\int_{[a,b)} q \omega^{f_1} d\mu_\Delta) \int_{[a,b)} q d\mu_\Delta + \log(\int_{[a,b)} q \nu^{f_2} d\mu_\Delta) \int_{[a,b)} q d\mu_\Delta
\]
is superadditive, that is,
\[
\int_{[a,b)} (p + q) \lambda^{f_1 + f_2} d\mu_\Delta
\]
\[
\geq \int_{[a,b)} (p + q) d\mu_\Delta \left( \lambda^\log(\int_{[a,b)} (p + q) \omega^{f_1} d\mu_\Delta) \int_{[a,b)} (p + q) d\mu_\Delta \right)
\]
\[
\times \lambda^\log(\int_{[a,b)} (p + q) \nu^{f_2} d\mu_\Delta) \int_{[a,b)} (p + q) d\mu_\Delta.
\]
and increasing; that is, if \( p \geq q \) such that \( \int_{(a,b)} p \, d\mu_{\Delta} > \int_{(a,b)} q \, d\mu_{\Delta} \), then
\[
\int_{(a,b)} p \left( f_1^+ + f_2^+ \right) d\mu_{\Delta} - \int_{(a,b)} p \, d\mu_{\Delta} \\
\times \lambda \log_{\omega} \left( \int_{(a,b)} p \, d\mu_{\Delta} \right) + \log_{\omega} \left( \int_{(a,b)} p \, d\mu_{\Delta} \right) \\
\geq \int_{(a,b)} q \left( f_1^+ + f_2^+ \right) d\mu_{\Delta} - \int_{(a,b)} q \, d\mu_{\Delta} \\
\times \lambda \log_{\omega} \left( \int_{(a,b)} q \, d\mu_{\Delta} \right) + \log_{\omega} \left( \int_{(a,b)} q \, d\mu_{\Delta} \right).
\]  
(79)

If \( p \) attains its minimum and maximum values on its domain, then
\[
\max_{t \in (a,b)} p(t) \left( \int_{(a,b)} \lambda f_1^+ f_2^+ d\mu_{\Delta} - (b-a) \right) \\
\times \lambda \log_{\omega} \left( \int_{(a,b)} \lambda f_1^+ f_2^+ d\mu_{\Delta} \right) + \log_{\omega} \left( \int_{(a,b)} \lambda f_1^+ f_2^+ d\mu_{\Delta} \right) \\
\geq \int_{(a,b)} p \lambda f_1^+ f_2^+ d\mu_{\Delta} - \int_{(a,b)} p \, d\mu_{\Delta} \\
\times \lambda \log_{\omega} \left( \int_{(a,b)} p \lambda f_1^+ f_2^+ d\mu_{\Delta} \right) + \log_{\omega} \left( \int_{(a,b)} p \lambda f_1^+ f_2^+ d\mu_{\Delta} \right) \\
\geq \min_{t \in (a,b)} p(t) \left( \int_{(a,b)} \lambda f_1^+ f_2^+ d\mu_{\Delta} - (b-a) \right) \\
\times \lambda \log_{\omega} \left( \int_{(a,b)} \lambda f_1^+ f_2^+ d\mu_{\Delta} \right) + \log_{\omega} \left( \int_{(a,b)} \lambda f_1^+ f_2^+ d\mu_{\Delta} \right).
\]  
(80)

Moreover, the inequalities in (78), (79), and (80) are reversed provided that
\[
\begin{align*}
(a') & \quad \omega, \nu > \lambda > 1, \text{ for } 1 / \log \lambda \geq 1 / \log \omega + 1 / \log \nu; \\
(b') & \quad \omega, \nu < \lambda < 0, \text{ for } 1 / \log \lambda \leq 1 / \log \omega + 1 / \log \nu.
\end{align*}
\]

Proof. Let \( n = 2 \) in Theorem 16. By setting \( \varphi(x, y) = x + y \), \( \chi(t) = \lambda t \), \( \psi_1(t) = \omega t \), and \( \psi_2(t) = \nu t \), we have
\[
H(s_1, s_2) = \left( s_1^{1 / \log \omega} \cdot s_2^{1 / \log \nu} \right)^{\log \lambda}.
\]  
(81)

Now, the proof is similar to the proof of Corollary 24. \( \square \)

Corollary 26. Let \( \lambda, \omega, \nu \in \mathbb{R} \) be such that
\[
\begin{align*}
(a) & \quad 0 < \omega, \nu \leq \lambda < 1, \text{ for all } f_1, f_2 > 0; \\
(b) & \quad 0 < \nu \leq \lambda \leq \omega < 1, \text{ for } f_2 \geq (((\nu - \lambda)(1 - \nu))/((\lambda - \nu)(1 - \omega))) f_1 \geq 0; \\
(c) & \quad 0 < \omega \leq \lambda \leq \nu < 1, \text{ for } (((\lambda - \omega)(1 - \nu))/((\nu - \lambda)(1 - \omega))) f_1 \geq f_2 \geq 0.
\end{align*}
\]

Suppose \( f_1, f_2 : [a, b] \to \mathbb{R} \) are \( \Delta \)-integrable and \( p, q : [a, b] \to \mathbb{R} \) are nonnegative and \( \Delta \)-integrable such that \( p(f_1 + f_2)^\lambda \), \( q(f_1 + f_2)^\lambda \), \( p f_1^\omega \), \( q f_1^\omega \), \( p f_2^\nu \), and \( q f_2^\nu \) are \( \Delta \)-integrable and \( \int_{(a,b)} p \, d\mu_{\Delta} > 0, \int_{(a,b)} q \, d\mu_{\Delta} > 0 \). Then the functional
\[
\int_{(a,b)} p \cdot (f_1 + f_2)^\lambda d\mu_{\Delta} - \int_{(a,b)} p \, d\mu_{\Delta} \\
\times \left[ \frac{\int_{(a,b)} p f_1^\omega d\mu_{\Delta}}{\int_{(a,b)} p \, d\mu_{\Delta}} \right]^{1/\omega} + \left( \frac{\int_{(a,b)} p f_2^\nu d\mu_{\Delta}}{\int_{(a,b)} p \, d\mu_{\Delta}} \right)^{1/\nu}
\]  
(82)

is superadditive, that is,
\[
\int_{(a,b)} (p + q) \cdot (f_1 + f_2)^\lambda d\mu_{\Delta} \\
- \int_{(a,b)} (p + q) d\mu_{\Delta} \left[ \frac{\int_{(a,b)} (p + q) f_1^\omega d\mu_{\Delta}}{\int_{(a,b)} (p + q) \, d\mu_{\Delta}} \right]^{1/\omega} \\
+ \left( \frac{\int_{(a,b)} (p + q) f_2^\nu d\mu_{\Delta}}{\int_{(a,b)} (p + q) \, d\mu_{\Delta}} \right)^{1/\nu}
\]  
(83)

and increasing; that is, if \( p \geq q \) such that \( \int_{(a,b)} p \, d\mu_{\Delta} > \int_{(a,b)} q \, d\mu_{\Delta} \), then
\[
\int_{(a,b)} p \cdot (f_1 + f_2)^\lambda d\mu_{\Delta} - \int_{(a,b)} p \, d\mu_{\Delta} \\
\times \left[ \frac{\int_{(a,b)} p f_1^\omega d\mu_{\Delta}}{\int_{(a,b)} p \, d\mu_{\Delta}} \right]^{1/\omega} + \left( \frac{\int_{(a,b)} p f_2^\nu d\mu_{\Delta}}{\int_{(a,b)} p \, d\mu_{\Delta}} \right)^{1/\nu}
\]  
(84)
If \( p \) attains its minimum and maximum values on its domain, then

\[
\max_{t \in [a,b]} p(t) \left[ \int_{[a,b]} (f_1 + f_2) \Delta d\mu - \int_{[a,b]} p \Delta d\mu \right] \\
- \int_{[a,b]} p \Delta d\mu \left[ \frac{\int_{[a,b]} p f_1^\omega d\mu}{b-a} \right]^{1/\omega} \\
+ \left( \frac{\int_{[a,b]} p f_2^\gamma d\mu}{b-a} \right)^{1/\gamma} \]
\geq \int_{[a,b]} p \cdot (f_1 + f_2) \Delta d\mu \Delta - \int_{[a,b]} p \Delta d\mu
\times \left[ \left( \frac{\int_{[a,b]} p f_1^\omega d\mu}{b-a} \right)^{1/\omega} + \left( \frac{\int_{[a,b]} p f_2^\gamma d\mu}{b-a} \right)^{1/\gamma} \right]
\geq \min_{t \in [a,b]} p(t) \left[ \int_{[a,b]} (f_1 + f_2) \Delta d\mu \\
- \int_{[a,b]} (p + q) \cdot (f_1 + f_2) \Delta d\mu \right] \\
\times \left[ \left( \frac{\int_{[a,b]} p f_1^\omega d\mu}{b-a} \right)^{1/\omega} \right]
+ \left[ \left( \frac{\int_{[a,b]} p f_2^\gamma d\mu}{b-a} \right)^{1/\gamma} \right].
\]

Moreover, the inequalities in (83), (84), and (85) are reversed provided that

(a') \( 1 < \lambda \leq \omega, \nu \), for all \( f_1, f_2 > 0 \);

(b') \( 1 < \nu \leq \lambda \leq \omega, \) for \( 0 \leq f_2 \leq ((\omega - \lambda)(\nu - 1))/((\lambda - \nu)(\omega - 1)) \); \( f_1 \);

(c') \( 1 < \omega \leq \lambda \leq \nu, \) for \( f_2 \geq ((\lambda - \omega)(\nu - 1))/((\nu - \lambda)(\omega - 1)) \); \( f_1 \) \( \geq 0 \).

Proof. Let \( n = 2 \) in Theorem 16. By setting \( \psi(x, y) = x + y \), \( \chi(t) = t^\lambda, \) \( \psi_1(t) = t^\omega, \) and \( \psi_2(t) = t^\nu \), we have

\[
H(s_1, s_2) = (s_1^{1/\omega} + s_2^{1/\gamma})^\lambda.
\]

Now, the proof is similar to the proof of Corollary 22, with some extra considerations of the definitions of \( E, F, \) and \( G \).

\[\square\]

**Corollary 27.** Suppose \( f_1, f_2 : [a, b] \to [0, \pi/4] \) are \( \Delta \)-integrable. Moreover, let \( p, q : [a, b] \to \mathbb{R} \) be nonnegative and \( \Delta \)-integrable such that \( p \cos(f_1 + f_2), q \cos(f_1 + f_2), p \cos(f_1) \), and \( q \cos(f_1), i = 1, 2 \), are \( \Delta \)-integrable and \( \int_{[a,b]} p \Delta d\mu \Delta > 0, \int_{[a,b]} q \Delta d\mu \Delta > 0 \). Then the functional

\[
\int_{[a,b]} p \Delta d\mu \Delta \cdot \cos \left[ \arccos \left( \frac{\int_{[a,b]} p \cdot \cos(f_1) \Delta d\mu}{\int_{[a,b]} p \Delta d\mu} \right) \right]
+ \arccos \left( \frac{\int_{[a,b]} p \cdot \cos(f_2) \Delta d\mu}{\int_{[a,b]} p \Delta d\mu} \right)
- \int_{[a,b]} p \cos(f_1 + f_2) \Delta d\mu
\]

is subadditive, that is,

\[
\int_{[a,b]} (p + q) \Delta d\mu
\cdot \cos \left[ \arccos \left( \frac{\int_{[a,b]} (p + q) \cdot \cos(f_1) \Delta d\mu}{\int_{[a,b]} (p + q) \Delta d\mu} \right) \right]
+ \arccos \left( \frac{\int_{[a,b]} (p + q) \cdot \cos(f_2) \Delta d\mu}{\int_{[a,b]} (p + q) \Delta d\mu} \right)
- \int_{[a,b]} (p + q) \cos(f_1 + f_2) \Delta d\mu
\]

and \( q \cos(f_1), i = 1, 2 \), are \( \Delta \)-integrable and \( \int_{[a,b]} p \Delta d\mu \Delta > 0, \int_{[a,b]} q \Delta d\mu \Delta > 0 \). Then the functional

\[
\int_{[a,b]} p \Delta d\mu \Delta \cdot \cos \left[ \arccos \left( \frac{\int_{[a,b]} p \cdot \cos(f_1) \Delta d\mu}{\int_{[a,b]} p \Delta d\mu} \right) \right]
+ \arccos \left( \frac{\int_{[a,b]} p \cdot \cos(f_2) \Delta d\mu}{\int_{[a,b]} p \Delta d\mu} \right)
- \int_{[a,b]} p \cos(f_1 + f_2) \Delta d\mu
\]
5. Applications to Hölder’s Inequality

Suppose \( f_i, i = 1, 2, \ldots, n \), are nonnegative \( \Delta \)-integrable functions on \([a, b)\) such that \( \prod_{i=1}^{n} f_i^{\alpha_i} \) is \( \Delta \)-integrable, where \( \alpha_i \geq 0, i = 1, \ldots, n \), are such that \( \sum_{i=1}^{n} \alpha_i = 1 \). Then, by using Theorem 3 (Hölder’s inequality on time scales), we have

\[
\int_{[a, b)} \prod_{i=1}^{n} f_i^{\alpha_i} d\mu \leq \left( \int_{[a, b)} f d\mu \right)^{\sum_{i=1}^{n} \alpha_i}.
\]

If \( \sum_{i=1}^{n} \alpha_i = A_n > 0 \), then (92) implies

\[
\int_{[a, b)} \prod_{i=1}^{n} f_i^{\alpha_i/A_n} d\mu \leq \left( \int_{[a, b)} f d\mu \right)^{\alpha_i/A_n}.
\]

In this section, we discuss properties of the functional, deduced from the Hölder inequality (93), defined in the following way.

Definition 28. Suppose \( f = (f_1, \ldots, f_n) \) is such that \( f_i, i = 1, \ldots, n \), are nonnegative \( \Delta \)-integrable functions on \([a, b)\). Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) be such that \( \alpha_i \geq 0 \) and \( \sum_{i=1}^{n} \alpha_i = A_n > 0 \). Then one defines the functional \( H_\Delta \) by

\[
H_\Delta (f, \alpha) = \prod_{i=1}^{n} \left( \int_{[a, b)} f_i d\mu \right)^{\alpha_i}.
\]

Theorem 29. Let \( \alpha = (\alpha_1, \ldots, \alpha_n) \) and \( \beta = (\beta_1, \ldots, \beta_n) \) be real \( n \)-tuples with \( \alpha_i \geq 0, \beta_i \geq 0 \) and \( \sum_{i=1}^{n} \alpha_i = A_n > 0, \sum_{i=1}^{n} \beta_i = B_n > 0 \). Suppose \( f_i, i = 1, \ldots, n \), are nonnegative \( \Delta \)-integrable on \([a, b)\) such that \( \prod_{i=1}^{n} f_i^{\alpha_i} \) and \( \prod_{i=1}^{n} f_i^{\beta_i} \) are \( \Delta \)-integrable. Then

\[
H_\Delta (f, \alpha + \beta) \geq H_\Delta (f, \alpha) \cdot H_\Delta (f, \beta),
\]

and \( H_\Delta (f, \cdot, \mu) \) is increasing; that is, if \( \alpha \geq \beta \) such that \( A_n > B_n \), then

\[
H_\Delta (f, \alpha) \geq H_\Delta (f, \beta).
\]
where

\[
\left( \int_{(a,b)} \prod_{i=1}^{n} f_{i}^{(a_i + b_i)/(A_n + B_n)} d\mu_{\Delta} \right)^{A_n+B_n} = \left[ \int_{(a,b)} \left( \prod_{i=1}^{n} f_{i}^{a_i/A_n} \right)^{A_n} d\mu_{\Delta} \right]^{A_n+B_n} \times \left( \prod_{i=1}^{n} f_{i}^{b_i/(A_n + B_n)} d\mu_{\Delta} \right)^{A_n+B_n} \leq \left( \int_{(a,b)} \prod_{i=1}^{n} f_{i}^{a_i/A_n} d\mu_{\Delta} \right)^{A_n} \left( \int_{(a,b)} \prod_{i=1}^{n} f_{i}^{b_i/(A_n + B_n)} d\mu_{\Delta} \right)^{B_n}. \tag{99}
\]

Now, by combining (98) and (99), we have

\[
H_{\Delta}(f, \alpha + \beta) \geq \frac{\prod_{i=1}^{n} \left( \int_{(a,b)} f_{i} d\mu_{\Delta} \right)^{a_i} \prod_{i=1}^{n} \left( \int_{(a,b)} f_{i} d\mu_{\Delta} \right)^{b_i}}{\left( \int_{(a,b)} \prod_{i=1}^{n} f_{i}^{a_i/A_n} d\mu_{\Delta} \right)^{A_n} \left( \int_{(a,b)} \prod_{i=1}^{n} f_{i}^{b_i/(A_n + B_n)} d\mu_{\Delta} \right)^{B_n}} = H_{\Delta}(f, \alpha) \cdot H_{\Delta}(f, \beta). \tag{100}
\]

If \( \alpha \geq \beta \), then \( \alpha - \beta \geq 0 \), and therefore

\[
H_{\Delta}(f, \alpha) = H_{\Delta}(f, (\alpha - \beta) + \beta) \geq H_{\Delta}(f, \alpha - \beta) \cdot H_{\Delta}(f, \beta) \tag{101}
\]

This completes the proof.

Corollary 30. Let \( f \) and \( \alpha \) satisfy the hypothesis of Theorem 29. Then

\[
\left[ \frac{\prod_{i=1}^{n} \int_{[a,b]} f_{i} d\mu_{\Delta}}{\left( \int_{[a,b]} \prod_{i=1}^{n} f_{i}^{1/n} d\mu_{\Delta} \right)^{n}} \right]^{\max_{\alpha \in \{\alpha_{i}\}}} \geq \left[ \frac{\prod_{i=1}^{n} \int_{[a,b]} f_{i} d\mu_{\Delta}}{\left( \int_{[a,b]} \prod_{i=1}^{n} f_{i}^{1/n} d\mu_{\Delta} \right)^{n}} \right]^{\min_{\alpha \in \{\alpha_{i}\}}} \geq H_{\Delta}(f, \alpha) \tag{102}
\]

Proof. Let

\[
\alpha_{\text{max}} = \left( \max_{1 \leq i \leq n} \{ \alpha_{i} \}, \ldots, \max_{1 \leq i \leq n} \{ \alpha_{i} \} \right), \quad \alpha_{\text{min}} = \left( \min_{1 \leq i \leq n} \{ \alpha_{i} \}, \ldots, \min_{1 \leq i \leq n} \{ \alpha_{i} \} \right). \tag{103}
\]

By Definition 28, we have

\[
H_{\Delta}(f, \alpha_{\text{max}}) = \left[ \frac{\prod_{i=1}^{n} \int_{[a,b]} f_{i} d\mu_{\Delta}}{\left( \int_{[a,b]} \prod_{i=1}^{n} f_{i}^{1/n} d\mu_{\Delta} \right)^{n}} \right]^{\max_{\alpha \in \{\alpha_{i}\}}},
\]

\[
H_{\Delta}(f, \alpha_{\text{min}}) = \left[ \frac{\prod_{i=1}^{n} \int_{[a,b]} f_{i} d\mu_{\Delta}}{\left( \int_{[a,b]} \prod_{i=1}^{n} f_{i}^{1/n} d\mu_{\Delta} \right)^{n}} \right]^{\min_{\alpha \in \{\alpha_{i}\}}}.
\]

Since \( \alpha_{\text{max}} \geq \alpha \geq \alpha_{\text{min}} \), the result follows from the second property of Theorem 29.

Corollary 31. Let \( f \), \( \alpha \), and \( \beta \) satisfy the hypothesis of Theorem 29 with \( A_n = B_n = 1 \). If there exist constants \( M > 1 > \alpha > \beta \), then

\[
H_{\Delta}(f, M\beta) \geq H_{\Delta}(f, \alpha) \geq H_{\Delta}(f, m\beta). \tag{105}
\]

Proof. By Definition 28, we have

\[
H_{\Delta}(f, M\beta) = MH_{\Delta}(f, \beta), \quad H_{\Delta}(f, m\beta) = mH_{\Delta}(f, \beta). \tag{106}
\]

Now the result follows from the second property of Theorem 29.

Remark 32. Some results for isotonic linear functionals related to the results given in this paper can be found in [16].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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