Research Article

On Hermite-Hadamard Type Inequalities for Riemann-Liouville Fractional Integrals via Two Kinds of Convexity

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We obtain some Hermite-Hadamard type inequalities for products of two $m$-convex functions via Riemann-Liouville integrals. The analogous results for $(\alpha, m)$-convex functions are also established.

1. Introduction

If $f : I \to \mathbb{R}$ is a convex function on the interval $I$, then for any $a, b \in I$ with $a \neq b$ we have the following double inequality:

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} f(t) \, dt \leq \frac{f(a) + f(b)}{2}. \quad (1)$$

This remarkable result is well known in the literature as the Hermite-Hadamard inequality.

Since then, some refinements of the Hermite-Hadamard inequality for convex functions have been extensively obtained by a number of authors (e.g., [1–7]).

In [8], Toader defined the concept of $m$-convexity as follows.

Definition 1 (see [8]). The function $f : [0, b] \to \mathbb{R}$ is said to be $m$-convex, where $m \in (0, 1]$, if for every $x, y \in [0, b]$ and $t \in (0, 1)$ one has

$$f \left( tx + m(1 - t) y \right) \leq tf(x) + m(1 - t)f(y). \quad (2)$$

In [3], Dragomir and Toader proved the following inequality of Hermite-Hadamard type for $m$-convex functions.

Theorem 2 (see [3]). Let $f : [0, \infty) \to \mathbb{R}$ be a $m$-convex function with $m \in (0, 1]$; if $0 \leq a < b < \infty$ and $f \in L_{1}[a, b]$, then one has the following inequality:

$$\frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \min \left\{ \frac{f(a) + mf(b/m)}{2}, \frac{f(b) + mf(a/m)}{2} \right\}. \quad (3)$$

The notion of $m$-convexity has been further generalized in [9] as it is stated in the following definition.

Definition 3 (see [9]). The function $f : [0, b] \to \mathbb{R}$ is said to be $(\alpha, m)$-convex, where $(\alpha, m) \in [0, 1]^{2}$, if for every $x, y \in [0, b]$ and $t \in [0, 1]$ one has

$$f \left( tx + m(1 - t) y \right) \leq t^{\alpha} f(x) + m(1 - t^{\alpha}) f(y). \quad (4)$$

In [10], Pachpatte established two new Hermite-Hadamard type inequalities for products of convex functions as follows.
Theorem 4 (see [10]). Let $f$ and $g$ be real-valued, nonnegative, and convex functions on $[a, b]$. Then

$$\frac{1}{b-a} \int_a^b f(x)g(x)\,dx \leq \frac{1}{3} M(a, b) + \frac{1}{6} N(a, b),$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Some Hermite-Hadamard type inequalities for products of two $m$-convex and $(a, m)$-convex functions are established in [11].

Theorem 5 (see [11]). Let $f, g : [0, \infty) \to [0, \infty)$ be functions such that $fg \in L_1[a,b]$, where $0 \leq a < b < \infty$. If $f$ is $m_1$-convex and $g$ is $m_2$-convex on $[a, b]$ for some fixed $m_1, m_2 \in (0, 1]$, then

$$\frac{1}{b-a} \int_a^b f(x)g(x)\,dx \leq \min \{M_1, M_2\},$$

where

$$M_1 = \frac{1}{3} \left[ f(a)g(a) + m_1m_2f\left(\frac{b}{m_1}\right)g\left(\frac{b}{m_2}\right) \right] + \frac{1}{6} \left[ m_2f(a)g\left(\frac{b}{m_2}\right) + m_1f\left(\frac{b}{m_1}\right)g(a) \right],$$

$$M_2 = \frac{1}{3} \left[ f(b)g(b) + m_1m_2f\left(\frac{a}{m_1}\right)g\left(\frac{a}{m_2}\right) \right] + \frac{1}{6} \left[ m_2f(b)g\left(\frac{a}{m_2}\right) + m_1f\left(\frac{a}{m_1}\right)g(b) \right].$$

Theorem 6 (see [11]). Let $f, g : [0, \infty) \to [0, \infty)$ be functions such that $fg \in L_1[a,b]$, where $0 \leq a < b < \infty$. If $f$ is $(\alpha_1, m_1)$ convex and $g$ is $(\alpha_2, m_2)$-convex on $[a, b]$ for some fixed $\alpha_1, m_1, \alpha_2, m_2 \in (0, 1]$, then

$$\frac{1}{b-a} \int_a^b f(x)g(x)\,dx \leq \min \{N_1, N_2\},$$

where

$$N_1 = \frac{f(a)g(a)}{\alpha_1 + \alpha_2 + 1} + m_2 \left[ \frac{1}{1 + \alpha_1} - \frac{1}{1 + \alpha_1 + \alpha_2} \right] f\left(\frac{b}{m_2}\right) g\left(\frac{b}{m_1}\right),$$

$$N_2 = \frac{f(b)g(b)}{\alpha_1 + \alpha_2 + 1} + m_1 \left[ \frac{1}{1 + \alpha_1} - \frac{1}{1 + \alpha_1 + \alpha_2} \right] f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right).$$

Some new integral inequalities involving two nonnegative and integrable functions that are related to the Hermite-Hadamard type are also proposed by many authors. In [12], Pachpatte established some Hermite-Hadamard type inequalities involving two log-convex functions. An analogous result for s-convex functions is obtained by Kirmaci et al. in [13]. In [14], Sarikaya et al. presented some integral inequalities for two $h$-convex functions.

It is remarkable that Sarikaya et al. [15] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 7 (see [15]). Let $f : [a, b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If $f$ is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[ f_a^\alpha f (b) + f_b^\alpha f (a) \right] \leq \frac{f(a) + f(b)}{2},$$

with $a > 0$.

We remark that the symbols $f_a^\alpha$ and $f_b^\alpha f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha \geq 0$ with $a \geq 0$ which are defined by

$$f_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt, \quad x > a,$$

$$f_b^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \, dt, \quad x < b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \, dt$. 


In this paper, we obtain some new Hermite-Hadamard type inequalities for products of two \( m \)-convex functions via Riemann-Liouville integrals. The analogous results for \((a, m)\)-convex functions are also given.

2. Inequalities for Products of Two Functions for Riemann-Liouville Fractional Integrals

**Theorem 8.** Let \( f, g : [0, \infty) \to [0, \infty) \), \( 0 \leq a < b \), be functions such that \( fg \in L_1[a, b] \). If \( f \) is \( m_1 \)-convex and \( g \) is \( m_2 \)-convex on \([a, b]\) with \( m_1, m_2 \in (0, 1] \), then one has

\[
\frac{\Gamma(\alpha)}{(b-a)^\alpha} f_a^\alpha f (b) g(b) \leq \frac{f(a) g(a)}{\alpha + 2} + m_2 \frac{f(a) g\left(\frac{b}{m_2}\right)}{(\alpha+1)(\alpha+2)} \int_0^1 t^\alpha (1-t)^{\alpha-1} dt
\]

(13)

Multiplying both sides of the above inequality by \( t^{\alpha-1} \) and then integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we obtain

\[
\int_0^1 t^{\alpha-1} f \left( ta + (1-t) b \right) g \left( ta + (1-t) b \right) dt
\]

\[
= \int_b^a \left( \frac{b-u}{b-a} \right)^{\alpha-1} f(u) g(u) \frac{du}{a-b}
\]

\[
= \frac{\Gamma(\alpha)}{(b-a)^\alpha} f_b^\alpha f (a) g(b)
\]

\[
\leq f(a) g(a) \int_0^1 t^{\alpha+1} dt + m_2 f(a) g\left(\frac{b}{m_2}\right) \int_0^1 t^\alpha (1-t)^{\alpha-1} dt
\]

\[
+ m_1 g(a) \int_0^1 t^{\alpha-1} dt + m_2 g\left(\frac{b}{m_2}\right) \int_0^1 t^\alpha (1-t)^2 dt.
\]

(16)

Analogously, we obtain

\[
\int_0^1 f \left( (1-t) a + tb \right) g \left( (1-t) a + tb \right) dt
\]

\[
\leq t^2 f(b) g(b) + m_2 f(b) g\left(\frac{a}{m_2}\right) t (1-t)
\]

\[
+ m_1 g(b) f\left(\frac{a}{m_1}\right) t (1-t)
\]

\[
+ m_1 m_2 f\left(\frac{a}{m_1}\right) g\left(\frac{a}{m_2}\right) (1-t)^2.
\]

(17)

From (14), we get

\[
f \left( ta + (1-t) b \right) \leq f(a) + m_1 f\left(\frac{b}{m_1}\right) t (1-t)
\]

\[
g \left( ta + (1-t) b \right) \leq g(a) + m_2 g\left(\frac{b}{m_2}\right) (1-t)^2.
\]

(14)
\[ \leq f(b)g(b) \int_{0}^{1} t^{\alpha+1} dt + m_2 f(b) \int_{0}^{1} t^\alpha (1-t) dt \\
+ m_1 g(b) \int_{0}^{1} t^\alpha (1-t) dt \\
+ m_1 m_2 f \left( \frac{a}{m_1} \right) g \left( \frac{a}{m_2} \right) \int_{0}^{1} t^{\alpha-1} (1-t)^2 dt \]
\[= \frac{f(b)g(b)}{\alpha + 2} + \frac{m_2}{(\alpha + 1)(\alpha + 2)} f(b) g \left( \frac{a}{m_2} \right) \]
\[+ \frac{m_1}{(\alpha + 1)(\alpha + 2)} g(b) f \left( \frac{a}{m_1} \right) \]
\[+ \frac{2m_1 m_2}{\alpha(\alpha + 1)(\alpha + 2)} f \left( \frac{a}{m_1} \right) g \left( \frac{a}{m_2} \right). \] (18)

which completes the proof. \(\square\)

**Corollary 9.** With assumptions in Theorem 8, if \(\alpha = 1\), one gets

\[ \frac{1}{b-a} \int_{a}^{b} f(x) g(x) \, dx \]
\[\leq \frac{1}{5} \left[ f(a) g(a) + m_1 m_2 f \left( \frac{b}{m_1} \right) g \left( \frac{b}{m_2} \right) \right] \]
\[+ \frac{1}{6} \left[ m_2 f(a) g \left( \frac{b}{m_2} \right) + m_1 f \left( \frac{b}{m_1} \right) g(a) \right], \] (19)

which is just the result in Theorem 5.

**Corollary 10.** With assumptions in Theorem 8, one gets

\[ \frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \left[ \int_{a}^{b} f(b) g(b) + \int_{a}^{b} f(a) g(a) \right] \]
\[\leq \frac{1}{2(\alpha + 2)} \left[ f(a) g(a) + f(b) g(b) \right] \]
\[+ \frac{m_2}{2(\alpha + 1)(\alpha + 2)} \left[ f(b) g \left( \frac{a}{m_2} \right) + f(a) g \left( \frac{b}{m_2} \right) \right]. \] (20)

**Corollary 11.** With assumptions in Theorem 8, if one chooses \(g: [a, b] \to \mathbb{R}\) as \(g(x) = 1\) and \(m_2 = 1\) for all \(x \in [a, b]\), one has

\[ \frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \left[ \int_{a}^{b} f(b) g(b) + \int_{a}^{b} f(a) g(a) \right] \]
\[\leq \frac{1}{\alpha_1 + \alpha_2 + \alpha} f(a) g(a) \]
\[+ \frac{\alpha_2}{(\alpha + \alpha_1)(\alpha + \alpha_1 + \alpha_2)} m_2 f(a) g \left( \frac{b}{m_2} \right) \]
\[+ \frac{\alpha_1}{(\alpha + \alpha_2)(\alpha + \alpha_1 + \alpha_2)} m_1 g(a) f \left( \frac{b}{m_1} \right) \]
\[+ \frac{1}{\alpha - 1} \left( - \frac{1}{\alpha + \alpha_1} + \frac{1}{\alpha + \alpha_1 + \alpha_2} \right) \]
\[\times m_1 m_2 f \left( \frac{b}{m_1} \right) g \left( \frac{b}{m_2} \right). \] (21)

**Theorem 12.** Let \(f, g: [0, \infty) \to [0, \infty)\), \(0 \leq a < b\), be functions such that \(fg \in L^1_{[a, b]}\). If \(f\) is \((\alpha_1, m_1)\)-convex and \(g\) is \((\alpha_2, m_2)\)-convex on \([a, b]\) with \((\alpha_1, m_1), (\alpha_2, m_2) \in (0, 1)^2\), respectively, then one has

\[ \frac{\Gamma(\alpha)}{(b-a)^{\alpha}} \int_{a}^{b} f(b) g(b) \]
\[\leq \frac{1}{\alpha_1 + \alpha_2 + \alpha} f(a) g(a) \]
\[+ \frac{\alpha_2}{(\alpha + \alpha_1)(\alpha + \alpha_1 + \alpha_2)} m_2 f(a) g \left( \frac{b}{m_2} \right) \]
\[+ \frac{\alpha_1}{(\alpha + \alpha_2)(\alpha + \alpha_1 + \alpha_2)} m_1 g(a) f \left( \frac{b}{m_1} \right) \]
\[+ \frac{1}{\alpha - 1} \left( - \frac{1}{\alpha + \alpha_1} + \frac{1}{\alpha + \alpha_1 + \alpha_2} \right) \]
\[\times m_1 m_2 f \left( \frac{b}{m_1} \right) g \left( \frac{b}{m_2} \right). \] (22)
Proof. Since \( f \) is \((\alpha_1, m_1)\)-convex and \( g \) is \((\alpha_2, m_2)\)-convex on \([a, b]\), then for \( t \in [0, 1] \) we get

\[
\begin{align*}
 f(ta + (1-t)b) & \leq \ell^{\alpha_1} f(a) + m_1 f(a) + m_2 f(a) g \left( \frac{b}{m_2} \right) (1-\ell^{\alpha_2}), \\
 g(ta + (1-t)b) & \leq \ell^{\alpha_2} g(a) + m_2 g(a) f \left( \frac{b}{m_1} \right) (1-\ell^{\alpha_1}).
\end{align*}
\]

From (23), we get

\[
\begin{align*}
 f(ta + (1-t)b) g(ta + (1-t)b) & \leq \ell^{\alpha_1+\alpha_2} f(a) g(a) + m_1 f(a) g \left( \frac{b}{m_2} \right) (1-\ell^{\alpha_2}) + m_1 g(a) f \left( \frac{b}{m_1} \right) (1-\ell^{\alpha_1}) + m_1 m_2 f \left( \frac{b}{m_1} \right) g \left( \frac{b}{m_2} \right) (1-\ell^{\alpha_2}) (1-\ell^{\alpha_1}).
\end{align*}
\]

Multiplying both sides of above inequality by \( \ell^{\alpha_1-1} \) and then integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we obtain

\[
\begin{align*}
\int_0^1 \ell^{\alpha_1-1} f(ta + (1-t)b) g(ta + (1-t)b) \, dt & = \int_0^1 \ell^{\alpha_1-1} f(u) g(u) \, du + \frac{\Gamma(\alpha_1)}{(b-a)^{\alpha_1}} f_b^a f(a) g(b) \\
& \leq f(a) g(a) \int_0^1 \ell^{\alpha_1+\alpha_2+\alpha_1-1} \, dt + m_2 f(a) g \left( \frac{b}{m_2} \right) \int_0^1 \ell^{\alpha_1+\alpha_2+\alpha_1-1} \, dt \\
& \quad + m_1 g(a) f \left( \frac{b}{m_1} \right) \int_0^1 \ell^{\alpha_1+\alpha_2+\alpha_1-1} \, dt + m_1 m_2 f \left( \frac{b}{m_1} \right) g \left( \frac{b}{m_2} \right) \int_0^1 \ell^{\alpha_1+\alpha_2+\alpha_1-1} \, dt.
\end{align*}
\]

Similarly, we have

\[
\begin{align*}
 f((1-t)a + tb) g((1-t)a + tb) & \leq \ell^{\alpha_1+\alpha_2} f(b) g(b) + m_2 f(b) g \left( \frac{a}{m_2} \right) (1-\ell^{\alpha_2}) + m_1 g(b) f \left( \frac{a}{m_1} \right) (1-\ell^{\alpha_1}) \\
& \quad + m_1 m_2 f \left( \frac{a}{m_1} \right) g \left( \frac{a}{m_2} \right) (1-\ell^{\alpha_1})(1-\ell^{\alpha_2}) + m_1 g(b) f \left( \frac{a}{m_1} \right) g \left( \frac{a}{m_2} \right) (1-\ell^{\alpha_1})(1-\ell^{\alpha_2}) \\
& \quad + m_1 m_2 f \left( \frac{a}{m_1} \right) g \left( \frac{a}{m_2} \right) (1-\ell^{\alpha_1})(1-\ell^{\alpha_2}) \\
& \quad + m_1 m_2 f \left( \frac{a}{m_1} \right) g \left( \frac{a}{m_2} \right) (1-\ell^{\alpha_1})(1-\ell^{\alpha_2})
\end{align*}
\]
\[
\Gamma (\alpha ) \left( \frac{1}{b-a} \left[ f_a b (b) + f_a a (a) \right] \right) \\
\leq \frac{1}{\alpha_1 + \alpha_2 + \alpha} \left[ f(a) g(a) + f(b) g(b) \right] \\
+ \frac{\alpha_2 m_2}{(\alpha + \alpha_1 + \alpha_2)} \\
+ \frac{\alpha_1 m_1}{(\alpha + \alpha_1 + \alpha_2)} \\
\times m_1 m_2 \left[ f \left( \frac{a}{m_1} \right) g \left( \frac{a}{m_2} \right) + f \left( \frac{b}{m_1} \right) g \left( \frac{b}{m_2} \right) \right].
\]
\]

(27)

We get the desired result.

**Corollary 13.** With assumptions in Theorem 12, if \( \alpha = 1 \), then

\[
\frac{1}{b-a} \int_a^b f(x) g(x) \, dx \\
\leq \frac{f(a) g(a)}{\alpha_1 + \alpha_2 + 1} + m_2 \left[ \frac{1}{1 + \alpha_1} - \frac{1}{1 + \alpha_1 + \alpha_2} \right] \\
\times f(a) g \left( \frac{b}{m_1} \right) \\
+ m_1 \left[ \frac{1}{1 + \alpha_2} - \frac{1}{1 + \alpha_1 + \alpha_2} \right] f \left( \frac{b}{m_1} \right) g(a) \\
+ m_1 m_2 \left[ \frac{1}{1 + \alpha_1} - \frac{1}{1 + \alpha_1 + \alpha_2} + \frac{1}{1 + \alpha_1 + \alpha_2} \right] \\
\times f \left( \frac{b}{m_1} \right) g \left( \frac{b}{m_2} \right),
\]

(28)

which is just the result in Theorem 6.

**Corollary 14.** With assumptions in Theorem 12, one gets

\[
\frac{1}{(b-a)^\alpha} \left[ \int_a^b f(b) g(b) + f(a) g(a) \right] \\
\leq \frac{1}{\alpha_1 + \alpha_2 + \alpha} \left[ f(a) g(a) + f(b) g(b) \right] \\
+ \frac{\alpha_2 m_2}{(\alpha + \alpha_1 + \alpha_2)} \\
+ \frac{\alpha_1 m_1}{(\alpha + \alpha_1 + \alpha_2)} \\
\times m_1 m_2 \left[ f \left( \frac{a}{m_1} \right) g \left( \frac{a}{m_2} \right) + f \left( \frac{b}{m_1} \right) g \left( \frac{b}{m_2} \right) \right].
\]

(29)

**3. Conclusion**

In this paper, we obtain some new Hermite-Hadamard type inequalities for products of two m-convex functions via Riemann-Liouville integrals. The analogous results for (a, m)-convex functions are also established. An interesting topic is whether we can use the methods in this paper to establish the Hermite-Hadamard inequalities for products of two convex functions on the coordinates via Riemann-Liouville integrals.

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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