Research Article

An Efficient Method for Time-Fractional Coupled Schrödinger System

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We present a new technique to obtain the solution of time-fractional coupled Schrödinger system. The fractional derivatives are considered in Caputo sense. The proposed scheme is based on Laplace transform and new homotopy perturbation method. To illustrate the power and reliability of the method some examples are provided. The results obtained by the proposed method show that the approach is very efficient and simple and can be applied to other partial differential equations.

1. Introduction

The intuitive idea of fractional order calculus is as old as integer order calculus. It can be observed from a letter that was written by Leibniz to *L'Hôpital*. The fractional order calculus is a generalization of the integer order calculus to a real or complex number. Fractional differential equations are used in many branches of sciences, mathematics, physics, chemistry, and engineering. Applications of fractional calculus and fractional-order differential equations include dielectric relaxation phenomena in polymeric materials [1], transport of passive tracers carried by fluid flow in a porous medium in groundwater hydrology [2], transport dynamics in systems governed by anomalous diffusion [3, 4], and long-time memory in financial time series [5] and so on [6, 7]. In particular, recently, much attention has been paid to the distributed-order differential equations and their applications in engineering fields that both integer-order systems and fractional-order systems are special cases of distributed-order systems. The reader may refer to [8–10].

Several schemes have been developed for the numerical solution of differential equations. The homotopy perturbation method was proposed by He [11] in 1999. This method has been used by many mathematicians and engineers to solve various functional equations. Homotopy method

was further developed and improved by He and applied to nonlinear oscillators with discontinuities [12], nonlinear wave equations [13], and boundary value problems [14]. It can be said that He's homotopy perturbation method is a universal one and is able to solve various kinds of nonlinear functional equations. For example, it was applied to nonlinear Schrödinger equations [15], to nonlinear equations arising in heat transfer [16], and to other equations [17–20]. In this method, the solution is considered to be an infinite series which usually converges rapidly to exact solutions. In this paper we introduce a new form of homotopy perturbation and Laplace transform methods by extending the idea of [21].

We extend the homotopy perturbation and Laplace transform method to solve the time-fractional coupled Schrödinger system. The nonlinear time-fractional coupled Schrödinger partial differential system is as [22]

$$\begin{aligned} iD_t^\mu u + iu_x + u_{xx} + u + v + v_1 f(|u|^2, |v|^2) u &= 0, \\ iD_t^\mu v - iv_x + v_{xx} + u - v + v_2 g(|u|^2, |v|^2) v &= 0, \end{aligned} \quad (1)$$

where u, v are unknown functions, v_1, v_2 are real constants, and $0 < \mu \leq 1$ is a parameter describing the order of the fractional Caputo derivative. f and g are arbitrary (smooth) nonlinear real functions. Nonlinear Schrödinger system is

one of the canonical nonlinear equations in physics, arising in various fields such as nonlinear optics, plasma physics, and surface waves [23].

This paper is organized as follows. In Section 2, we recall some basic definitions and results dealing with the fractional calculus and Laplace transform which are later used in this paper. In Section 3 the homotopy perturbation method is described. The basic idea behind the new method is illustrated in Section 4. Finally, in Section 5, the application of homotopy perturbation and Laplace transform method for solving time-fractional coupled Schrödinger systems are presented.

2. Preliminaries and Notations

Some basic definitions and properties of the fractional calculus theory are used in this paper.

Definition 1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ , $\mu \in \mathbb{R}$, if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$ and it is said to be in the space C_μ^n if and only if $f^{(n)} \in C_\mu$, $n \in \mathbb{N}$. Clearly $C_\mu \in C_\beta$ if $\beta > \mu$.

Definition 2. The left-sided Riemann-Liouville fractional integral operator of order $\mu > 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as follows:

$$J^\mu f(t) = \begin{cases} \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} f(\tau) d\tau, & \mu > 0, \\ f(t), & \mu = 0, \end{cases} \quad (2)$$

where $\Gamma(\cdot)$ is the well-known Gamma function.

Some of the most important properties of operator J^μ , for $f(t) \in C_\mu$, $\mu, \beta \geq 0$, and $\gamma > -1$ are as follows:

$$\begin{aligned} (1) \quad J^\mu J^\beta f(t) &= J^{(\mu+\beta)} f(t), \\ (2) \quad J^\mu J^\beta f(t) &= J^\beta J^\mu f(t), \\ (3) \quad J^\mu t^\gamma &= \frac{\Gamma(\gamma+1)}{\Gamma(\mu+\gamma+1)} t^{\mu+\gamma}. \end{aligned} \quad (3)$$

Definition 3. Amongst a variety of definitions for fractional order derivatives, Caputo fractional derivative has been used [24, 25] as it is suitable for describing various phenomena, since the initial values of the function and its integer order derivatives have to be specified, so Caputo fractional derivative of function $f(t)$ is defined as

$$D^\mu f(t) = J^{n-\mu} f(t) = \frac{1}{\Gamma(n-\mu)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\mu-n+1}} d\tau, \quad (4)$$

where $n-1 < \mu \leq n$, $n \in \mathbb{N}$, $t \geq 0$, and $f(t) \in C_{-1}^n$.

In this paper, we have considered time-fractional coupled Schrödinger system, where the unknown function $u(x, t)$ is assumed to be a causal function of fractional derivatives which are taken in Caputo sense as follows.

Definition 4. The Caputo time-fractional derivative operator of order $\mu > 0$ is defined as

$$\begin{aligned} D_t^\mu u(x, t) &= J_t^{n-\mu} u(x, t) \\ &= \frac{1}{\Gamma(n-\mu)} \int_0^t (t-\tau)^{n-\mu-1} \frac{\partial^n u(x, \tau)}{\partial \tau^n} d\tau, \quad n-1 < \mu \leq n. \end{aligned} \quad (5)$$

Definition 5. The Laplace transform of a function $u(x, t)$, $t > 0$ is defined as

$$\mathcal{L}[u(x, t)] = \int_0^\infty e^{-st} u(x, t) dt, \quad (6)$$

where s can be either real or complex. The Laplace transform $\mathcal{L}[u(x, t)]$ of the Caputo derivative is defined as

$$\begin{aligned} \mathcal{L}\{D_t^\mu u(x, t)\} &= s^\mu \mathcal{L}\{u(x, t)\} \\ &\quad - \sum_{k=0}^{n-1} u^{(k)}(x, 0^+) s^{\mu-1-k}, \quad n-1 < \mu \leq n. \end{aligned} \quad (7)$$

Lemma 6. If $n-1 < \mu \leq n$, $n \in \mathbb{N}$, and $k \geq 0$, then we have

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \mathcal{L} \left\{ \frac{t^{k\mu}}{\Gamma(k\mu+1)} \right\} \right\} = \frac{t^{\mu(k+1)}}{\Gamma(\mu(k+1)+1)}, \quad (8)$$

where $\mathcal{L}^{-1}$ is the inverse Laplace transform.

The Mittag-Leffler function plays a very important role in the fractional differential equations and in fact it was introduced by Mittag-Leffler in 1903 [26]. The Mittag-Leffler function $E_\mu(z)$ with $\mu > 0$ is defined by the following series representation:

$$E_\mu(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\mu n + 1)}, \quad (9)$$

where $z \in \mathbb{C}$. For $\mu = 1$, (9) becomes

$$E_1(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z. \quad (10)$$

The key result that indicates why Mittag-Leffler functions are so important in fractional calculus is the following theorem. It essentially states that the eigenfunctions of Caputo differential operators may be expressed in terms of Mittag-Leffler functions.

Theorem 7 (see [27]). If $\mu > 0$ and $\lambda \in \mathbb{R}$, then we have

$$D_t^\mu (E_\mu(\lambda z^\mu)) = \lambda E_\mu(\lambda z^\mu). \quad (11)$$

3. The Homotopy Perturbation Method

For the convenience of the reader, we will first present a brief account of homotopy perturbation method. Let us consider the following differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \quad (12)$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \quad (13)$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, and Γ is the boundary of the domain Ω .

The operator A can be generally divided into two parts L and N , where L is linear, while N is nonlinear. Therefore, (12) can be written as follows:

$$L(u) + N(u) - f(r) = 0. \quad (14)$$

By using homotopy technique, one can construct a homotopy $y(r, p) : \Omega \times [0, 1] \rightarrow R$ which satisfies

$$\begin{aligned} H(y, p) &= (1-p)[L(y) + L(u_0)] + p[A(y) - f(r)] \\ &= 0, \quad p \in [0, 1], \quad r \in \Omega, \end{aligned} \quad (15)$$

which is equivalent to

$$H(y, p) = L(y) - L(u_0) + pL(u_0) + p[N(y) - f(r)] = 0, \quad (16)$$

where $p \in [0, 1]$ is an embedding parameter and u_0 is an initial guess approximation of (12) which satisfies the boundary conditions. Clearly, we have

$$\begin{aligned} H(y, 0) &= L(y) - L(u_0) = 0, \\ H(y, 1) &= A(y) - f(r) = 0. \end{aligned} \quad (17)$$

Thus, the changing process of p from 0 to 1 is just that of $y(r, p)$ from $u_0(r)$ to $y(r)$. In topology this is called deformation and $L(y) - L(u_0)$ and $A(y) - f(r)$ are called homotopic. If, the embedding parameter p , ($0 \leq p \leq 1$) is considered as a small parameter, applying the classical perturbation technique, we can naturally assume that the solution of (15) and (16) can be given as a power series in p ; that is,

$$y = y_0 + py_1 + p^2y_2 + \dots \quad (18)$$

According to homotopy perturbation method, the approximation solution of (12) can be expressed as a series of the power of p ; that is,

$$u = \lim_{p \rightarrow 1} y = y_0 + y_1 + y_2 + \dots \quad (19)$$

The convergence of series (19) has been proved by He in his paper [11]. It is worth noting that the major advantage of homotopy perturbation method is that the perturbation equation can be freely constructed in many ways by homotopy in topology and the initial approximation can also be freely selected. Moreover, the construction of the homotopy for the perturbed problem plays a very important role for obtaining desired accuracy [28].

4. Basic Ideas of the Homotopy Perturbation and Laplace Transform Method

To illustrate the basic ideas of this method, we consider the general form of a system of nonlinear fractional partial differential equations:

$$\begin{aligned} D_t^\mu u_i(x, t) - A_i(u_i(x, t)) - f_i(x, t) \\ = 0, \quad \mu \in (0, 1], \quad i = 1, 2, \dots, n, \end{aligned} \quad (20)$$

with initial conditions

$$u_i(x, 0) = u_i, \quad i = 1, 2, \dots, n, \quad (21)$$

where A_i are operators and $f_i(x, t)$ are known analytical functions. The operators A_i can be divided into two parts, L_i and N_i , where L_i are the linear operators and N_i are nonlinear operators. Therefore, (20) can be rewritten as

$$\begin{aligned} D_t^\mu u_i(x, t) - L_i(u_i(x, t)) - N_i(u_i(x, t)) \\ - f_i(x, t) = 0, \quad i = 1, \dots, n. \end{aligned} \quad (22)$$

By the new homotopy perturbation method [29], we construct the following homotopies:

$$\begin{aligned} H(V_i(x, t), p) \\ = (1-p)[D_t^\mu V_i(x, t) - u_{i,0}(x, t)] \\ + p[D_t^\mu V_i(x, t) - L_i(V_i(x, t)) \\ - N_i(V_i(x, t)) - f_i(x, t)] = 0, \quad i = 1, \dots, n, \end{aligned} \quad (23)$$

or equivalently

$$\begin{aligned} H(V_i(x, t), p) \\ = D_t^\mu V_i(x, t) - u_{i,0}(x, t) \\ + p[u_{i,0}(x, t) - L_i(V_i(x, t)) \\ - N_i(V_i(x, t)) - f_i(x, t)] = 0, \quad i = 1, \dots, n, \end{aligned} \quad (24)$$

where $p \in [0, 1]$ is an embedding parameter and $u_{i,0}(x, t)$ are initial approximations for the solution of (20). Clearly, we have from (23) and (24)

$$\begin{aligned} H(V_i(x, t), 0) &= D_t^\mu V_i(x, t) - u_{i,0}(x, t) \\ &= 0, \quad i = 1, \dots, n, \\ H(V_i(x, t), 1) &= D_t^\mu V_i(x, t) \\ &\quad - L_i(V_i(x, t)) - N_i(V_i(x, t)) - f_i(x, t) \\ &= 0, \quad i = 1, \dots, n. \end{aligned} \quad (25)$$

By applying Laplace transform on both sides of (24), we have

$$\begin{aligned} \mathcal{L}\{D_t^\mu V_i(x, t) - u_{i,0}(x, t) \\ + p[u_{i,0}(x, t) - L_i(V_i(x, t)) - N_i(V_i(x, t)) \\ - f_i(x, t)]\} = 0, \quad i = 1, \dots, n. \end{aligned} \quad (26)$$

Using (7), we derive

$$\begin{aligned} & s^\mu \mathcal{L} \{V_i(x, t)\} - s^{\mu-1} V_i(x, 0) \\ &= \mathcal{L} \{u_{i,0}(x, t) - p [u_{i,0}(x, t) - L_i(V_i(x, t)) \\ &\quad - N_i(V_i(x, t)) - f_i(x, t)]\}, \end{aligned} \quad (27)$$

$$i = 1, \dots, n,$$

or

$$\begin{aligned} & \mathcal{L} \{V_i(x, t)\} \\ &= \frac{1}{s} V_i(x, 0) \\ &+ \frac{1}{s^\mu} \{ \mathcal{L} \{u_{i,0}(x, t) - p [u_{i,0}(x, t) - L_i(V_i(x, t)) \\ &\quad - N_i(V_i(x, t)) - f_i(x, t)]\} \}, \end{aligned} \quad (28)$$

$$i = 1, \dots, n.$$

By applying inverse Laplace transform on both sides of (28), we have

$$\begin{aligned} & V_i(x, t) \\ &= V_i(x, 0) \\ &+ \mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \mathcal{L} \{u_{i,0}(x, t) \right. \\ &\quad \left. - p [u_{i,0}(x, t) - L_i(V_i(x, t)) \right. \\ &\quad \left. - N_i(V_i(x, t)) - f_i(x, t)] \} \right\}, \end{aligned} \quad (29)$$

$$i = 1, \dots, n.$$

According to the homotopy perturbation method, we can first use the embedding parameter p as a small parameter and assume that the solution of (29) can be written as a power series in p as follows:

$$V_i(x, t) = \sum_{j=0}^{\infty} p^j V_{i,j}(x, t), \quad i = 1, \dots, n, \quad (30)$$

where $V_{i,j}(x, t)$, $i = 0, \dots, n$, $j = 1, \dots$ are functions which should be determined. Suppose that the initial approximation of the solutions of (20) is in the following form:

$$u_{i,0}(x, t) = \sum_{j=0}^{\infty} \frac{a_{i,j}(x)}{\Gamma(j\mu + 1)} t^{j\mu}, \quad i = 1, \dots, n, \quad (31)$$

where $a_{i,j}(x)$, for $j = 1, 2, \dots$, $i = 1, \dots, n$, are functions which must be computed. Substituting (30) and (31) into (29)

and equating terms with identical powers of p we obtain the following set of equations:

$$\begin{aligned} p^0: & V_{i,0}(x, t) = V_i(x, 0) \\ &+ \mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \mathcal{L} \left\{ \sum_{j=0}^{\infty} \frac{a_{i,j}(x)}{\Gamma(j\mu + 1)} t^{j\mu} \right\} \right\}, \\ p^1: & V_{i,1}(x, t) \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \mathcal{L} \right. \\ &\quad \times \left\{ - \sum_{n=0}^{\infty} \frac{a_{i,j}(x)}{\Gamma(j\mu + 1)} t^{j\mu} \right. \\ &\quad \left. \left. + [L_i(V_{i,0}(x, t)) + N_i(V_{i,0}(x, t)) + f_i(x, t)] \right\} \right\}, \\ p^2: & V_{i,2}(x, t) = \mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \mathcal{L} \{L_i(V_{i,0}(x, t), V_{i,1}(x, t)) \right. \\ &\quad \left. + N_i(V_{i,0}(x, t), V_{i,1}(x, t))\} \right\}, \\ p^3: & V_{i,3}(x, t) \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \mathcal{L} \{L_i(V_{i,0}(x, t), V_{i,1}(x, t), V_{i,2}(x, t)) \right. \\ &\quad \left. + N_i(V_{i,0}(x, t), V_{i,1}(x, t), V_{i,2}(x, t))\} \right\}, \\ &\vdots \\ p^k: & V_{i,k}(x, t) \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \mathcal{L} \{L_i(V_{i,0}(x, t), V_{i,1}(x, t), \dots, V_{i,k-1}(x, t)) \right. \\ &\quad \left. + N_i(V_{i,0}(x, t), V_{i,1}(x, t), \dots, V_{i,k-1}(x, t))\} \right\}, \\ &\vdots \end{aligned} \quad (32)$$

Now if we solve these equations in such a way that $V_{i,1}(x, t) = 0$, then (32) yield

$$V_{i,2}(x, t) = V_{i,3}(x, t) = \dots = 0, \quad i = 1, \dots, n. \quad (33)$$

Therefore the exact solution is obtained by

$$\begin{aligned} & u_i(x, t) = V_i(x, t) = V_i(x, 0) \\ &+ \mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \mathcal{L} \left\{ \sum_{j=0}^{\infty} \frac{a_{i,j}(x)}{\Gamma(j\mu + 1)} t^{j\mu} \right\} \right\} \end{aligned}$$

$$= V_i(x, 0) + \sum_{j=0}^{\infty} \frac{a_{i,j}(x)}{\Gamma(\mu(j+1))} t^{\mu(j+1)}, \quad i = 1, \dots, n. \quad (34)$$

5. Example

To illustrate the power and reliability of the method for the time-fractional coupled Schrödinger system some examples are provided. The results reveal that the method is very effective and simple.

Example 8. Consider the following linear time-fractional coupled Schrödinger system:

$$\begin{aligned} iD_t^\mu u + iu_x + u_{xx} + 2u + v &= f_1(x, t), \\ iD_t^\mu v - iv_x + v_{xx} + u - 3v &= f_2(x, t), \end{aligned} \quad (35)$$

subject to the following initial conditions:

$$u(x, 0) = v(x, 0) = 0, \quad (36)$$

where $f_1(x, t)$ and $f_2(x, t)$ are of the form

$$\begin{aligned} f_1(x, t) &= \frac{i2^{2\mu+1}t^\mu e^{2i\pi x} \Gamma(\mu + 3/2)}{\sqrt{\pi}(2\mu + 1)} \\ &\quad + (-2\pi e^{2i\pi x} - 4\pi^2 e^{2i\pi x} + 3e^{2i\pi x}) t^{2\mu}, \\ f_2(x, t) &= \frac{i2^{2\mu+1}t^\mu e^{2i\pi x} \Gamma(\mu + 3/2)}{\sqrt{\pi}(2\mu + 1)} \\ &\quad + (2\pi e^{2i\pi x} - 4\pi^2 e^{2i\pi x} - 2e^{2i\pi x}) t^{2\mu}, \end{aligned} \quad (37)$$

with the exact solutions

$$u(x, t) = v(x, t) = e^{2i\pi x} t^{2\mu}, \quad (38)$$

where $0 < \mu \leq 1$.

To solve (35) by the homotopy perturbation and Laplace transform method, we construct the following homotopy:

$$\begin{aligned} &iD_t^\mu U \\ &= u_0(x, t) \\ &\quad - p[u_0(x, t) + iU_x + U_{xx} + 2U + V - f_1(x, t)], \\ &iD_t^\mu V \\ &= v_0(x, t) \\ &\quad - p[v_0(x, t) - iV_x + V_{xx} + U - 3V - f_2(x, t)]. \end{aligned} \quad (39)$$

Applying the Laplace transform on both sides of (39), we have

$$\begin{aligned} &s^\mu \mathcal{L}\{U\} - s^{\mu-1} U(x, 0) \\ &= -i\mathcal{L}\{u_0(x, 0) \\ &\quad - p[u_0(x, t) + iU_x + U_{xx} + 2U + V - f_1(x, t)]\}, \\ &s^\mu \mathcal{L}\{V\} - s^{\mu-1} V(x, 0) \\ &= -i\mathcal{L}\{v_0(x, 0) \\ &\quad - p[v_0(x, t) - iV_x + V_{xx} + U - 3V - f_2(x, t)]\}, \end{aligned} \quad (40)$$

or

$$\begin{aligned} \mathcal{L}\{U\} &= \frac{1}{s} U(x, 0) \\ &\quad - \frac{i}{s^\mu} \{\mathcal{L}\{u_0(x, 0) \\ &\quad - p[u_0(x, t) + iU_x + U_{xx} \\ &\quad + 2U + V - f_1(x, t)]\}\}, \\ \mathcal{L}\{V\} &= \frac{1}{s} V(x, 0) \\ &\quad - \frac{i}{s^\mu} \{\mathcal{L}\{v_0(x, 0) \\ &\quad - p[v_0(x, t) - iV_x + V_{xx} \\ &\quad + U - 3V - f_2(x, t)]\}\}. \end{aligned} \quad (41)$$

The inverse Laplace transform of (41) and the initial conditions $U(x, 0) = V(x, 0) = 0$ lead us to

$$\begin{aligned} U &= -i\mathcal{L}^{-1} \\ &\quad \left\{ \frac{1}{s^\mu} \{\mathcal{L}\{u_0(x, 0) \right. \\ &\quad \left. - p[u_0(x, t) + iU_x + U_{xx} \right. \\ &\quad \left. + 2U + V - f_1(x, t)]\}\} \right\}, \\ V &= -i\mathcal{L}^{-1} \end{aligned} \quad (42)$$

$$\begin{aligned} &\left\{ \frac{1}{s^\mu} \{\mathcal{L}\{v_0(x, 0) \right. \\ &\quad \left. - p[v_0(x, t) - iV_x + V_{xx} \right. \\ &\quad \left. + U - 3V - f_2(x, t)]\}\} \right\}. \end{aligned}$$

Suppose that the solution is expanded as (30); substituting (30) into (42), collecting the same powers of p , and equating each coefficient of p to zero yield

$$p^0: \begin{cases} U_0(x, t) = -i\mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \{\mathcal{L}\{u_0(x, t)\}\} \right\}, \\ V_0(x, t) = -i\mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \{\mathcal{L}\{v_0(x, t)\}\} \right\}, \end{cases}$$

$$\begin{aligned}
p^1: & \begin{cases} U_1(x, t) \\ = i\mathcal{L}^{-1} \\ \left\{ \frac{1}{s^\mu} \right. \\ \quad \times \left\{ \mathcal{L} \left\{ u_0(x, t) + iU_{0_x} \right. \right. \\ \quad \left. \left. + U_{0_{xx}} + 2U_0 + V_0 - f_1(x, t) \right\} \right\} \Big\}, \\ V_1(x, t) \\ = i\mathcal{L}^{-1} \\ \left\{ \frac{1}{s^\mu} \right. \\ \quad \times \left\{ \mathcal{L} \left\{ v_0(x, t) - iV_{0_x} + V_{0_{xx}} \right. \right. \\ \quad \left. \left. + U_0 - 3V_0 - f_2(x, t) \right\} \right\} \Big\}, \end{cases} \\
p^2: & \begin{cases} U_2(x, t) \\ = i\mathcal{L}^{-1} \\ \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \left\{ iU_{1_x} + U_{1_{xx}} + 2U_1 + V_1 \right\} \right\} \right\}, \\ V_2(x, t) \\ = i\mathcal{L}^{-1} \\ \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \left\{ -iV_{1_x} + V_{1_{xx}} + U_1 - 3V_1 \right\} \right\} \right\}, \end{cases} \\
& \vdots \\
p^j: & \begin{cases} U_j(x, t) \\ = i\mathcal{L}^{-1} \\ \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \left\{ iU_{j-1_x} + U_{j-1_{xx}} + 2U_{j-1} + V_{j-1} \right\} \right\} \right\}, \\ V_j(x, t) \\ = i\mathcal{L}^{-1} \\ \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \left\{ -iV_{j-1_x} + V_{j-1_{xx}} + U_{j-1} - 3V_{j-1} \right\} \right\} \right\}, \end{cases} \\
& \vdots
\end{aligned} \tag{43}$$

Assume $u_0(x, t) = \sum_{n=0}^{\infty} ((a_n(x)/\Gamma(n\mu + 1))t^{n\mu})$, $v_0(x, t) = \sum_{n=0}^{\infty} ((b_n(x)/\Gamma(n\mu + 1))t^{n\mu})$. Solving the above equations, for $U_1(x, t)$, $V_1(x, t)$, leads to the result

$$\begin{aligned}
U_1(x, t) &= \left(\frac{ia_0(x)}{\Gamma(\mu + 1)} \right) t^\mu \\
&+ \left(\left(e^{2i\pi x} \Gamma(2\mu + 1) + \frac{d^2}{dx^2} a_0(x) + b_0(x) \right. \right. \\
&\quad \left. \left. + ia_1(x) + i\frac{d}{dx} a_0(x) + 2a_0(x) \right) \right. \\
&\quad \left. \times (\Gamma(2\mu + 1))^{-1} \right) t^{2\mu}
\end{aligned}$$

$$\begin{aligned}
&+ \left(i \left(-i\frac{d^2}{dx^2} a_1(x) + \frac{d}{dx} a_1(x) + a_2(x) - 2ia_1(x) \right. \right. \\
&\quad \left. \left. - ib_1(x) + e^{2i\pi x} \Gamma(2\mu + 1) (-3 + 4\pi^2 + 2\pi) \right) \right. \\
&\quad \left. \times (\Gamma(3\mu + 1))^{-1} \right) t^{3\mu} \\
&+ \left(i \left(-i\frac{d^2}{dx^2} a_2(x) + a_3(x) + \frac{d}{dx} a_2(x) \right. \right. \\
&\quad \left. \left. - 2ia_2(x) - ib_2(x) \right) \times (\Gamma(4\mu + 1))^{-1} \right) t^{4\mu} \\
&+ \dots, \\
V_1(x, t) &= \left(\frac{ib_0(x)}{\Gamma(\mu + 1)} \right) t^\mu \\
&+ \left(\left(e^{2i\pi x} \Gamma(2\mu + 1) + \frac{d^2}{dx^2} b_0(x) + ib_1(x) + a_0(x) \right. \right. \\
&\quad \left. \left. - i\frac{d}{dx} b_0(x) - 3b_0(x) \right) \times (\Gamma(2\mu + 1))^{-1} \right) t^{2\mu} \\
&+ \left(i \left(-\frac{d}{dx} b_1(x) + b_2(x) - ia_1(x) + 3ib_1(x) \right. \right. \\
&\quad \left. \left. - i\frac{d^2}{dx^2} b_1(x) + 2e^{2i\pi x} \Gamma(2\mu + 1) (1 + 2\pi^2 - \pi) \right) \right. \\
&\quad \left. \times (\Gamma(3\mu + 1))^{-1} \right) t^{3\mu} \\
&+ \left(i \left(-\frac{d}{dx} b_2(x) + b_3(x) - ia_2(x) + 3ib_2(x) \right. \right. \\
&\quad \left. \left. - i\frac{d^2}{dx^2} b_2(x) \right) \times (\Gamma(4\mu + 1))^{-1} \right) t^{4\mu} + \dots.
\end{aligned} \tag{44}$$

By the vanishing of $U_1(x, t)$, $V_1(x, t)$ the coefficients $a_n(x)$, $b_n(x)$ ($n = 0, 1, 2, \dots$) are determined to be

$$\begin{aligned}
a_0(x) &= 0, \quad a_1(x) = ie^{2i\pi x} \Gamma(2\mu + 1), \\
a_2(x) &= a_3(x) = \dots = 0, \\
b_0(x) &= 0, \quad b_1(x) = ie^{2i\pi x} \Gamma(2\mu + 1), \\
b_2(x) &= b_3(x) = \dots = 0.
\end{aligned} \tag{45}$$

Therefore, the solutions of (35) are

$$\begin{aligned}
u(x, t) &= U_0(x, t) \\
&= -i \left(\frac{a_0(x) t^\mu}{\Gamma(\mu + 1)} + \frac{a_1(x) t^{2\mu}}{\Gamma(2\mu + 1)} + \frac{a_2(x) t^{3\mu}}{\Gamma(3\mu + 1)} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{a_3(x)t^{4\mu}}{\Gamma(4\mu+1)} + \dots \Big) = e^{2i\pi x} t^{2\mu}, \\
v(x, t) &= V_0(x, t) \\
&= -i \left(\frac{b_0(x)t^\mu}{\Gamma(\mu+1)} + \frac{b_1(x)t^{2\mu}}{\Gamma(2\mu+1)} + \frac{b_2(x)t^{3\mu}}{\Gamma(3\mu+1)} \right. \\
&\quad \left. + \frac{b_3(x)t^{4\mu}}{\Gamma(4\mu+1)} + \dots \right) = e^{2i\pi x} t^{2\mu}, \quad (46)
\end{aligned}$$

which are the exact solutions. Now, if we put $\mu = 1$ in (46), we obtain $u(x, t) = e^{2i\pi x} t^2$, $v(x, t) = e^{2i\pi x} t^2$ which is the exact solution of the given coupled Schrödinger system (35).

Example 9. Consider the following nonlinear time-fractional coupled Schrödinger system:

$$\begin{aligned}
iD_t^\mu u + iu_x + u_{xx} + u + v + 2(|u|^2 + |v|^2)u &= f_1(x, t), \\
iD_t^\mu v - iv_x + v_{xx} + u - v + 4(|u|^2 + |v|^2)v &= f_2(x, t), \quad (47)
\end{aligned}$$

subject to the following initial conditions:

$$u(x, 0) = v(x, 0) = 0, \quad (48)$$

where $f_1(x, t)$ and $f_2(x, t)$ have the following form:

$$\begin{aligned}
f_1(x, t) &= \left(\frac{i2^{2\mu+1}t^\mu\Gamma(\mu+3/2)}{\sqrt{\pi}(2\mu+1)} + 4t^{6\mu} \right) e^{ix}, \\
f_2(x, t) &= \left(\frac{i2^{2\mu+1}t^\mu\Gamma(\mu+3/2)}{\sqrt{\pi}(2\mu+1)} + 8t^{6\mu} \right) e^{ix}, \quad (49)
\end{aligned}$$

with the exact solutions

$$u(x, t) = v(x, t) = e^{ix} t^{2\mu}, \quad (50)$$

where $0 < \mu \leq 1$.

To solve (47) by the homotopy perturbation and Laplace transform method, we construct the following homotopy:

$$\begin{aligned}
iD_t^\mu U &= u_0(x, t) \\
&- p \left[u_0(x, t) + iU_x + U_{xx} + U + V \right. \\
&\quad \left. + 2(|U|^2 + |V|^2)U - f_1(x, t) \right], \\
iD_t^\mu V &= v_0(x, t) \\
&- p \left[v_0(x, t) - iV_x + V_{xx} + U - V \right. \\
&\quad \left. + 4(|U|^2 + |V|^2)V - f_2(x, t) \right]. \quad (51)
\end{aligned}$$

Applying Laplace transform on both sides of (51), we have

$$\begin{aligned}
s^\mu \mathcal{L}\{U\} - s^{\mu-1}U(x, 0) &= -i\mathcal{L}\{u_0(x, 0) \\
&- p[u_0(x, t) + iU_x + U_{xx} + U + V \\
&\quad + 2(|U|^2 + |V|^2)U - f_1(x, t)]\}, \\
s^\mu \mathcal{L}\{V\} - s^{\mu-1}V(x, 0) &= -i\mathcal{L}\{v_0(x, 0) \\
&- p[v_0(x, t) - iV_x + V_{xx} + U - V \\
&\quad + 4(|U|^2 + |V|^2)V - f_2(x, t)]\}, \quad (52)
\end{aligned}$$

or

$$\begin{aligned}
\mathcal{L}\{U\} &= \frac{1}{s}U(x, 0) \\
&- \frac{i}{s^\mu} \left\{ \mathcal{L}\{u_0(x, 0) \right. \\
&\quad \left. - p[u_0(x, t) + iU_x + U_{xx} + U + V \right. \\
&\quad \left. + 2(|U|^2 + |V|^2)U - f_1(x, t)]\right\}, \\
\mathcal{L}\{V\} &= \frac{1}{s}V(x, 0) \\
&- \frac{i}{s^\mu} \left\{ \mathcal{L}\{v_0(x, 0) \right. \\
&\quad \left. - p[v_0(x, t) - iV_x + V_{xx} + U - V \right. \\
&\quad \left. + 4(|U|^2 + |V|^2)V - f_2(x, t)]\right\}. \quad (53)
\end{aligned}$$

The inverse Laplace transform of (53) and the initial conditions $U(x, 0) = V(x, 0) = 0$ lead us to

$$\begin{aligned}
U &= -i\mathcal{L}^{-1} \\
&\left\{ \frac{1}{s^\mu} \left\{ \mathcal{L}\{u_0(x, 0) \right. \right. \\
&\quad \left. - p[u_0(x, t) + iU_x + U_{xx} + U + V \right. \\
&\quad \left. + 2(|U|^2 + |V|^2)U - f_1(x, t)]\right\} \right\},
\end{aligned}$$

$$\begin{aligned}
 V &= -i\mathcal{L}^{-1} \\
 &\left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \left\{ v_0(x, 0) \right. \right. \right. \\
 &\quad \left. \left. \left. - p \left[v_0(x, t) - iV_x + V_{xx} + U - V \right. \right. \right. \right. \\
 &\quad \left. \left. \left. + 4(|U|^2 + |V|^2) V - f_2(x, t) \right] \right\} \right\} \right\}.
 \end{aligned} \tag{54}$$

Suppose that the solution is expanded as (30); substituting (30) into (54), collecting the same powers of p , and equating each coefficient of p to zero yield

$$\begin{aligned}
 p^0: &\begin{cases} U_0(x, t) = -i\mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \{u_0(x, t)\} \right\} \right\}, \\ V_0(x, t) = -i\mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \{v_0(x, t)\} \right\} \right\}, \end{cases} \\
 p^1: &\begin{cases} U_1(x, t) \\ = i\mathcal{L}^{-1} \\ \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \left\{ u_0(x, t) + iU_{0x} + U_{0xx} + U_0 + V_0 \right. \right. \right. \right. \\ \quad \left. \left. \left. + 2(U_0\bar{U}_0 + V_0\bar{V}_0) U_0 - f_1(x, t) \right\} \right\} \right\}, \\ V_1(x, t) \\ = i\mathcal{L}^{-1} \\ \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \left\{ v_0(x, t) - iV_{0x} + V_{0xx} + U_0 - V_0 \right. \right. \right. \right. \\ \quad \left. \left. \left. + 4(U_0\bar{U}_0 + V_0\bar{V}_0) V_0 - f_2(x, t) \right\} \right\} \right\}, \end{cases} \\
 p^2: &\begin{cases} U_2(x, t) \\ = i\mathcal{L}^{-1} \\ \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \left\{ iU_{1x} + U_{1xx} + 2U_1 + V_1 \right. \right. \right. \right. \\ \quad \left. \left. \left. + 2(U_1\bar{U}_1 + V_1\bar{V}_1) U_1 \right\} \right\} \right\}, \\ V_2(x, t) \\ = i\mathcal{L}^{-1} \\ \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \left\{ -iV_{1x} + V_{1xx} + U_1 - 3V_1 \right. \right. \right. \right. \\ \quad \left. \left. \left. + 4(U_1\bar{U}_1 + V_1\bar{V}_1) V_1 \right\} \right\} \right\}, \end{cases} \\
 &\vdots \\
 p^j: &\begin{cases} U_j(x, t) \\ = i\mathcal{L}^{-1} \\ \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \times \left\{ iU_{j-1x} + U_{j-1xx} + 2U_{j-1} + V_{j-1} \right. \right. \right. \right. \\ \quad \left. \left. \left. + 2(U_{j-1}\bar{U}_{j-1} + V_{j-1}\bar{V}_{j-1}) U_{j-1} \right\} \right\} \right\}, \\ V_j(x, t) \\ = i\mathcal{L}^{-1} \\ \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \left\{ -iV_{j-1x} + V_{j-1xx} + U_{j-1} - 3V_{j-1} \right. \right. \right. \right. \\ \quad \left. \left. \left. + 4(U_{j-1}\bar{U}_{j-1} + V_{j-1}\bar{V}_{j-1}) V_{j-1} \right\} \right\} \right\}, \end{cases} \\
 &\vdots
 \end{aligned} \tag{55}$$

Assume $u_0(x, t) = \sum_{n=0}^{\infty} ((a_n(x)/\Gamma(n\mu + 1))t^{n\mu})$, $v_0(x, t) = \sum_{n=0}^{\infty} ((b_n(x)/\Gamma(n\mu + 1))t^{n\mu})$. Solving the above equations, for $U_1(x, t)$, $V_1(x, t)$, leads to the result

$$\begin{aligned}
 U_1(x, t) &= \left(\frac{ia_0(x)}{\Gamma(\mu + 1)} \right) t^\mu \\
 &+ \left(\left(e^{ix} \mu 4^\mu \Gamma(\mu) \Gamma\left(\mu + \frac{1}{2}\right) + ia_1(x) \sqrt{\pi} + a_0(x) \sqrt{\pi} \right. \right. \\
 &\quad \left. \left. + \sqrt{\pi} \frac{d^2}{dx^2} a_0(x) + \sqrt{\pi} b_0(x) + i \left(\frac{d}{dx} a_0(x) \right) \sqrt{\pi} \right) \right. \\
 &\quad \left. \times (\Gamma(2\mu + 1) \sqrt{\pi})^{-1} \right) t^{2\mu} \\
 &+ \left(i \left(a_2(x) + \frac{d}{dx} a_1(x) - i \frac{d^2}{dx^2} a_1(x) - ia_1(x) \right. \right. \\
 &\quad \left. \left. - ib_1(x) \right) \times (\Gamma(3\mu + 1))^{-1} \right) t^{3\mu} \\
 &+ i \left(\left(a_3(x) + \frac{d}{dx} a_2(x) - i \frac{d^2}{dx^2} a_2(x) - ia_2(x) \right. \right. \\
 &\quad \left. \left. - ib_2(x) \right) \times (\Gamma(4\mu + 1))^{-1} \right. \\
 &\quad \left. - i\Gamma\left(\mu + \frac{1}{3}\right) \Gamma\left(\mu + \frac{2}{3}\right) \right. \\
 &\quad \left. \times (\bar{a}_0(x) a_0(x) + b_0(x) \bar{b}_0(x)) 3^{1/2+3\mu} a_0(x) \right. \\
 &\quad \left. \times (\Gamma(4\mu + 1) (\Gamma(\mu + 1))^2 \pi)^{-1} \right) t^{4\mu} + \dots, \\
 V_1(x, t) &= \left(\frac{ib_0(x)}{\Gamma(\mu + 1)} \right) t^\mu \\
 &+ \left(\left(e^{ix} \mu 4^\mu \Gamma(\mu) \Gamma\left(\mu + \frac{1}{2}\right) + a_0(x) \sqrt{\pi} - \sqrt{\pi} b_0(x) \right. \right. \\
 &\quad \left. \left. + \sqrt{\pi} \frac{d^2}{dx^2} b_0(x) + ib_1(x) \sqrt{\pi} - i \left(\frac{d}{dx} b_0(x) \right) \sqrt{\pi} \right) \right. \\
 &\quad \left. \times (\Gamma(2\mu + 1) \sqrt{\pi})^{-1} \right) t^{2\mu} \\
 &+ \left(i \left(b_2(x) - ia_1(x) + ib_1(x) - \frac{d}{dx} b_1(x) \right. \right. \\
 &\quad \left. \left. - i \frac{d^2}{dx^2} b_1(x) \right) \times (\Gamma(3\mu + 1))^{-1} \right) t^{3\mu} \\
 &+ i \left(\left(-\frac{d}{dx} b_2(x) - ia_2(x) - i \frac{d^2}{dx^2} b_2(x) + ib_2(x) \right. \right. \\
 &\quad \left. \left. + b_3(x) \right) \times (\Gamma(4\mu + 1))^{-1} \right.
 \end{aligned}$$

$$\begin{aligned}
& -2i\Gamma\left(\mu + \frac{1}{3}\right)\Gamma\left(\mu + \frac{2}{3}\right) \\
& \times \left(a_0(x)\bar{a}_0(x) + b_0(x)\bar{b}_0(x)\right)3^{1/2+3\mu}b_0(x) \\
& \times \left(\Gamma(4\mu+1)(\Gamma(\mu+1))^2\pi\right)^{-1}t^{4\mu} + \dots.
\end{aligned} \tag{56}$$

By the vanishing of $U_1(x, t)$, $V_1(x, t)$ the coefficients $a_n(x)$, $b_n(x)$ ($n = 0, 1, 2, \dots$) are determined to be

$$\begin{aligned}
a_0(x) &= 0, & a_1(x) &= ie^{ix}\Gamma(2\mu+1), \\
a_2(x) &= a_3(x) = \dots = 0, \\
b_0(x) &= 0, & b_1(x) &= ie^{ix}\Gamma(2\mu+1), \\
b_2(x) &= b_3(x) = \dots = 0.
\end{aligned} \tag{57}$$

Therefore, the exact solutions of the system of (47) can be expressed as

$$\begin{aligned}
u(x, t) &= U_0(x, t) \\
&= -i\left(\frac{a_0(x)t^\mu}{\Gamma(\mu+1)} + \frac{a_1(x)t^{2\mu}}{\Gamma(2\mu+1)} + \frac{a_2(x)t^{3\mu}}{\Gamma(3\mu+1)} \right. \\
&\quad \left. + \frac{a_3(x)t^{4\mu}}{\Gamma(4\mu+1)} + \dots\right) \\
&= e^{ix}t^{2\mu}, \\
v(x, t) &= V_0(x, t) \\
&= -i\left(\frac{b_0(x)t^\mu}{\Gamma(\mu+1)} + \frac{b_1(x)t^{2\mu}}{\Gamma(2\mu+1)} + \frac{b_2(x)t^{3\mu}}{\Gamma(3\mu+1)} \right. \\
&\quad \left. + \frac{b_3(x)t^{4\mu}}{\Gamma(4\mu+1)} + \dots\right) \\
&= e^{ix}t^{2\mu}.
\end{aligned} \tag{58}$$

Now, if we put $\mu = 1$ in (58), we obtain $u(x, t) = e^{ix}t^2$, $v(x, t) = e^{ix}t^2$ which is the exact solution of the given coupled Schrödinger system (47).

Example 10. Consider the following nonlinear time-fractional coupled Schrödinger system:

$$\begin{aligned}
iD_t^\mu u + iu_x + u_{xx} + u + v + 2(|u|^2 + |v|^2)u &= f_1(x, t), \\
iD_t^\mu v - iv_x + v_{xx} + u - v - 3(|u|^2 - |v|^2)v &= f_2(x, t),
\end{aligned} \tag{59}$$

subject to the following initial conditions:

$$u(x, 0) = e^{ix}, \quad v(x, 0) = e^{-ix}, \tag{60}$$

where $f_1(x, t)$ and $f_2(x, t)$ have the following form:

$$\begin{aligned}
f_1(x, t) &= (ie^{ix} + 4e^{ix}E_\mu^2(t^\mu) - 2i\sin x)E_\mu(t^\mu), \\
f_2(x, t) &= (ie^{-ix} - 2e^{-ix} + 2i\sin x)E_\mu(t^\mu),
\end{aligned} \tag{61}$$

with the exact solutions

$$u(x, t) = e^{ix}E_\mu(t^\mu), \quad v(x, t) = e^{-ix}E_\mu(t^\mu), \tag{62}$$

where $0 < \mu \leq 1$.

To solve (59) by the LTNHPM, we construct the following homotopy:

$$\begin{aligned}
iD_t^\mu U &= u_0(x, t) \\
&- p[u_0(x, t) + iU_x + U_{xx} + U + V \\
&\quad + 2(|U|^2 + |V|^2)U - f_1(x, t)], \\
iD_t^\mu V &= v_0(x, t) \\
&- p[v_0(x, t) - iV_x + V_{xx} + U - V \\
&\quad - 3(|U|^2 - |V|^2)V - f_2(x, t)].
\end{aligned} \tag{63}$$

Applying Laplace transform on both sides of (63), we have

$$\begin{aligned}
s^\mu \mathcal{L}\{U\} - s^{\mu-1}U(x, 0) \\
&= -i\mathcal{L}\{u_0(x, 0) - p[u_0(x, t) + iU_x + U_{xx} + U + V \\
&\quad + 2(|U|^2 + |V|^2)U - f_1(x, t)]\}, \\
s^\mu \mathcal{L}\{V\} - s^{\mu-1}V(x, 0) \\
&= -i\mathcal{L}\{v_0(x, 0) - p[v_0(x, t) - iV_x + V_{xx} + U - V \\
&\quad - 3(|U|^2 - |V|^2)V - f_2(x, t)]\},
\end{aligned} \tag{64}$$

or

$$\begin{aligned}
\mathcal{L}\{U\} &= \frac{1}{s}U(x, 0) \\
&- \frac{i}{s^\mu} \left\{ \mathcal{L}\{u_0(x, 0) \right. \\
&\quad \left. - p[u_0(x, t) + iU_x + U_{xx} + U + V \right. \\
&\quad \left. + 2(|U|^2 + |V|^2)U - f_1(x, t)]\} \right\}, \\
\mathcal{L}\{V\} &= \frac{1}{s}V(x, 0) \\
&- \frac{i}{s^\mu} \left\{ \mathcal{L}\{v_0(x, 0) \right. \\
&\quad \left. - p[v_0(x, t) - iV_x + V_{xx} + U - V \right. \\
&\quad \left. - 3(|U|^2 - |V|^2)V - f_2(x, t)]\} \right\}.
\end{aligned} \tag{65}$$

The inverse Laplace transform (65) and the initial conditions $U(x, 0) = e^{ix}$, $V(x, 0) = e^{-ix}$, lead us to

$$U = e^{ix} - i\mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \{u_0(x, 0)\} - p \left[u_0(x, t) + iU_x + U_{xx} + U + V + 2(|U|^2 + |V|^2)U - f_1(x, t) \right] \right\} \right\},$$

$$V = e^{-ix} - i\mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \{v_0(x, 0)\} - p \left[v_0(x, t) - iV_x + V_{xx} + U - V - 3(|U|^2 - |V|^2)V - f_2(x, t) \right] \right\} \right\}. \quad (66)$$

Suppose that the solution is expanded as (30); substituting (30) into (66), collecting the same powers of p , and equating each coefficient of p to zero yield

$$\begin{aligned} p^0: & \begin{cases} U_0(x, t) = e^{ix} + \mathcal{L}^{-1} \left\{ \frac{-i}{s^\mu} \left\{ \mathcal{L} \{u_0(x, t)\} \right\} \right\}, \\ V_0(x, t) = e^{-ix} + \mathcal{L}^{-1} \left\{ \frac{-i}{s^\mu} \left\{ \mathcal{L} \{v_0(x, t)\} \right\} \right\}, \end{cases} \\ p^1: & \begin{cases} U_1(x, t) = i\mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \{u_0(x, t) + iU_{0x} + U_{0xx} + U_0 + V_0 + 2(U_0\bar{U}_0 + V_0\bar{V}_0)U_0 - f_1(x, t)\} \right\} \right\}, \\ V_1(x, t) = i\mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \{v_0(x, t) - iV_{0x} + V_{0xx} + U_0 - V_0 - 3(U_0\bar{U}_0 - V_0\bar{V}_0)V_0 - f_2(x, t)\} \right\} \right\}, \end{cases} \\ p^2: & \begin{cases} U_2(x, t) = i\mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \{iU_{1x} + U_{1xx} + 2U_1 + V_1 + 2(U_1\bar{U}_1 + V_1\bar{V}_1)U_1\} \right\} \right\}, \\ V_2(x, t) = i\mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \{-iV_{1x} + V_{1xx} + U_1 - 3V_1 - 3(U_1\bar{U}_1 - V_1\bar{V}_1)V_1\} \right\} \right\}, \end{cases} \\ & \vdots \end{aligned}$$

$$p^j: \begin{cases} U_j(x, t) = i\mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \{iU_{j-1x} + U_{j-1xx} + 2U_{j-1} + V_{j-1} + 2(U_{j-1}\bar{U}_{j-1} + V_{j-1}\bar{V}_{j-1})U_{j-1}\} \right\} \right\}, \\ V_j(x, t) = i\mathcal{L}^{-1} \left\{ \frac{1}{s^\mu} \left\{ \mathcal{L} \{-iV_{j-1x} + V_{j-1xx} + U_{j-1} - 3V_{j-1} - 3(U_{j-1}\bar{U}_{j-1} - V_{j-1}\bar{V}_{j-1})V_{j-1}\} \right\} \right\}, \end{cases} \quad (67)$$

Assume $u_0(x, t) = \sum_{n=0}^{\infty} ((a_n(x)/\Gamma(n\mu + 1))t^{n\mu})$, $v_0(x, t) = \sum_{n=0}^{\infty} ((b_n(x)/\Gamma(n\mu + 1))t^{n\mu})$. Solving the above equations, for $U_1(x, t)$, $V_1(x, t)$, we obtain the result

$$\begin{aligned} U_1(x, t) &= \left(\frac{i(a_0(x) - ie^{ix})}{\Gamma(\mu + 1)} \right) t^\mu \\ &+ \left(i \left(a_1(x) - 7ia_0(x) + 2i\bar{b}_0(x) - ib_0(x) - i\frac{d^2}{dx^2}a_0(x) + \frac{d}{dx}a_0(x) - e^{-ix} - (11 + i)e^{ix} + 2ie^{2ix}(-b_0(x) + \bar{a}_0(x)) \right) \times (\Gamma(2\mu + 1))^{-1} \right) t^{2\mu} + \dots, \\ V_1(x, t) &= \left(\frac{i(b_0(x) - ie^{-ix})}{\Gamma(\mu + 1)} \right) t^\mu \\ &+ \left(i \left(b_1(x) - 2ib_0(x) - \frac{d}{dx}b_0(x) - 3i\bar{a}_0(x) - ia_0(x) - i\frac{d^2}{dx^2}b_0(x) - e^{ix} + 3ie^{-2ix}(\bar{b}_0(x) + a_0(x)) + (3 - i)e^{-ix} \right) \times (\Gamma(2\mu + 1))^{-1} \right) t^{2\mu} + \dots. \end{aligned} \quad (68)$$

By the vanishing of $U_1(x, t)$, $V_1(x, t)$ the coefficients $a_n(x)$, $b_n(x)$ ($n = 0, 1, 2, \dots$) are determined to be

$$\begin{aligned} a_0(x) &= a_1(x) = a_2(x) = a_3(x) = \dots = ie^{ix}, \\ b_0(x) &= b_1(x) = b_2(x) = b_3(x) = \dots = ie^{-ix}. \end{aligned} \quad (69)$$

Therefore, the solutions of (59) are

$$\begin{aligned}
 u(x, t) &= U_0(x, t) \\
 &= e^{ix} \\
 &\quad - i \left(\frac{a_0(x) t^\mu}{\Gamma(\mu+1)} + \frac{a_1(x) t^{2\mu}}{\Gamma(2\mu+1)} + \frac{a_2(x) t^{3\mu}}{\Gamma(3\mu+1)} \right. \\
 &\quad \left. + \frac{a_3(x) t^{4\mu}}{\Gamma(4\mu+1)} + \dots \right) = e^{ix} E_\mu(t^\mu), \\
 v(x, t) &= V_0(x, t) \\
 &= e^{-ix} \\
 &\quad - i \left(\frac{b_0(x) t^\mu}{\Gamma(\mu+1)} + \frac{b_1(x) t^{2\mu}}{\Gamma(2\mu+1)} + \frac{b_2(x) t^{3\mu}}{\Gamma(3\mu+1)} \right. \\
 &\quad \left. + \frac{b_3(x) t^{4\mu}}{\Gamma(4\mu+1)} + \dots \right) = e^{-ix} E_\mu(t^\mu).
 \end{aligned} \tag{70}$$

Now, if we put $\mu = 1$ in (70), we obtain $u(x, t) = e^{ix+t}$, $v(x, t) = e^{-ix+t}$ which is the exact solution of the given coupled Schrödinger system (59).

6. Conclusion

In this paper, we have introduced a combination of Laplace transform and homotopy perturbation methods for solving fractional Schrödinger equations which we called homotopy perturbation and Laplace transform method. In this scheme, the solution considered to be a Taylor series which converges rapidly to the exact solution of the nonlinear equation. As shown in the three examples of this paper, a clear conclusion can be drawn from the results that the homotopy perturbation and Laplace transform method provide an efficient method to handle nonlinear partial differential equations of fractional order. The computations associated with the examples were performed using Maple 13.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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