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Research Article

Uniform Approximation of Periodical Functions by Trigonometric Sums of Special Type

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The approximation characteristics of trigonometric sums $U_{n,p}^{\psi}$ of special type on the class $C_{\beta,\infty}^{\psi}$ of (ψ,β)-differentiable (in the sense of A. I. Stepanets) periodical functions are studied. Because of agreement between parameters of approximative sums and approximated classes, the solution of Kolmogorov-Nikol'skii problem is obtained in a sufficiently general case. It is shown that in a number of important cases these sums provide higher order of approximation in comparison with Fourier sums, de la Vallée Poussin sums, and others on the class $C_{\beta,\infty}^{\psi}$ in the uniform metric. The range of parameters in which the sums $U_{n,p}^{\psi}$ give the order of the best uniform approximation on the classes $C_{\beta,\infty}^{\psi}$ is indicated.

1. The Introduction and Problem Definition

Let C be the space of continuous 2π-periodical functions f where the norm is defined by $\|f\|_C = \max_x |f(x)|$.

Let us consider the class $C_{\beta,\infty}^{\psi}$ [1] of functions $f \in C$ such that for given $\beta \in \mathbb{R}$ and fixed sequence $\psi(k)$ $(k \in \mathbb{N})$ of real numbers the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \left(a_k(f) \cos\left(kx + \frac{\beta\pi}{2}\right) + b_k \sin\left(kx + \frac{\beta\pi}{2}\right)\right) \quad (1)$$

are the Fourier series of some function $\varphi \in S_\infty$, where

$$S_\infty = \left\{ \varphi : \operatorname*{ess\,sup}_t |\varphi(t)| \leqslant 1 \right\}. \quad (2)$$

The function φ is called the (ψ,β)-derivative of f and denoted by f_β^ψ. For $\psi(k) = k^{-r}$, $r > 0$, the class $C_{\beta,\infty}^{\psi}$ coincides with Weyl-Nagy class W_β^r, and for $\beta = r$ coincides with Weyl class W_r^r. In the case of natural r and $\beta = r$ the class $C_{\beta,\infty}^{\psi}$ is a class W^r of periodical functions whose rth derivatives nearly everywhere do not exceed unity in absolute value. If $\psi(k)$ is a sequence such that

$$\lim_{k\to\infty} k^r \psi(k) = 0 \quad \forall r \in \mathbb{R}, \quad (3)$$

then $C_{\beta,\infty}^{\psi}$ consists of infinitely differentiable functions (see [2, Chapter 1, Section 8]). The example of a sequence satisfying condition (3) is $\psi(k) = e^{-\alpha k^r}$, $\alpha > 0$, $r > 0$. In this case the class $C_{\beta,\infty}^{\psi}$ will be denoted by $C_{\beta,\infty}^{\alpha,r}$. If $\psi(k)$ satisfies the condition

$$|\psi(k)| \leqslant K e^{-\alpha k}, \quad k \in \mathbb{N},\ \alpha > 0, \quad (4)$$

$C_{\beta,\infty}^{\psi}$ consists of analytical functions regularly continuing into the strip $|\operatorname{Im} z| < \alpha$ of the complex plane.

Following [3, page 147], we set by $\mathfrak{M}$ the set of all continuous convex downwards functions $\psi(t)$, $t \geqslant 1$, satisfying the condition

$$\lim_{t\to\infty} \psi(t) = 0 \quad (5)$$

and associate each $\psi \in \mathfrak{M}$ with the characteristic

$$\eta(t) = \eta(\psi;t) = \psi^{-1}\left(\frac{1}{2}\psi(t)\right), \quad t \geqslant 1, \quad (6)$$

where $\psi^{-1}(\cdot)$ is an inverse function of $\psi(\cdot)$. By using $\eta(t)$ we define the next subset of $\mathfrak{M}$ as

$$F = \left\{ \psi \in \mathfrak{M} : \eta'(\psi;t) \leqslant K,\ \forall t \geqslant 1 \right\}, \qquad \eta'(\psi;t) := \eta'(\psi;t+0). \quad (7)$$

As is shown in [3, page 153] all functions $\psi \in \mathfrak{M}$ for which

$$0 < C_1 \leqslant \frac{t}{T(t)} \leqslant C_2, \quad \forall t \geqslant 1,\ C_1, C_2 = \text{const}, \tag{8}$$

where $T(t) = T(\psi; t) = \eta(\psi; t) - t$, belong to F. The set of such functions is denoted by $\mathfrak{M}_C$. The quantity $T(t)$ has a simple geometric interpretation. It is equal to the length of interval where the value of the function $\psi(t)$ decreases by two times. Thereby it is natural to call the function $T(t)$ the half-decay period of ψ. The examples of $\psi(\cdot)$ from $\mathfrak{M}_C$ are the functions $\psi_1(t) = t^{-r}$, $r > 0$, $\psi_2(t) = 1/t^r \ln(t+\gamma)$, $r > 0$, $\gamma \geqslant 1$, and others. The set F also includes (see [3, page 153]) the subset $\mathfrak{M}^+_\infty$ of all functions $\psi \in \mathfrak{M}$ for which the characteristic

$$\mu(t) = \mu(\psi; t) = \frac{t}{\eta(t) - t}, \quad t \geqslant 1, \tag{9}$$

called the modulus of half-decay, tends monotonically to infinity as $t \to \infty$. If $\psi_3(t) = e^{-\alpha t^r}$, $\alpha > 0$, $r > 0$, then $\eta(\psi_3; t) = t((\ln 2/\alpha t^r) + 1)^{1/r}$ and that is why

$$\begin{aligned} \mu(\psi_3; t) &= \frac{t}{\eta(\psi_3; t) - t} \\ &= \frac{1}{((\ln 2/\alpha t^r) + 1)^{1/r} - 1} \uparrow \infty, \quad t \longrightarrow \infty. \end{aligned} \tag{10}$$

Therefore, $\psi_3 \in \mathfrak{M}^+_\infty \subset F$.

Thus, in F there exist functions that tend to zero according to the power law as well as the functions that tend to zero faster than any power function. However, the example of the function $\psi_4(t) = 1/\ln(t+1)(\eta(\psi_4; t) = (t+1)^2 - 1$, so $\eta'(\psi_4; t) = 2(t+1))$ shows that the set F may not contain functions tending to zero lower than any power function.

In what follows, we assume that a sequence $\psi(k)$ defining class $C^\psi_{\beta,\infty}$ is a restriction on the set $\mathbb{N}$ of natural numbers of some function $\psi(t)$, $t \geqslant 1$, from F.

We consider for an arbitrary function $f \in C$ the following sum:

$$U^\varphi_{n,p}(f; x) = \sum_{k=0}^{n-1} \lambda_{n,p}(k) A_k(f; x), \tag{11}$$

where

$$\begin{aligned} \lambda_{n,p}(k) &= \lambda_{n,p}(k; \varphi) \\ &= \begin{cases} 1, & 0 \leqslant k \leqslant n-p, \\ 1 - \dfrac{\varphi(k)}{\varphi(n)}, & n-p+1 \leqslant k \leqslant n-1, \end{cases} \end{aligned} \tag{12}$$

$p \in \mathbb{N}$, $p \in [1, n]$, $\varphi(k)\,(k \in \mathbb{N})$ is a given monotonically increasing to infinity sequence of real numbers,

$$\begin{aligned} A_k(f; x) &:= a_k(f) \cos kx + b_k(f) \sin kx, \\ A_0(f; x) &:= \frac{a_0(f)}{2}, \end{aligned} \tag{13}$$

and $a_0(f)$, $a_k(f)$, $b_k(f)$ are the Fourier coefficients of the function f.

When the parameters p and $\varphi(k)$ are chosen in a certain way, the sum $U^\varphi_{n,p}(f; x)$ coincides with some classical sums such as Zygmund sum [4] (for $p = n$ and $\varphi(k) = k^s$, $s > 0$)

$$Z^s_n(f; x) = \sum_{k=0}^{n-1} \left(1 - \frac{k^s}{n^s}\right) A_k(f; x), \quad s > 0, \tag{14}$$

Fejér sum [5] (for $p = n$ and $\varphi(k) = k$)

$$\sigma_{n-1}(f; x) = \sum_{k=0}^{n-1} \left(1 - \frac{k}{n}\right) A_k(f; x), \tag{15}$$

de la Vallée Poussin sum [6] (for $p \in \mathbb{N}$, $1 \leqslant p \leqslant n$, and $\varphi(k) = k - n + p$)

$$V_{n,p}(f; x) = \sum_{k=0}^{n-1} \lambda_{n,p}(k) A_k(f; x), \tag{16}$$

where

$$\lambda_{n,p}(k) = \begin{cases} 1, & 0 \leqslant k \leqslant n-p, \\ 1 - \dfrac{k-n+p}{p}, & n-p+1 \leqslant k \leqslant n-1, \end{cases} \tag{17}$$

and Fourier sum (for $p = 1$)

$$S_{n-1}(f; x) = \sum_{k=0}^{n-1} A_k(f; x). \tag{18}$$

For $p = n$, the sum $U^\varphi_{n,p}(f; x)$ coincides with the so-called generalized Zygmund sum [7] (see also [8, 9])

$$Z^\varphi_n(f; x) = \sum_{k=0}^{n-1} \left(1 - \frac{\varphi(k)}{\varphi(n)}\right) A_k(f; x), \tag{19}$$

where $\varphi(k)$ $(k \in \mathbb{N})$ is a given monotonically increasing to infinity sequence of real numbers.

The aim of the current paper is an investigation of asymptotic behavior as $n \to \infty$ of the quantity

$$\mathscr{E}\left(C^\psi_{\beta,\infty}; U^\psi_{n,p}\right) = \sup_{f \in C^\psi_{\beta,\infty}} \left\| f(\cdot) - U^\psi_{n,p}(f; \cdot) \right\|_C, \quad \psi \in F,\ \beta \in \mathbb{R}, \tag{20}$$

where $U^\psi_{n,p}(f; \cdot)$ is the sum $U^\varphi_{n,p}(f; \cdot)$ of type (11) with $\varphi(k) = (k - n + p)/\psi(k)$, $n, p \in \mathbb{N}$, $p \leqslant n$. Note that this research area for Fourier sum, de la Vallée Poussin sum, and Zygmund sum has a long history on different functional classes. It is connected with Kolmogorov, Nikol'skii, Timan, Dzyadyk, Stechkin, Korneichuk, Efimov, Telyakovskii, Stepanets, Motornii, Trigub, Rukasov, and others. For more information on this subject see, for example, [10–13].

Following Stepanets [1], we call that the solution of Kolmogorov-Nikol'skii problem is found for sum $U^\psi_{n,p}(f; \cdot)$ on the class $C^\psi_{\beta,\infty}$ if the asymptotic equality

$$\mathscr{E}\left(C^\psi_{\beta,\infty}; U^\psi_{n,p}\right) = \nu(n) + o(1)\,\nu(n), \quad n \longrightarrow \infty, \tag{21}$$

is obtained, where $\nu(n) = \nu(n, p, \psi, \beta)$ is some specific sequence.

In the current work, the solution of Kolmogorov-Nikol'skii problem for sum $U_{n,p}^{\psi}(f;\cdot)$ is found on $C_{\beta,\infty}^{\psi}$ and it is shown that this sum provides higher order of approximation in the uniform metric than Fourier sum, de la Vallée Poussin sum, and other classical sums do.

2. The Main Results

Theorem 1. *Let $\psi \in F$, $\beta \in \mathbb{R}$, $n, p \in \mathbb{N}$, and $p \leqslant n$. Then, as $n \to \infty$*

$$\mathscr{E}\left(C_{\beta,\infty}^{\psi};U_{n,p}^{\psi}\right)=\psi(n)\left(\frac{4}{\pi^2}A_{n,p}^{\psi}+O(1)\right), \tag{22}$$

where

$$A_{n,p}^{\psi}=\begin{cases}\ln p, & \text{if } T(n)\leqslant 1,\\ \ln\dfrac{p}{T(n)}, & \text{if } 1\leqslant T(n)\leqslant p,\\ \ln\dfrac{T(n)}{p}, & \text{if } T(n)\geqslant p,\end{cases} \tag{23}$$

$T(n)=\eta(\psi;n)-n$, $\eta(\psi;n)=\psi^{-1}((1/2)\psi(n))$, and $O(1)$ is a quantity uniformly bounded in β, n, and p.

As is shown in [3, page 508], if $\psi \in F$ and $\beta \in \mathbb{R}$, then for the quantity of the best uniform approximation of the class $C_{\beta,\infty}^{\psi}$ by trigonometric polynomials of order not more than $n-1$:

$$E_n\left(C_{\beta,\infty}^{\psi}\right)=\sup_{f\in C_{\beta,\infty}^{\psi}}\inf_{t_{n-1}\in\mathcal{T}_{n-1}}\left\|f(\cdot)-t_{n-1}(\cdot)\right\|_{C}, \tag{24}$$

the estimate

$$E_n\left(C_{\beta,\infty}^{\psi}\right)\asymp\psi(n) \tag{25}$$

is true (the notation $\alpha(n)\asymp\beta(n)$ means that there exist constants $K_1,K_2>0$ such that $K_1\beta(n)\leqslant\alpha(n)\leqslant K_2\beta(n)$). Theorem 1 and (25) lead to the following statement.

Corollary 2. *Let $\psi \in F$, $\beta \in \mathbb{R}$, $p=p(n)$, $p,n \in \mathbb{N}$, $p \leqslant n$, and $T(n)\asymp p(n)$. Then the order estimate*

$$\mathscr{E}\left(C_{\beta,\infty}^{\psi};U_{n,p}^{\psi}\right)\asymp E_n\left(C_{\beta,\infty}^{\psi}\right)\asymp\psi(n) \tag{26}$$

holds, implying that the sum $U_{n,p}^{\psi}(f;x)$ provides the order of the best uniform approximation of the class $C_{\beta,\infty}^{\psi}$.

If $\lim_{\substack{n\to\infty\\ p\to\infty}}T(n)/p=\xi$, where $\xi=0$ or $\xi=\infty$, then equality (22) gives the solution of Kolmogorov-Nikol'skii problem for $U_{n,p}^{\psi}(f;x)$.

From (22) and (23) we can write the following.

Corollary 3. *Let the conditions of Theorem 1 be satisfied. Then, as $n \to \infty$*

$$\mathscr{E}\left(C_{\beta,\infty}^{\psi};U_{n,p}^{\psi}\right)=\psi(n)\left(\frac{4}{\pi^2}A_{n,p}^{\psi}+O(1)\right), \tag{27}$$

where

$$A_{n,p}^{\psi}=\begin{cases}\ln p, & \text{if } T(n) \text{ is bounded},\\ \ln^{+}\dfrac{p}{T(n)}, & \text{if } T(n)\ll p,\\ \ln^{+}\dfrac{T(n)}{p}, & \text{if } T(n)\gg p,\end{cases} \tag{28}$$

$T(n)=\eta(\psi;n)-n$, $\eta(\psi;n)=\psi^{-1}((1/2)\psi(n))$, $\ln^{+}t=\max\{\ln t,0\}$, $O(1)$ is a quantity uniformly bounded in β, n, and p, and the notation $T(n)\ll p(T(n)\gg p)$ means that there exists a constant $K>0$ such that $T(n)<Kp(T(n)>Kp)$.

Taking into account that, for $\psi_{\alpha,r}(t)=e^{-\alpha t^r}$, $\alpha>0$, $r>0$, the equality

$$T\left(\psi_{\alpha,r};t\right)=t^{1-r}\left(\frac{\ln 2}{\alpha r}+O(1)\right) \tag{29}$$

holds, from Corollary 3, we have the following.

Corollary 4. *Let $\alpha>0$, $r>0$, $\beta \in \mathbb{R}$, $n,p \in \mathbb{N}$, and $p \leqslant n$. Then, as $n \to \infty$*

$$\mathscr{E}\left(C_{\beta,\infty}^{\alpha,r};U_{n,p}^{\psi_{\alpha,r}}\right)=e^{-\alpha n^r}\left(\frac{4}{\pi^2}A_{n,p}^{\alpha,r}+O(1)\right), \tag{30}$$

where

$$A_{n,p}^{\alpha,r}=\begin{cases}\ln p, & \text{if } r\geqslant 1,\\ \ln^{+}\dfrac{p}{n^{1-r}}, & \text{if } r<1,\ n^{1-r}\ll p,\\ \ln^{+}\dfrac{n^{1-r}}{p}, & \text{if } r<1,\ n^{1-r}\gg p,\end{cases} \tag{31}$$

$\ln^{+}n=\max\{\ln n,0\}$, and $O(1)$ is a quantity uniformly bounded in β, n, and p.

Note that if $p\to\infty$ and $n-p\to\infty$, then the sum $U_{n,p}^{\psi_{\alpha,1}}(f;x)$ provides higher order of approximation on the class $C_{\beta,\infty}^{\alpha,1}$ than de la Vallée Poussin sum $V_{n,p}(f;x)$ does. Indeed, it is proved in [14, page 981] (see also [15, page 130] and [16, page 10]) that

$$\begin{aligned}\mathscr{E}\left(C_{\beta,\infty}^{\alpha,1};V_{n,p}\right)=&\frac{e^{-\alpha(n-p+1)}}{p}\\ &\times\left(\frac{4}{\pi\left(1-e^{-2\alpha}\right)}+O(1)\right.\\ &\left.\times\left(\frac{e^{-\alpha}}{\left(1-e^{-\alpha}\right)^3(n-p+1)}+\frac{e^{-\alpha p}}{1-e^{-\alpha}}\right)\right),\end{aligned} \tag{32}$$

where $O(1)$ is a quantity uniformly bounded in n, p, α, and β. Comparing (30) with (32), we find that if $p=p(n)$ satisfies the condition

$$p\longrightarrow\infty,\qquad n-p\longrightarrow\infty, \tag{33}$$

then

$$\lim_{\substack{n-p\to\infty\\ p\to\infty}} \frac{\mathscr{E}\left(C^{\alpha,1}_{\beta,\infty}; U^{\psi_{\alpha,1}}_{n,p}\right)}{\mathscr{E}\left(C^{\alpha,1}_{\beta,\infty}; V_{n,p}\right)} = 0. \tag{34}$$

As noted earlier, the sum $U^{\psi}_{n,p}(f;x)$ for $p = 1$ coincides with Fourier sum $S_{n-1}(f;x)$ of order $n-1$. For such a value of p, from Theorem 1 we get the following.

Corollary 5. *Let $\psi \in F$, $\beta \in \mathbb{R}$, and $n \in \mathbb{N}$. Then, as $n \to \infty$*

$$\mathscr{E}\left(C^{\psi}_{\beta,\infty}; S_{n-1}\right) = \psi(n)\left(\frac{4}{\pi^2}\ln^+ T(n) + O(1)\right), \tag{35}$$

where $\ln^+ n = \max\{\ln n, 0\}$, $T(n) = \eta(\psi;n) - n$, $\eta(\psi;n) = \psi^{-1}((1/2)\psi(n))$, and $O(1)$ is a quantity uniformly bounded in β and n.

Equality (35) was obtained by Stepanets (see, for example, [3, page 257]). It gives the solution of Kolmogorov-Nikol'skii problem for Fourier sum when

$$T(n) \longrightarrow \infty, \qquad n \longrightarrow \infty. \tag{36}$$

For $\psi(t) = t^{-r}$, $\beta = r$ (in this case $C^{\psi}_{\beta,\infty} = W^r_r$ and $T(n) = (2^{1/r} - 1)n$), equality (35) takes the from

$$\mathscr{E}\left(W^r; S_{n-1}\right) = n^{-r}\left(\frac{4}{\pi^2}\ln n + O(1)\right). \tag{37}$$

Asymptotic equality (37) was established by Kolmogorov [17] (for $r \in \mathbb{N}$) and Pinkewitch [18] (for $r > 0$).

Corollary 2 and equality (35) show that in the case when $T(n) \to \infty$ and $p = p(n)$ is chosen such that $p(n) \asymp T(n)$, the sum $U^{\psi}_{n,p}(f;x)$ provides higher order of approximation on $C^{\psi}_{\beta,\infty}$ in comparison with Fourier sum.

Setting $p = n$ in Corollary 3 and taking into account that $U^{\psi}_{n,n}(f;x)$ is a generalized Zygmund sum $Z^{\varphi}_n(f;x)$ ($\varphi(k) = k/\psi(k)$, $k \in \mathbb{N}$), we obtain the following.

Corollary 6. *Let $\in F$, $\varphi(k) = k/\psi(k)$, $\beta \in \mathbb{R}$, and $n \in \mathbb{N}$. Then, as $n \to \infty$*

$$\mathscr{E}\left(C^{\psi}_{\beta,\infty}; Z^{\varphi}_n\right) = \psi(n)\left(\frac{4}{\pi^2}A^{\psi}_n + O(1)\right), \tag{38}$$

where

$$A^{\psi}_n = \begin{cases} \ln n, & \text{if } T(n) \text{ is bounded},\\ \ln^+\dfrac{n}{T(n)}, & \text{if } T(n) \ll n,\\ \ln^+\dfrac{T(n)}{n}, & \text{if } T(n) \gg n, \end{cases} \tag{39}$$

$T(n) = \eta(\psi;n) - n$, $\eta(\psi;n) = \psi^{-1}((1/2)\psi(n))$, $\ln^+ n = \max\{\ln n, 0\}$, $O(1)$ is a quantity uniformly bounded in β, n, and the notation $T(n) \ll n$ ($T(n) \gg n$) means that there exists a constant $K > 0$ such that $T(n) < Kn$ ($T(n) > Kn$).

Since by definition (8)

$$T(n) \asymp n \quad \forall \psi \in \mathfrak{M}_C, \tag{40}$$

from (25), (38), and inclusion $\mathfrak{M}_C \subset F$, we have

$$\mathscr{E}\left(C^{\psi}_{\beta,\infty}; Z^{\varphi}_n\right) \asymp E_n\left(C^{\psi}_{\beta,\infty}\right) \asymp \psi(n), \quad \psi \in \mathfrak{M}_C. \tag{41}$$

This estimate follows also from Corollary 2 for $p = n$.

Taking $\psi(t) = t^{-r}$, $r > 0$, in (41) (in this case $C^{\psi}_{\beta,\infty} = W^r_{\beta,\infty}$ and $Z^{\varphi}_n(f;x) = Z^s_n(f;x)$, $s = r+1$), we get

$$\mathscr{E}\left(W^r_{\beta,\infty}; Z^s_n\right) \asymp n^{-r}. \tag{42}$$

Note that estimate (42) can be obtained from more precise results of Telyakovskii [19] (see Theorem 7).

3. Proof of Theorem 1

Consider the quantity

$$\rho_{n,p}(f;x) := f(x) - U^{\psi}_{n,p}(f;x), \quad x \in \mathbb{R}, \tag{43}$$

where $f \in C^{\psi}_{\beta,\infty}$ and ψ belongs to the set

$$\mathfrak{M}' = \left\{\psi \in \mathfrak{M} : \int_1^{\infty} \frac{\psi(t)}{t}dt < \infty\right\} \tag{44}$$

(as shown in [3, page 155], $F \subset \mathfrak{M}'$). By Theorem 4.1 [2, page 71], the following equality holds at any point x:

$$\rho_{n,p}(f;x) = \int_{-\infty}^{\infty} f^{\psi}_{\beta}\left(x + \frac{t}{n}\right)\widehat{\tau}_{n,p}(t)\,dt, \quad n \in \mathbb{N}, \tag{45}$$

where

$$\widehat{\tau}_{n,p}(t) := \frac{1}{\pi}\int_0^{\infty} \tau_{n,p}(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du, \tag{46}$$

$$\tau_{n,p}(u) = \begin{cases} 0, & 0 \leqslant u \leqslant \dfrac{n-p}{n},\\ \psi(n)\dfrac{nu - n + p}{p}, & \dfrac{n-p}{n} \leqslant u \leqslant 1,\\ \psi(nu), & u \geqslant 1. \end{cases} \tag{47}$$

Let us simplify the right-hand side of (45) to obtain the principal term of the quantity $\rho_{n,p}(f;x)$. To do this, set

$$\widehat{\tau}_{n,p+}(t) := \frac{1}{\pi}\int_0^{\infty} \tau_{n,p}(u)\cos ut\,du, \tag{48}$$

$$\widehat{\tau}_{n,p-}(t) := \frac{1}{\pi}\int_0^{\infty} \tau_{n,p}(u)\sin ut\,du. \tag{49}$$

Taking (48) and (49) into account, equality (45) can be written in the following form:

$$\begin{aligned}\rho_{n,p}(f;x) &= \cos\frac{\beta\pi}{2}\int_{-\infty}^{\infty} f^{\psi}_{\beta}\left(x + \frac{t}{n}\right)\widehat{\tau}_{n,p+}(t)\,dt\\ &\quad - \sin\frac{\beta\pi}{2}\int_{-\infty}^{\infty} f^{\psi}_{\beta}\left(x + \frac{t}{n}\right)\widehat{\tau}_{n,p-}(t)\,dt\\ &= \cos\frac{\beta\pi}{2}\rho_{n,p+}(f;x) - \sin\frac{\beta\pi}{2}\rho_{n,p-}(f;x).\end{aligned} \tag{50}$$

Since the class $C^{\psi}_{\beta,\infty}$ is invariant under shifts of the argument (if $f \in C^{\psi}_{\beta,\infty}$, then the function $f_1(\cdot) = f(\cdot + h)$, $h \in \mathbb{R}$, also belongs to $C^{\psi}_{\beta,\infty}$), for quantity (20) we have

$$\mathscr{E}\left(C^{\psi}_{\beta,\infty}; U^{\psi}_{n,p}\right) = \sup_{f \in C^{\psi}_{\beta,\infty}} \left|\rho_{n,p}(f;0)\right|. \tag{51}$$

It follows from (51) that we may restrict our attention to $\rho_{n,p}(f;x)$ at $x = 0$.

We prove the following result.

Lemma 7. *Assume that $\psi \in \mathfrak{M}'$, $\beta \in \mathbb{R}$, $n, p \in \mathbb{N}$, $p \leqslant n$, and $\alpha(n)$ is an arbitrary sequence of real numbers satisfying the condition*

$$\alpha(n) \geqslant \frac{K}{n}, \quad n \in \mathbb{N}, \tag{52}$$

where K is some positive constant. Then for any function $f \in C^{\psi}_{\beta,\infty}$, as $n \to \infty$, one has

$$\rho_{n,p}(f;0) = (-1)^s \frac{\psi(n)}{\pi} \int_{\mathcal{I}_{\alpha_n}} f^{\psi}_{\beta}\left(\frac{t}{n}\right) \frac{\sin(t + (\beta\pi/2))}{t} dt + O(1) r_n, \tag{53}$$

where

$$\mathcal{I}_{\alpha_n} = \left(-\frac{n}{p}, -\alpha(n) n\right) \bigcup \left(\alpha(n) n, \frac{n}{p}\right), \quad \text{if } \alpha(n) \leqslant \frac{1}{p},$$

$$\mathcal{I}_{\alpha_n} = \left(-\alpha(n) n, -\frac{n}{p}\right) \bigcup \left(\frac{n}{p}, \alpha(n) n\right), \quad \text{if } \frac{1}{p} < \alpha(n) \leqslant 1,$$

$$\mathcal{I}_{\alpha_n} = \left(-n, -\frac{n}{p}\right) \bigcup \left(\frac{n}{p}, n\right), \quad \text{if } \alpha(n) > 1,$$

$$s = s(\alpha, p) = \begin{cases} 1, & \alpha(n) \leqslant \dfrac{1}{p}, \\ 0, & \alpha(n) > \dfrac{1}{p}, \end{cases}$$

$$r_n = \psi(n) + \int_{1/\alpha(n)}^{\infty} \frac{\psi(t+n)}{t} dt + \int_{\alpha(n)}^{\infty} \frac{\psi(n) - \psi(n + 1/t)}{t} dt, \tag{54}$$

and $O(1)$ is a quantity uniformly bounded in β, n, and p.

Proof. By (50) Lemma 7 will be established if we show that

$$\rho_{n,p+}(f;0) = (-1)^s \frac{\psi(n)}{\pi} \int_{\mathcal{I}_{\alpha_n}} f^{\psi}_{\beta}\left(\frac{t}{n}\right) \frac{\sin t}{t} dt + O(1) r_n, \tag{55}$$

$$\rho_{n,p-}(f;0) = (-1)^{s+1} \frac{\psi(n)}{\pi} \int_{\mathcal{I}_{\alpha_n}} f^{\psi}_{\beta}\left(\frac{t}{n}\right) \frac{\cos t}{t} dt + O(1) r_n. \tag{56}$$

Since estimate (56) can be obtained in the same way as (55) we only prove estimate (55).

Assume that $\alpha(n) \leqslant 1/p$. Represent the quantity $\rho_{n,p+}(f;0)$ in the form

$$\begin{aligned} \rho_{n,p+}(f;0) = {} & \int_{|t| \leqslant \alpha(n)n} f^{\psi}_{\beta}\left(\frac{t}{n}\right) \widehat{\tau}_{n,p+}(t)\, dt \\ & + \int_{\alpha(n)n \leqslant |t| \leqslant n/p} f^{\psi}_{\beta}\left(\frac{t}{n}\right) \widehat{\tau}_{n,p+}(t)\, dt \\ & + \int_{|t| \geqslant n/p} f^{\psi}_{\beta}\left(\frac{t}{n}\right) \widehat{\tau}_{n,p+}(t)\, dt. \end{aligned} \tag{57}$$

Using for $|t| \leqslant \alpha(n)n$ the expression

$$\begin{aligned} \widehat{\tau}_{n,p+}(t) = {} & \frac{\psi(n)}{\pi} \int_{(n-p)/n}^{1} \frac{nu - n + p}{p} \cos ut\, du \\ & + \frac{1}{\pi} \int_{1}^{\infty} \psi(nu) \cos ut\, du, \end{aligned} \tag{58}$$

obtained from (47) and (48), we have

$$\begin{aligned} & \int_{|t| \leqslant \alpha(n)n} f^{\psi}_{\beta}\left(\frac{t}{n}\right) \widehat{\tau}_{n,p+}(t)\, dt \\ & = \frac{\psi(n)}{\pi} \int_{|t| \leqslant \alpha(n)n} f^{\psi}_{\beta}\left(\frac{t}{n}\right) \int_{(n-p)/n}^{1} \frac{nu - n + p}{p} \cos ut\, du\, dt \\ & \quad + \frac{1}{\pi} \int_{|t| \leqslant \alpha(n)n} f^{\psi}_{\beta}\left(\frac{t}{n}\right) \int_{1}^{\infty} \psi(nu) \cos ut\, du\, dt. \end{aligned} \tag{59}$$

Taking into account the inclusion $f^{\psi}_{\beta} \in S_{\infty}$, the estimate

$$\begin{aligned} & \left| \int_{(n-p)/n}^{1} \frac{nu - n + p}{p} \cos ut\, du \right| \\ & \leqslant \int_{(n-p)/n}^{1} \frac{nu - n + p}{p} du = \frac{p}{2n}, \end{aligned} \tag{60}$$

and condition $\alpha(n) \leqslant 1/p$, it follows from (59) that

$$\begin{aligned} & \int_{|t| \leqslant \alpha(n)n} f^{\psi}_{\beta}\left(\frac{t}{n}\right) \widehat{\tau}_{n,p+}(t)\, dt \\ & = \frac{1}{\pi} \int_{|t| \leqslant \alpha(n)n} f^{\psi}_{\beta}\left(\frac{t}{n}\right) \int_{1}^{\infty} \psi(nu) \cos ut\, du\, dt \\ & \quad + O(1) \psi(n). \end{aligned} \tag{61}$$

As proved in [3, Chapter 5, Section 11] (see (11.9) and (11.17)), if $f^{\psi}_{\beta} \in S_{\infty}$, then

$$\begin{aligned} & \int_{|t| \leqslant \alpha(n)n} f^{\psi}_{\beta}\left(\frac{t}{n}\right) \int_{1}^{\infty} \psi(nu) \cos ut\, du\, dt \\ & = O(1) \left(\psi(n) + \int_{1/\alpha(n)}^{\infty} \frac{\psi(t+n)}{t} dt \right). \end{aligned} \tag{62}$$

Therefore, from (61) we obtain

$$\int_{|t|\leqslant\alpha(n)n} f_\beta^\psi\left(\frac{t}{n}\right)\widehat{\tau}_{n,p+}(t)\,dt = O(1)\left(\psi(n)+\int_{1/\alpha(n)}^{\infty}\frac{\psi(t+n)}{t}dt\right). \tag{63}$$

Consider the second integral on the right-hand side of (57). To find its principal term represent the quantity $\widehat{\tau}_{n,p+}(t)$ in the form

$$\widehat{\tau}_{n,p+}(t) = \frac{\psi(n)}{\pi}\int_{(n-p)/n}^{1}\frac{nu-n+p}{p}\cos ut\,du - \frac{\psi(n)}{\pi t}\sin t - \frac{n}{\pi t}\int_1^\infty \psi'(nu)\sin ut\,du, \tag{64}$$

obtained from (58) by integrating the second integral by parts:

$$\int_1^\infty \psi(nu)\cos ut\,du = -\frac{\psi(n)}{t}\sin t - \frac{n}{t}\int_1^\infty \psi'(nu)\sin ut\,du. \tag{65}$$

As is easy to see, estimate (60) implies that

$$\int_{\alpha(n)n\leqslant|t|\leqslant n/p} f_\beta^\psi\left(\frac{t}{n}\right)\int_{(n-p)/n}^{1}\frac{nu-n+p}{p}\cos ut\,du\,dt = O(1). \tag{66}$$

Then by (64)

$$\begin{aligned}\int_{\alpha(n)n\leqslant|t|\leqslant n/p} f_\beta^\psi\left(\frac{t}{n}\right)\widehat{\tau}_{n,p+}(t)\,dt &= -\frac{\psi(n)}{\pi}\int_{\alpha(n)n\leqslant|t|\leqslant n/p} f_\beta^\psi\left(\frac{t}{n}\right)\frac{\sin t}{t}dt \\ &\quad - \frac{n}{\pi}\int_{\alpha(n)n\leqslant|t|\leqslant n/p} f_\beta^\psi\left(\frac{t}{n}\right)\frac{1}{t}\int_1^\infty \psi'(nu)\sin ut\,du\,dt \\ &\quad + O(1)\,\psi(n).\end{aligned} \tag{67}$$

Now we consider the third integral in (57). Integration by parts in the first integral of (64) yields

$$\widehat{\tau}_{n,p+}(t) = \frac{\psi(n)}{\pi t^2}\frac{n}{p}\left(\cos t - \cos\frac{n-p}{n}t\right) - \frac{n}{\pi t}\int_1^\infty \psi'(nu)\sin ut\,du. \tag{68}$$

Since

$$\frac{n}{p}\int_{|t|\geqslant n/p} f_\beta^\psi\left(\frac{t}{n}\right)\frac{1}{t^2}\left(\cos t - \cos\frac{n-p}{n}t\right)dt = O(1), \tag{69}$$

from (68) and (69) we have

$$\begin{aligned}\int_{|t|\geqslant n/p} f_\beta^\psi\left(\frac{t}{n}\right)\widehat{\tau}_{n,p+}(t)\,dt &= -\frac{n}{\pi}\int_{|t|\geqslant n/p} f_\beta^\psi\left(\frac{t}{n}\right)\frac{1}{t}\int_1^\infty \psi'(nu)\sin ut\,du\,dt \\ &\quad + O(1)\,\psi(n).\end{aligned} \tag{70}$$

Combining (63), (67), (70), and the estimate

$$n\int_{|t|\geqslant\alpha(n)n} f_\beta^\psi\left(\frac{t}{n}\right)\frac{1}{t}\int_1^\infty \psi'(nu)\sin ut\,du\,dt = O(1)\int_{\alpha(n)}^{\infty}\frac{\psi(n)-\psi(n+1/t)}{t}dt, \tag{71}$$

proved in [3] (see (11.31)), from (57) we get

$$\begin{aligned}\rho_{n,p+}(f;0) &= -\frac{\psi(n)}{\pi}\int_{\alpha(n)n\leqslant|t|\leqslant n/p} f_\beta^\psi\left(\frac{t}{n}\right)\frac{\sin t}{t}dt \\ &\quad + O(1)\left(\psi(n)+\int_{1/\alpha(n)}^{\infty}\frac{\psi(t+n)}{t}dt \right. \\ &\quad \left. + \int_{\alpha(n)}^{\infty}\frac{\psi(n)-\psi(n+1/t)}{t}dt\right).\end{aligned} \tag{72}$$

Assume that $\alpha(n) > 1/p$. The following equality holds:

$$\begin{aligned}\rho_{n,p+}(f;0) &= \int_{|t|\leqslant n/p} f_\beta^\psi\left(\frac{t}{n}\right)\widehat{\tau}_{n,p+}(t)\,dt \\ &\quad + \int_{n/p\leqslant|t|\leqslant\alpha(n)n} f_\beta^\psi\left(\frac{t}{n}\right)\widehat{\tau}_{n,p+}(t)\,dt \\ &\quad + \int_{|t|\geqslant\alpha(n)n} f_\beta^\psi\left(\frac{t}{n}\right)\widehat{\tau}_{n,p+}(t)\,dt.\end{aligned} \tag{73}$$

By using representation (58) and estimate (60) it is not hard to establish that

$$\int_{|t|\leqslant n/p} f_\beta^\psi\left(\frac{t}{n}\right)\widehat{\tau}_{n,p+}(t)\,dt = \frac{1}{\pi}\int_{|t|\leqslant n/p} f_\beta^\psi\left(\frac{t}{n}\right)\int_1^\infty \psi(nu)\cos ut\,du\,dt + O(1)\,\psi(n). \tag{74}$$

In view of (65) and (68), we have

$$\widehat{\tau}_{n,p+}(t) = \frac{\psi(n)}{\pi}\frac{\sin t}{t} + \frac{1}{\pi}\int_1^\infty \psi(nu)\cos ut\,du + \frac{\psi(n)}{\pi t^2}\frac{n}{p}\left(\cos t - \cos\frac{n-p}{n}t\right). \tag{75}$$

This and

$$\frac{n}{p}\int_{n/p\leqslant|t|\leqslant\alpha(n)n} f_\beta^\psi\left(\frac{t}{n}\right)\frac{1}{t^2}\left(\cos t - \cos\frac{n-p}{n}t\right)dt = O(1)\frac{n}{p}\int_{n/p}^{\alpha(n)n}\frac{dt}{t^2} = O(1) \tag{76}$$

give us the estimate for the second integral in (73):

$$\begin{aligned}&\int_{n/p\leqslant|t|\leqslant\alpha(n)n} f_\beta^\psi\left(\frac{t}{n}\right)\widehat{\tau}_{n,p+}(t)\,dt\\&\quad=\frac{\psi(n)}{\pi}\int_{n/p\leqslant|t|\leqslant\alpha(n)n} f_\beta^\psi\left(\frac{t}{n}\right)\frac{\sin t}{t}dt\\&\qquad+\frac{1}{\pi}\int_{n/p\leqslant|t|\leqslant\alpha(n)n} f_\beta^\psi\left(\frac{t}{n}\right)\int_1^\infty\psi(nu)\cos ut\,du\,dt\\&\qquad+O(1)\,\psi(n).\end{aligned}\tag{77}$$

To find the estimate of the third integral on the right-hand side of (73) we use representation (68). We have

$$\begin{aligned}&\int_{|t|\geqslant\alpha(n)n} f_\beta^\psi\left(\frac{t}{n}\right)\widehat{\tau}_{n,p+}(t)\,dt\\&\quad=\frac{\psi(n)}{\pi}\frac{n}{p}\int_{|t|\geqslant\alpha(n)n} f_\beta^\psi\left(\frac{t}{n}\right)\frac{1}{t^2}\left(\cos t-\cos\frac{n-p}{n}t\right)dt\\&\qquad-\frac{n}{\pi}\int_{|t|\geqslant\alpha(n)n} f_\beta^\psi\left(\frac{t}{n}\right)\frac{1}{t}\int_1^\infty\psi'(nu)\sin ut\,du\,dt.\end{aligned}\tag{78}$$

Since

$$\begin{aligned}&\frac{n}{p}\int_{|t|\geqslant\alpha(n)n} f_\beta^\psi\left(\frac{t}{n}\right)\frac{1}{t^2}\left(\cos t-\cos\frac{n-p}{n}t\right)dt\\&\quad=O(1)\,\frac{n}{p}\int_{\alpha(n)n}^\infty\frac{dt}{t^2}=O(1),\end{aligned}\tag{79}$$

we hence obtain from (78) by using (71) that

$$\begin{aligned}&\int_{|t|\geqslant\alpha(n)n} f_\beta^\psi\left(\frac{t}{n}\right)\widehat{\tau}_{n,p+}(t)\,dt\\&\quad=O(1)\left(\psi(n)+\int_{\alpha(n)}^\infty\frac{\psi(n)-\psi(n+1/t)}{t}dt\right).\end{aligned}\tag{80}$$

Combining (73), (74), (77), and (80) and taking into account (62), we get

$$\begin{aligned}\rho_{n,p+}(f;0)=&\frac{\psi(n)}{\pi}\int_{n/p\leqslant|t|\leqslant\alpha(n)n} f_\beta^\psi\left(\frac{t}{n}\right)\frac{\sin t}{t}dt\\&+O(1)\left(\psi(n)+\int_{1/\alpha(n)}^\infty\frac{\psi(t+n)}{t}dt\right.\\&\left.+\int_{\alpha(n)}^\infty\frac{\psi(n)-\psi(n+1/t)}{t}dt\right).\end{aligned}\tag{81}$$

For the proof of lemma in the case when $\alpha(n)>1$, it is enough to apply the estimate [1, page 119] (see Lemma 5)

$$\begin{aligned}&\left|\int_{|t|\geqslant\alpha^*(n)}\varphi\left(\frac{t}{n}\right)\frac{\sin(t+(\gamma\pi/2))}{t}dt\right|\\&\quad=O(1),\quad\varphi\in S_\infty,\ n\longrightarrow\infty,\ \gamma\in\mathbb{R},\end{aligned}\tag{82}$$

where $\{\alpha^*(n)\}=\{\alpha(n):\alpha(n)\geqslant n\pi\}$, to the first integral of (81). Indeed, using (82) we can write

$$\begin{aligned}&\int_{n/p\leqslant|t|\leqslant\alpha(n)n} f_\beta^\psi\left(\frac{t}{n}\right)\frac{\sin t}{t}dt\\&\quad=\int_{n/p\leqslant|t|\leqslant n} f_\beta^\psi\left(\frac{t}{n}\right)\frac{\sin t}{t}dt+O(1).\end{aligned}\tag{83}$$

Thus, for $\alpha(n)>1$ estimate (55) follows from (81) and (83). Lemma 7 is proved. □

Setting in Lemma 7

$$\alpha(n)=\alpha(\psi;n)=\frac{1}{\eta(\psi;n)-n},\tag{84}$$

where $\eta(\psi;n)$ is defined by (6) and $\psi\in F$ ($F\subset\mathfrak{M}'$), and taking into account the following estimates from [3, pages 155, 156]:

$$\begin{aligned}&\int_{\eta(\psi;n)}^\infty\frac{\psi(t)}{t-n}dt=O(1)\,\psi(n),\quad\forall\psi\in F,\\&\int_n^{\eta(\psi;n)}\frac{\psi(n)-\psi(t)}{t-n}dt=O(1)\,\psi(n),\quad\forall\psi\in F,\end{aligned}\tag{85}$$

$$\mu(n)=\mu(\psi;n)=\frac{n}{\eta(\psi;n)-n}\geqslant K>0,\quad\forall\psi\in F,\tag{86}$$

we obtain the following statement.

Corollary 8. *Assume that $\psi\in F$, $\beta\in\mathbb{R}$, $n,p\in\mathbb{N}$, and $p\leqslant n$. Then for any $f\in C_{\beta,\infty}^\psi$, as $n\to\infty$, the following equality holds:*

$$\begin{aligned}\rho_{n,p}(f;0)=&(-1)^s\frac{\psi(n)}{\pi}\\&\times\int_{\mathcal{I}_n} f_\beta^\psi\left(\frac{t}{n}\right)\frac{\sin(t+(\beta\pi/2))}{t}dt+O(1)\,\psi(n),\end{aligned}\tag{87}$$

where

$$\begin{aligned}&\mathcal{I}_n=\left(-\frac{n}{p},-\mu(n)\right)\bigcup\left(\mu(n),\frac{n}{p}\right),\quad\text{if }\mu(n)\leqslant\frac{n}{p},\\&\mathcal{I}_n=\left(-\mu(n),-\frac{n}{p}\right)\bigcup\left(\frac{n}{p},\mu(n)\right),\quad\text{if }\frac{n}{p}<\mu(n)\leqslant n,\\&\mathcal{I}_n=\left(-n,-\frac{n}{p}\right)\bigcup\left(\frac{n}{p},n\right),\quad\text{if }\mu(n)>n,\\&\mu(n)=\mu(\psi;n)=\frac{n}{\eta(\psi;n)-n},\\&s=s(\psi,n,p)=\begin{cases}1, & \mu(n)\leqslant\dfrac{n}{p},\\ 0, & \mu(n)>\dfrac{n}{p},\end{cases}\end{aligned}\tag{88}$$

and $O(1)$ is a quantity uniformly bounded in β, n, and p.

We now set

$$t_k = (2k+1-\beta)\frac{\pi}{2}, \quad k \in \mathbb{Z},$$
$$x_k = t_k + \frac{\pi}{2}, \quad k \in \mathbb{Z}, \tag{89}$$

$$l_n(t) = \begin{cases} \dfrac{1}{x_k}, & t \in [t_k, t_{k+1}],\ k = k_0, k_0+1, \ldots, k_1-1, \\ & \qquad k_2, k_2+1, \ldots, k_3-1, \\ 0, & t \in (-\infty, t_{k_0}) \cup (t_{k_3}, \infty), \end{cases} \tag{90}$$

where k_0, k_1, k_2, and k_3 are selected in such a way that

$$\begin{aligned} t_{k_0-1} &< -\frac{n}{p} \leqslant t_{k_0}, \\ t_{k_1} &< -\mu(n) \leqslant t_{k_1+1}, \\ t_{k_2-1} &< \mu(n) \leqslant t_{k_2}, \\ t_{k_3} &< \frac{n}{p} \leqslant t_{k_3+1}, \end{aligned} \tag{91}$$

if $\mu(n) \leqslant n/p$,

$$\begin{aligned} t_{k_0-1} &< -\mu(n) \leqslant t_{k_0}, \\ t_{k_1} &< -\frac{n}{p} \leqslant t_{k_1+1}, \\ t_{k_2-1} &< \frac{n}{p} \leqslant t_{k_2}, \\ t_{k_3} &< \mu(n) \leqslant t_{k_3+1}, \end{aligned} \tag{92}$$

if $n/p < \mu(n) \leqslant n$, and

$$\begin{aligned} t_{k_0-1} &< -n \leqslant t_{k_0}, \\ t_{k_1} &< -\frac{n}{p} \leqslant t_{k_1+1}, \\ t_{k_2-1} &< \frac{n}{p} \leqslant t_{k_2}, \\ t_{k_3} &< n \leqslant t_{k_3+1}, \end{aligned} \tag{93}$$

if $\mu(n) > n$.

In the above notation the integral on the right-hand side of (87) can be represented in the form

$$\begin{aligned} &\int_{\mathcal{I}_n} f_\beta^\psi\left(\frac{t}{n}\right) \frac{\sin(t + (\beta\pi/2))}{t} dt \\ &\quad = \int_{(t_{k_0}, t_{k_1}) \cup (t_{k_2}, t_{k_3})} f_\beta^\psi\left(\frac{t}{n}\right) l_n(t) \sin\left(t + \frac{\beta\pi}{2}\right) dt \\ &\qquad + R_{n,p}^{(1)} + R_{n,p}^{(2)}, \end{aligned} \tag{94}$$

where

$$R_{n,p}^{(1)} = \int_{(t_{k_0}, t_{k_1}) \cup (t_{k_2}, t_{k_3})} f_\beta^\psi\left(\frac{t}{n}\right)\left(\frac{1}{t} - l_n(t)\right) \sin\left(t + \frac{\beta\pi}{2}\right) dt, \tag{95}$$

$$\begin{aligned} R_{n,p}^{(2)} = &\left(\int_{-a}^{t_{k_0}} + \int_{t_{k_1}}^{-b} + \int_{b}^{t_{k_2}} + \int_{t_{k_3}}^{a}\right) f_\beta^\psi\left(\frac{t}{n}\right) \\ &\times \frac{\sin(t + (\beta\pi/2))}{t} dt, \end{aligned} \tag{96}$$

$$a = \begin{cases} \dfrac{n}{p}, & \mu(n) \leqslant \dfrac{n}{p}, \\ \mu(n), & \dfrac{n}{p} < \mu(n) \leqslant n, \\ n, & \mu(n) > n, \end{cases} \tag{97}$$

$$b = \begin{cases} \mu(n), & \mu(n) \leqslant \dfrac{n}{p}, \\ \dfrac{n}{p}, & \mu(n) > \dfrac{n}{p}. \end{cases} \tag{98}$$

We claim that

$$R_{n,p}^{(1)} = O(1), \tag{99}$$

$$R_{n,p}^{(2)} = O(1). \tag{100}$$

Making elementary transformations in (95) and taking into account that $|f_\beta^\psi(\cdot)| \leqslant 1$, we have

$$\begin{aligned} R_{n,p}^{(1)} &= O(1) \\ &\quad \times \left(\sum_{k=k_0}^{k_1-1} \int_{t_k}^{t_{k+1}} \left|\frac{1}{t} - \frac{1}{x_k}\right| dt + \sum_{k=k_2}^{k_3-1} \int_{t_k}^{t_{k+1}} \left|\frac{1}{t} - \frac{1}{x_k}\right| dt\right). \end{aligned} \tag{101}$$

Since

$$\int_{t_k}^{x_k} \left|\frac{1}{t} - \frac{1}{x_k}\right| dt > \int_{x_k}^{t_{k+1}} \left|\frac{1}{t} - \frac{1}{x_k}\right| dt, \quad k \in \mathbb{Z}, \tag{102}$$

we can write the estimate

$$\begin{aligned} &\int_{t_k}^{t_{k+1}} \left|\frac{1}{t} - \frac{1}{x_k}\right| dt \\ &\quad < 2\int_{t_k}^{x_k} \left|\frac{1}{t} - \frac{1}{x_k}\right| dt \\ &\quad = 2\int_{t_k}^{x_k} \frac{x_k - t}{|t x_k|} dt \leqslant \pi \int_{t_k}^{x_k} \frac{dt}{t^2} < \pi \int_{t_k}^{t_{k+1}} \frac{dt}{t^2}. \end{aligned} \tag{103}$$

Combining (101) and (103), we obtain

$$\begin{aligned} R_{n,p}^{(1)} &= O(1)\left(\int_{t_{k_0}}^{t_{k_1}} \frac{dt}{t^2} + \int_{t_{k_2}}^{t_{k_3}} \frac{dt}{t^2}\right) \\ &= O(1)\left(\frac{1}{|t_{k_1}|} + \frac{1}{t_{k_2}}\right) = O(1) \max\left\{\frac{1}{\mu(n)}, 1\right\}. \end{aligned} \tag{104}$$

By (86) $\mu(n)$ is bounded below; therefore, (99) follows from (104).

Since

$$\begin{aligned} t_{k_0} + a &< t_{k_0} - t_{k_0-1} = \pi, \\ -b - t_{k_1} &\leqslant t_{k_1+1} - t_{k_1} = \pi, \\ t_{k_2} - b &< t_{k_2} - t_{k_2-1} = \pi, \\ a - t_{k_3} &\leqslant t_{k_3+1} - t_{k_3} = \pi, \end{aligned} \tag{105}$$

it follows from (96) that

$$\begin{aligned} R_{n,p}^{(2)} &= O(1)\left(\frac{t_{k_0}+a}{|t_{k_0}|} + \frac{-b-t_{k_1}}{b} + \frac{t_{k_2}-b}{b} + \frac{a-t_{k_3}}{t_{k_3}}\right) \\ &= O(1)\left(\frac{1}{|t_{k_0}|} + \frac{1}{b} + \frac{1}{t_{k_3}}\right) \\ &= O(1)\max\left\{\frac{1}{\mu(n)}, 1\right\} = O(1). \end{aligned} \tag{106}$$

Combining (87), (94), (99), and (100), we obtain

$$\begin{aligned} &\rho_{n,p}(f;0) \\ &= (-1)^s \frac{\psi(n)}{\pi} \int_{(t_{k_0},t_{k_1})\cup(t_{k_2},t_{k_3})} f_\beta^\psi\left(\frac{t}{n}\right) l_n(t) \sin\left(t + \frac{\beta\pi}{2}\right) dt \\ &\quad + O(1)\psi(n). \end{aligned} \tag{107}$$

We proceed to the finding of the estimate of $\mathscr{E}(C_{\beta,\infty}^\psi; U_{n,p}^\psi)$. Substituting (107) in (51), we have

$$\begin{aligned} &\mathscr{E}\left(C_{\beta,\infty}^\psi; U_{n,p}^\psi\right) \\ &\quad \leqslant \frac{\psi(n)}{\pi} \int_{(t_{k_0},t_{k_1})\cup(t_{k_2},t_{k_3})} \left| l_n(t) \sin\left(t + \frac{\beta\pi}{2}\right)\right| dt \\ &\quad + O(1)\psi(n). \end{aligned} \tag{108}$$

We shall show that

$$\int_{(t_{k_0},t_{k_1})\cup(t_{k_2},t_{k_3})} \left| l_n(t) \sin\left(t + \frac{\beta\pi}{2}\right)\right| dt = \frac{4}{\pi} A_{n,p}^\psi + O(1), \tag{109}$$

where

$$A_{n,p}^\psi = \begin{cases} \ln \dfrac{n}{p\mu(n)}, & \text{if } \mu(n) \leqslant \dfrac{n}{p}, \\ \ln \dfrac{p\mu(n)}{n}, & \text{if } \dfrac{n}{p} < \mu(n) \leqslant n, \\ \ln p, & \text{if } \mu(n) > n. \end{cases} \tag{110}$$

Bearing in mind the symmetry of the function $\sin(t + \beta\pi/2)$ relative to x_k of the segment $[t_k, t_{k+1}]$, $k \in \mathbb{Z}$, we obtain

$$\begin{aligned} &\int_{t_k}^{t_{k+1}} \left|\sin\left(t + \frac{\beta\pi}{2}\right)\right| dt \\ &\quad = 2\int_{t_k}^{x_k} \left|\sin\left(t + \frac{\beta\pi}{2}\right)\right| dt \\ &\quad = 2\int_{(k+(1/2))\pi}^{(k+1)\pi} |\sin t|\, dt = 2\int_0^{\pi/2} \cos t\, dt = 2. \end{aligned} \tag{111}$$

From this we get that

$$\begin{aligned} &\int_{(t_{k_0},t_{k_1})\cup(t_{k_2},t_{k_3})} \left| l_n(t) \sin\left(t + \frac{\beta\pi}{2}\right)\right| dt \\ &\quad = \sum_{k=k_0}^{k_1-1} \frac{1}{|x_k|} \int_{t_k}^{t_{k+1}} \left|\sin\left(t + \frac{\beta\pi}{2}\right)\right| dt \\ &\quad + \sum_{k=k_2}^{k_3-1} \frac{1}{x_k} \int_{t_k}^{t_{k+1}} \left|\sin\left(t + \frac{\beta\pi}{2}\right)\right| dt \\ &\quad = 2\left(\sum_{k=k_0}^{k_1-1} \frac{1}{|x_k|} + \sum_{k=k_2}^{k_3-1} \frac{1}{x_k}\right). \end{aligned} \tag{112}$$

Since

$$t_{k+1} - t_k = \pi, \quad k \in \mathbb{Z}, \tag{113}$$

this gives us the equality

$$\begin{aligned} \sum_{k=k_0}^{k_1-1} \frac{1}{|x_k|} &= \frac{1}{\pi} \sum_{k=k_0}^{k_1-1} \int_{t_k}^{t_{k+1}} \frac{1}{|x_k|} dt \\ &= -\frac{1}{\pi} \int_{t_{k_0}}^{t_{k_1}} \frac{dt}{t} + \frac{1}{\pi} \sum_{k=k_0}^{k_1-1} \int_{t_k}^{t_{k+1}} \left(-\frac{1}{x_k} + \frac{1}{t}\right) dt. \end{aligned} \tag{114}$$

Using (103), we have

$$\begin{aligned} &\left|\sum_{k=k_0}^{k_1-1} \int_{t_k}^{t_{k+1}} \left(-\frac{1}{x_k} + \frac{1}{t}\right) dt\right| \\ &\quad \leqslant \sum_{k=k_0}^{k_1-1} \int_{t_k}^{t_{k+1}} \left|\frac{1}{t} - \frac{1}{x_k}\right| dt < \pi \sum_{k=k_0}^{k_1-1} \int_{t_k}^{t_{k+1}} \frac{dt}{t^2} \\ &\quad = \pi \int_{t_{k_0}}^{t_{k_1}} \frac{dt}{t^2} < \frac{\pi}{|t_{k_1}|} \leqslant K \max\left\{\frac{1}{\mu(n)}, 1\right\} = O(1). \end{aligned} \tag{115}$$

Comparing (114) and (115), we find that

$$\sum_{k=k_0}^{k_1-1} \frac{1}{|x_k|} = -\frac{1}{\pi} \int_{t_{k_0}}^{t_{k_1}} \frac{dt}{t} + O(1). \tag{116}$$

Using the notations of (97) and (98), we represent (116) in the form

$$\begin{aligned}\sum_{k=k_0}^{k_1-1}\frac{1}{|x_k|} &= -\frac{1}{\pi}\int_{-a}^{-b}\frac{dt}{t}+\frac{1}{\pi}\int_{-a}^{-b}\frac{dt}{t}\\ &\quad -\frac{1}{\pi}\int_{t_{k_0}}^{t_{k_1}}\frac{dt}{t}+O(1)\\ &= -\frac{1}{\pi}\int_{-a}^{-b}\frac{dt}{t}+\int_{t_{k_1}}^{-b}\frac{dt}{t}+\int_{-a}^{t_{k_0}}\frac{dt}{t}+O(1).\end{aligned} \tag{117}$$

Since

$$\begin{aligned}-b-t_{k_1} &\leqslant \pi,\\ t_{k_0}+a &< \pi,\end{aligned} \tag{118}$$

we have

$$\begin{aligned}\int_{t_{k_1}}^{-b}\frac{dt}{t}+\int_{-a}^{t_{k_0}}\frac{dt}{t} &= O(1)\left(\frac{1}{b}+\frac{1}{|t_{k_0}|}\right)=O(1)\frac{1}{|t_{k_0}|}\\ &= O(1)\max\left\{\frac{1}{\mu(n)},1\right\}=O(1).\end{aligned} \tag{119}$$

Comparing (117) and (119), we obtain

$$\sum_{k=k_0}^{k_1-1}\frac{1}{|x_k|}=-\frac{1}{\pi}\int_{-a}^{-b}\frac{dt}{t}+O(1)=\frac{1}{\pi}\ln\frac{a}{b}+O(1). \tag{120}$$

Thus,

$$\sum_{k=k_0}^{k_1-1}\frac{1}{|x_k|}=\frac{1}{\pi}A_{n,p}^{\psi}+O(1), \tag{121}$$

where $A_{n,p}^{\psi}$ is defined by (110).

In an analogous way one can prove that

$$\sum_{k=k_2}^{k_3-1}\frac{1}{x_k}=\frac{1}{\pi}A_{n,p}^{\psi}+O(1). \tag{122}$$

Relation (109) follows from (112), (121), and (122).

Comparing (108) and (109), we find

$$\mathscr{E}\left(C_{\beta,\infty}^{\psi};U_{n,p}^{\psi}\right)\leqslant\psi(n)\left(\frac{4}{\pi^2}A_{n,p}^{\psi}+O(1)\right). \tag{123}$$

We shall show that the equality sign can be put in (123). Denote by $\varphi_0(t)$ the 2π-periodic function such that

$$\begin{aligned}&\int_{-\pi}^{\pi}\varphi_0(t)\,dt=0,\quad |\varphi_0(t)|\leqslant 1,\\ &\varphi_0(t)=\operatorname{sign}\left(l_n(nt)\sin\left(nt+\frac{\beta\pi}{2}\right)\right),\quad t\in[-1,1],\end{aligned} \tag{124}$$

where $l_n(t)$ is defined by (90). Obviously, such a function exists. According to Subsection 7.2 of [2, pages 136, 137] in $C_{\beta,\infty}^{\psi}$, $\psi\in F$, there exists a function $f_0(t)$ such that $\varphi_0(t)$ is its (ψ,β)-derivative. From (107) and (109), we have

$$\begin{aligned}&\left|\rho_{n,p}(f_0;0)\right|\\ &\quad=\frac{\psi(n)}{\pi}\left|\int_{(t_{k_0},t_{k_1})\cup(t_{k_2},t_{k_3})}\varphi_0\left(\frac{t}{n}\right)l_n(t)\sin\left(t+\frac{\beta\pi}{2}\right)dt\right|\\ &\qquad+O(1)\psi(n)\\ &\quad=\frac{\psi(n)}{\pi}\int_{(t_{k_0},t_{k_1})\cup(t_{k_2},t_{k_3})}\left|l_n(t)\sin\left(t+\frac{\beta\pi}{2}\right)\right|dt\\ &\qquad+O(1)\psi(n)\\ &\quad=\psi(n)\left(\frac{4}{\pi^2}A_{n,p}^{\psi}+O(1)\right).\end{aligned} \tag{125}$$

Since $\mathscr{E}(C_{\beta,\infty}^{\psi};U_{n,p}^{\psi})\geqslant|\rho_{n,p}(f_0,0)|$, it follows from (123) and (125) that if $\psi\in F$, $\beta\in\mathbb{R}$, $n,p\in\mathbb{N}$, $p\leqslant n$, and $n\to\infty$, then

$$\mathscr{E}\left(C_{\beta,\infty}^{\psi};U_{n,p}^{\psi}\right)=\psi(n)\left(\frac{4}{\pi^2}A_{n,p}^{\psi}+O(1)\right), \tag{126}$$

where

$$A_{n,p}^{\psi}=\begin{cases}\ln p, & \text{if } T(n)\leqslant 1,\\ \ln\dfrac{p}{T(n)}, & \text{if } 1\leqslant T(n)\leqslant p,\\ \ln\dfrac{T(n)}{p}, & \text{if } T(n)\geqslant p,\end{cases} \tag{127}$$

$T(n)=\eta(\psi;n)-n$, $\eta(\psi;n)=\psi^{-1}((1/2)\psi(n))$, and $O(1)$ is a quantity uniformly bounded in β, n, and p.

Theorem 1 is proved.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] A. I. Stepanets, "Classification of periodic functions and the rate of convergence of their fourier series," *Mathematics of the USSR-Izvestiya*, vol. 28, no. 1, pp. 99–132, 1987.

[2] A. I. Stepanets, *Classification and Approximation of Periodic Functions*, Kluwer Academic, Dodrecht, The Netherlands, 1995.

[3] A. I. Stepanets, *Methods of Approximation Theory*, VSP, Leiden, The Netherlands, 2005.

[4] A. Zygmund, "The approximation of functions by typical means of their Fourier series," *Duke Mathematical Journal*, vol. 12, pp. 695–704, 1945.

[5] L. Fejér, "Untersuchungen über Fouriersche Reihen," *Mathematische Annalen*, vol. 58, no. 1-2, pp. 51–69, 1903.

[6] Ch.-J. de la Vallée-Poussin, "Sur la meilleure approximation des fonctions d'une variable réelle par des expressions d'ordre donné," *Comptes Rendus de l'Académie des sciences*, vol. 166, no. 4, pp. 799–802, 1918.

[7] S. Aljančić, "Classe de saturation du procédé des moyennes typiques de Riesz," *Académie Serbe des Sciences, Publications de l'Institut Mathématique*, vol. 13, pp. 113–122, 1959.

[8] V. T. Gavrilyuk, "Saturation classes for linear methods of summation of Fourier series," *Ukrainian Mathematical Journal*, vol. 40, no. 5, pp. 486–492, 1988.

[9] A. S. Serdyuk and Ie. Yu. Ovsii, "Approximation of the classes $C^{\psi}_{\beta}H_{\omega}$ by generalized Zygmund sums," *Ukrainian Mathematical Journal*, vol. 61, no. 4, pp. 627–644, 2009.

[10] A. I. Stepanets, "Approximation properties of the Zygmund method," *Ukrainian Mathematical Journal*, vol. 51, no. 4, pp. 546–576, 1999.

[11] A. I. Stepanets, "Solution of the Kolmogorov-Nikol'skii problem for the Poisson integrals of continuous functions," *Sbornik Mathematics*, vol. 192, no. 1, pp. 113–138, 2001.

[12] O. I. Stepanets, "Approximation of convolution classes by Fourier sums: new results," *Ukrainian Mathematical Journal*, vol. 54, no. 5, pp. 713–740, 2002.

[13] A. I. Stepanets and V. I. Rukasov, "Approximation properties of the de la Vallée Poussin method," *Ukrainian Mathematical Journal*, vol. 54, no. 8, pp. 1324–1354, 2002.

[14] V. I. Rukasov, "Approximation of classes of analytic functions by de la Vallée-Poussin sums," *Ukrainian Mathematical Journal*, vol. 55, no. 6, pp. 974–986, 2003.

[15] A. S. Serdyuk, "Approximation of Poisson integrals by de la Vallée Poussin sums," *Ukrainian Mathematical Journal*, vol. 56, no. 1, pp. 122–134, 2004.

[16] A. S. Serdyuk, Ie. Yu. Ovsii, and A. P. Musienko, "Approximation of classes of analytic functions by de la Vallée Poussin sums in uniform metric," *Rendiconti di Matematica*, vol. 32, no. 1-2, pp. 1–15, 2012.

[17] A. Kolmogorov, "Zur Grössenordnung des Restgliedes Fourierscher Reihen differenzierbarer Funktionen," *Annals of Mathematics*, vol. 36, no. 2, pp. 521–526, 1935.

[18] W. T. Pinkewitch, "The order of the remainder in the Fourier series of Weyldi differentiable functions," *Izvestiya Akademii Nauk Matematicheskaya*, vol. 4, pp. 521–528, 1940 (Russian).

[19] S. A. Telyakovskii, "On norms of trigonometric polynomials and approximation of differentiable functions by linear means of their Fourier series. I," *American Mathematical Society Translations*, vol. 28, no. 2, pp. 283–322, 1963, Translation from *Trudy Matematicheskogo Instituta imeni V. A. Steklova*, vol. 62, pp. 61–97, 1961.