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Research Article

The Effect of Slow Invariant Manifold and Slow Flow Dynamics on the Energy Transfer and Dissipation of a Singular Damped System with an Essential Nonlinear Attachment

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We study the effect of slow flow dynamics and slow invariant manifolds on the energy transfer and dissipation of a dissipative system of two linear oscillators coupled with an essential nonlinear oscillator with a mass much smaller than the masses of the linear oscillators. We calculate the slow flow of the system, the slow invariant manifold, the total energy of the system, and the energy that is stored in the nonlinear oscillator for different sets of the parameters and show that the bifurcations of the SIM and the dynamics of the slow flow play an important role in the energy transfer from the linear to the nonlinear oscillator and the rate of dissipation of the total energy of the initial system.

1. Introduction

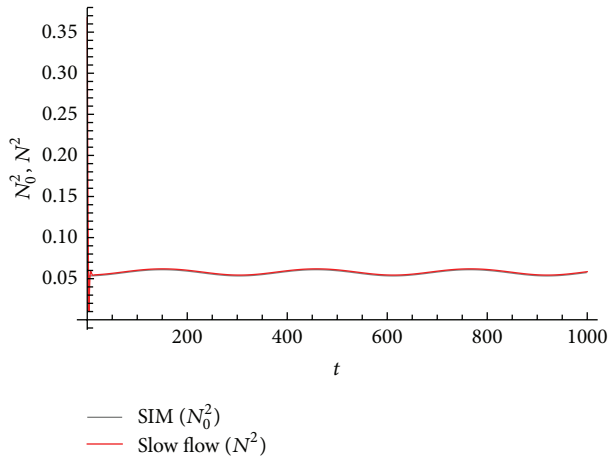
Mechanical structures where the nonlinear attachments have small masses in comparison to the structures to which they are attached and the systems are non conservative act as nonlinear energy sinks (NESs) and absorb, through irreversible transient transfer, energy from the linear parts due to resonance captures in vicinities of resonance manifolds of the underlying conservative systems for certain ranges of parameters and initial conditions [1–3].

Systems, where the masses of the nonlinear attachments are small in comparison to the linear oscillator, are singular and their dynamics are governed by different time scales. Such systems can be treated with the use of methods such as singularity analysis or multiple scale analysis [4, 5]. As it has been shown in previous works the slow flow of the system and the slow invariant manifolds (SIMs) obtained in the singular limit play a very important role in the dynamics under consideration [3, 6–8].

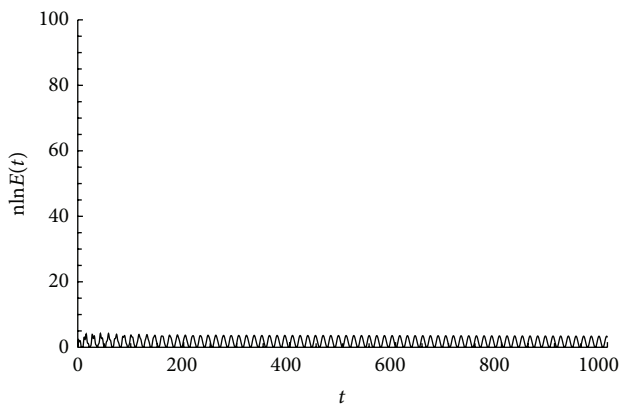
Furthermore, many works [1, 3, 9] have shown that energy transfer from a linear oscillator to its coupled nonlinear oscillator as well as the rate of dissipation of the energy is directly connected to the bifurcations of the SIM and the dynamics of the slow flow of the system.

In this work we study a dissipative system of two linear oscillators coupled with an essential nonlinear oscillator. In previous works [6, 7] we studied the SIM of the system and made a classification of its structure. Specifically, the structure of the SIM may be classified in three cases. First case when the SIM has always one stable branch. Second case when the SIM has always three branches, two of them stable and the one that lies between them unstable. In the above cases the SIM has no bifurcations. The third case is when the SIM Bifurcates. We also showed that the slow flow of the system has rich dynamics. It may oscillate, following the stable branch of the SIM, make relaxation oscillations, or have chaotic orbits.

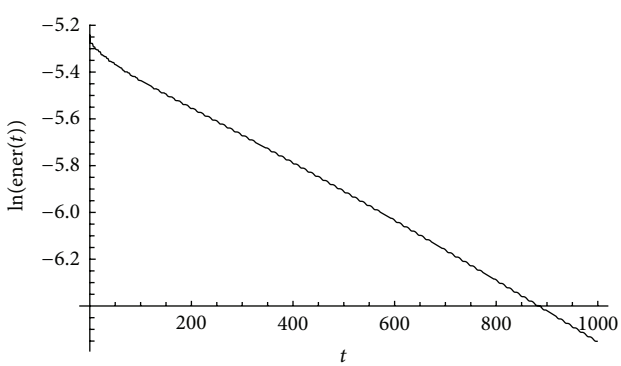
In this work we study the effect of the slow invariant manifold and the dynamics of the slow flow of the system on the energy transfer from the linear to the nonlinear oscillator and the dissipation of the total energy of the system. From our numerical results we confirm that the SIM, its bifurcations, and the dynamics of the slow flow play an essential role in the energy transfer and dissipation of the system.



(a) Gray line: SIM (N_0^2), red line: slow flow (N^2)

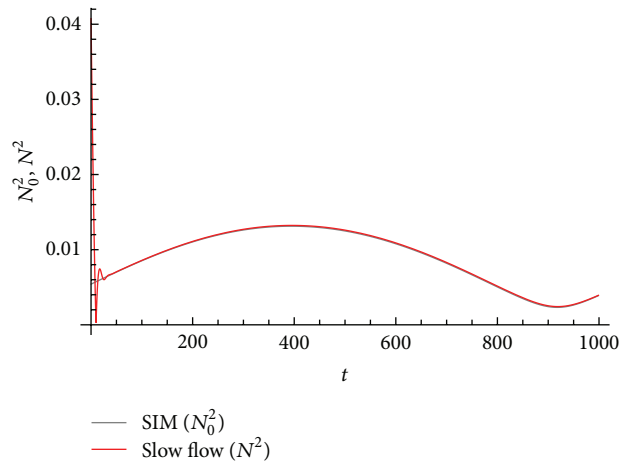


(b) Instantaneous energy (%) stored in the nonlinear oscillator

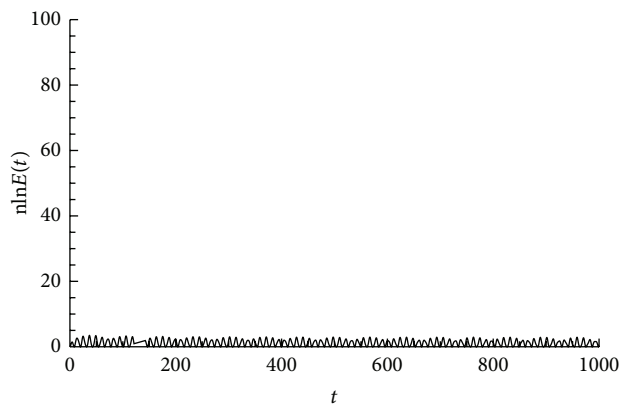


(c) Total energy

Figure 1: One stable branch of the SIM: $a = 0.1$, $d = 1.0$, $\widehat{\lambda} = 0.8$, $x_0 = 0.09$, and $x_1 = 0.05$.



(a) Gray line: SIM (N_0^2), red line: slow flow (N^2)



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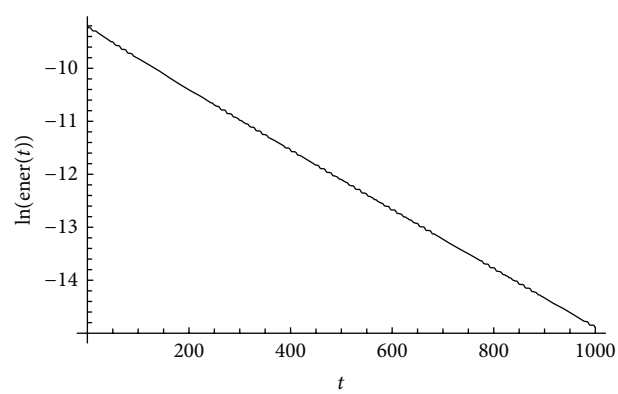


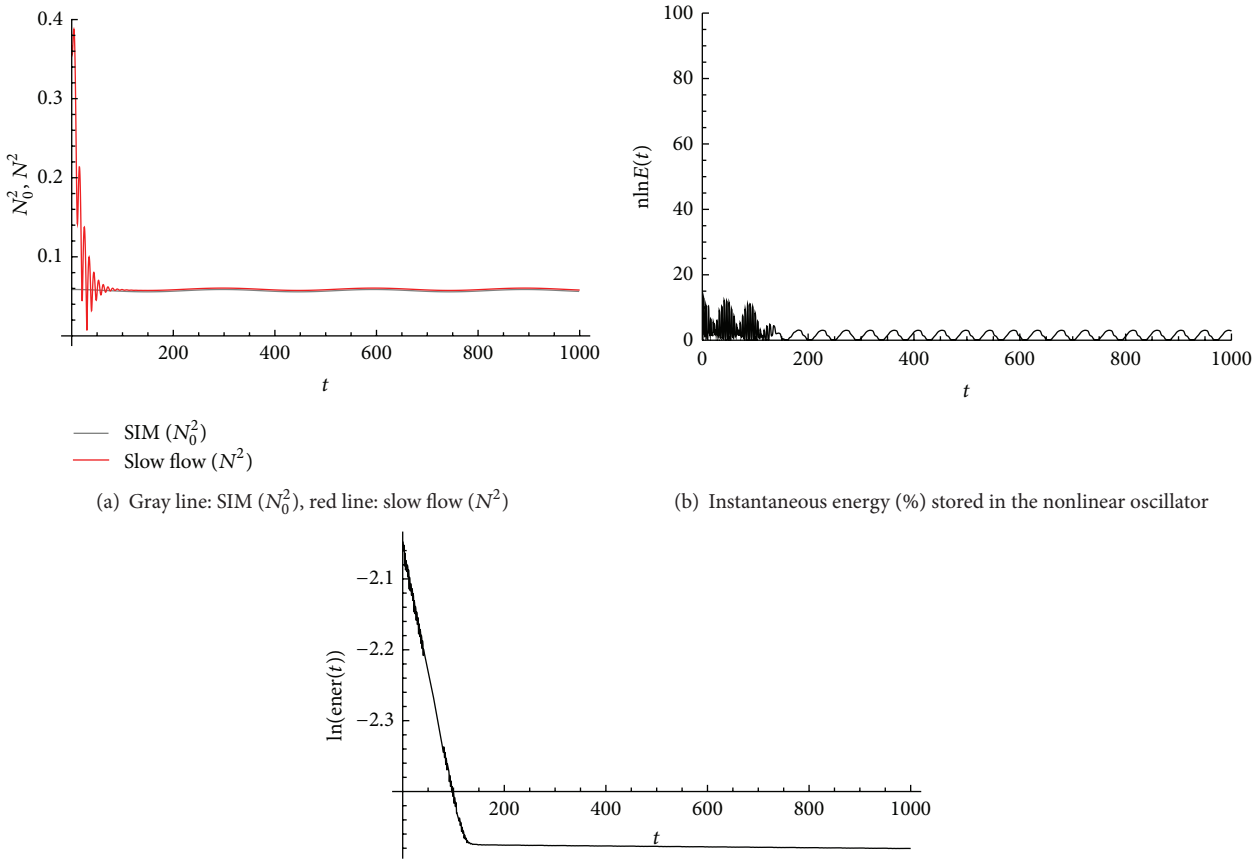
(c) Total energy

Figure 2: One stable branch of the SIM: $a = 0.2$, $d = 0.2$, $\widehat{\lambda} = 0.5$, $x_0 = 0.01$, and $x_1 = 0.01$.

The paper is organized as follows. In Section 2 we derive the slow flow and the SIM of the system. In Section 3 we calculate the energy of the system and the instantaneous energy that is stored in the nonlinear oscillator. In Section 4 we present numerical simulations for each case of the SIM and the behavior of the system's energy transfer and dissipation. Finally we conclude the paper in Section 5.

2. The Slow Flow and the SIM of the System

The initial system considered in this work is composed of two coupled linear oscillators and a nonlinear oscillator interacting through an essential nonlinearity with one of the linear oscillators and a small mass in comparison to the



(a) Gray line: SIM (N_0^2), red line: slow flow (N^2)

(b) Instantaneous energy (%) stored in the nonlinear oscillator

(c) Total energy

Figure 3: One stable branch of the SIM: $a = 0.01$, $d = 1.0$, $\widehat{\lambda} = 0.1$, $x_0 = 0.9$, and $x_1 = 0.5$.

masses of the linear oscillators. The equations of motion are given by

$$\begin{aligned} \epsilon\ddot{y} + \epsilon\lambda\left(\dot{y} - \dot{x}_0\right) + C\left(y - x_0\right)^3 &= 0, \\ \ddot{x}_0 + d\left(x_0 - x_1\right) &= \epsilon\lambda\left(\dot{y} - \dot{x}_0\right) + C\left(y - x_0\right)^3, \\ \ddot{x}_1 + ax_1 + d\left(x_1 - x_0\right) &= 0, \end{aligned} \tag{1}$$

where y, x_0, x_1 are the displacements, λ is the damping parameter, C is the nonlinear coefficient, a and d are the stiffness coefficients of the linear oscillators, and $\epsilon \ll 1$ the mass of the nonlinear attachment.

After applying the linear singular transformation $v = \epsilon^{-1/2}x_0 + \epsilon^{1/2}y$, $w = \epsilon^{-1/2}x_0 - \epsilon^{-1/2}y$ the system assumes the form:

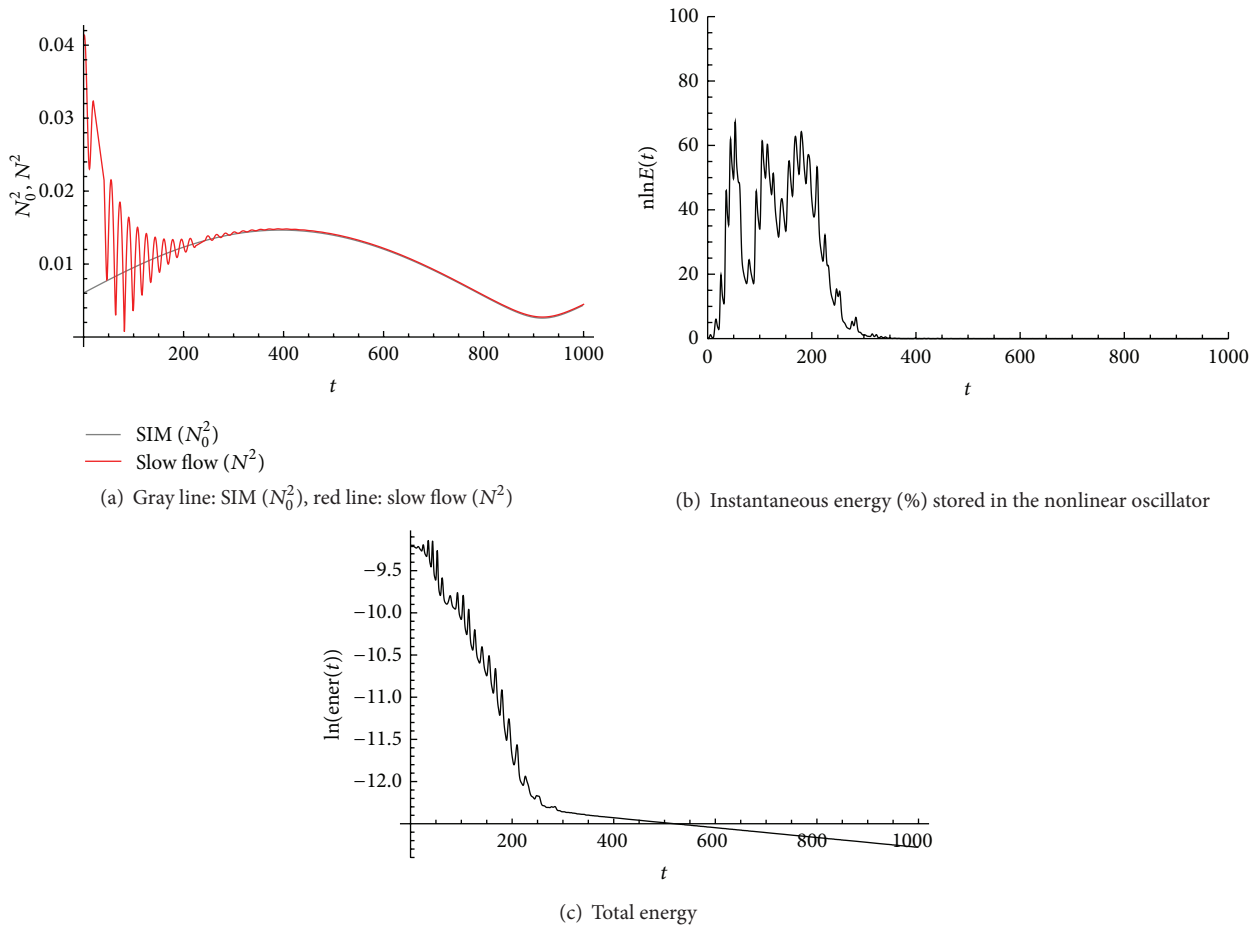
$$\begin{aligned} \ddot{w} + \lambda\left(1+\epsilon\right)\dot{w} + C\left(1+\epsilon\right)w^3 &= -d\frac{v + \epsilon w}{1+\epsilon} + dx_1, \\ \ddot{v} + \frac{d}{1+\epsilon}v - dx_1 &= -\frac{\epsilon d w}{1+\epsilon}, \\ \ddot{x}_1 + \left(a+d\right)x_1 - \frac{d}{1+\epsilon}v &= \frac{\epsilon d w}{1+\epsilon}. \end{aligned} \tag{2}$$

By just expanding the solution of v and x_1 with respect to ϵ and substituting them in the second and third equations of (2) we see that at zeroth order we have only the homogeneous part of the linear equations with eigenfrequencies

$$\begin{aligned} \omega_1^2 &= \Big(\left(1+\epsilon\right)a + \left(2+\epsilon\right)d \\ &\quad - \sqrt{-4ad\left(1+\epsilon\right) + \left(\left(1+\epsilon\right)a + \left(2+\epsilon\right)d\right)^2}\Big) \\ &\quad \times \left(2\left(1+\epsilon\right)\right)^{-1}, \\ \omega_2^2 &= \Big(\left(1+\epsilon\right)a + \left(2+\epsilon\right)d \\ &\quad + \sqrt{-4ad\left(1+\epsilon\right) + \left(\left(1+\epsilon\right)a + \left(2+\epsilon\right)d\right)^2}\Big) \\ &\quad \times \left(2\left(1+\epsilon\right)\right)^{-1}, \end{aligned} \tag{3}$$

and corresponding eigenvectors $(K_1, 1)$ and $(K_2, 1)$, with

$$\begin{aligned} K_1 &= \Big(\left(1+\epsilon\right)a + \epsilon d \\ &\quad - \sqrt{-4ad\left(1+\epsilon\right) + \left(\left(1+\epsilon\right)a + \left(2+\epsilon\right)d\right)^2}\Big) \\ &\quad \times \left(2d\right)^{-1}, \end{aligned}$$



(a) Gray line: SIM (N_0^2), red line: slow flow (N^2)

(b) Instantaneous energy (%) stored in the nonlinear oscillator

(c) Total energy

Figure 4: One stable branch of the SIM: $a = 0.2$, $d = 0.2$, $\widehat{\lambda} = 0.05$, $x_0 = 0.01$, and $x_1 = 0.01$.

$$K_2 = \Big((1+\epsilon)\, a + \epsilon d + \sqrt{-4ad\,(1+\epsilon) + ((1+\epsilon)\, a + (2+\epsilon)\, d)^2} \Big) \times (2d)^{-1}, \tag{4}$$

while the nonhomogeneous part appears in $O(\epsilon)$.

Notice that for $\epsilon = 0$ and therefore for small ϵ the eigenfrequencies are distinct and can not have a unity ratio. Their ordering is not important.

We introduce modal coordinates for the linear part of (2) through the coordinate transformations, $v = K_1 z_1 + K_2 z_2$, $x_1 = z_1 + z_2$, and assume zero initial displacements and nonzero initial velocities $\dot{z}_1(0) = \omega_1 z_{10}$, $\dot{z}_2(0) = \omega_2 z_{20}$. We replace the solutions of the nonhomogeneous linear oscillators in the first equation of (2) and after performing a time transformation $\widehat{t} \to \omega_{20} t$, where $\omega_2 = \omega_{20} + \epsilon \overline{B}$, we derive the following reduced system:

$$w'' + \widehat{\lambda} w' + \widehat{C} w^3 = \widehat{A} \sin\left(\frac{\omega_1}{\omega_{20}} \widehat{t}\right) + \widehat{B} \sin\left(1 + \frac{\epsilon \overline{B}}{\omega_{20}} \widehat{t}\right) + O(\epsilon), \tag{5}$$

where

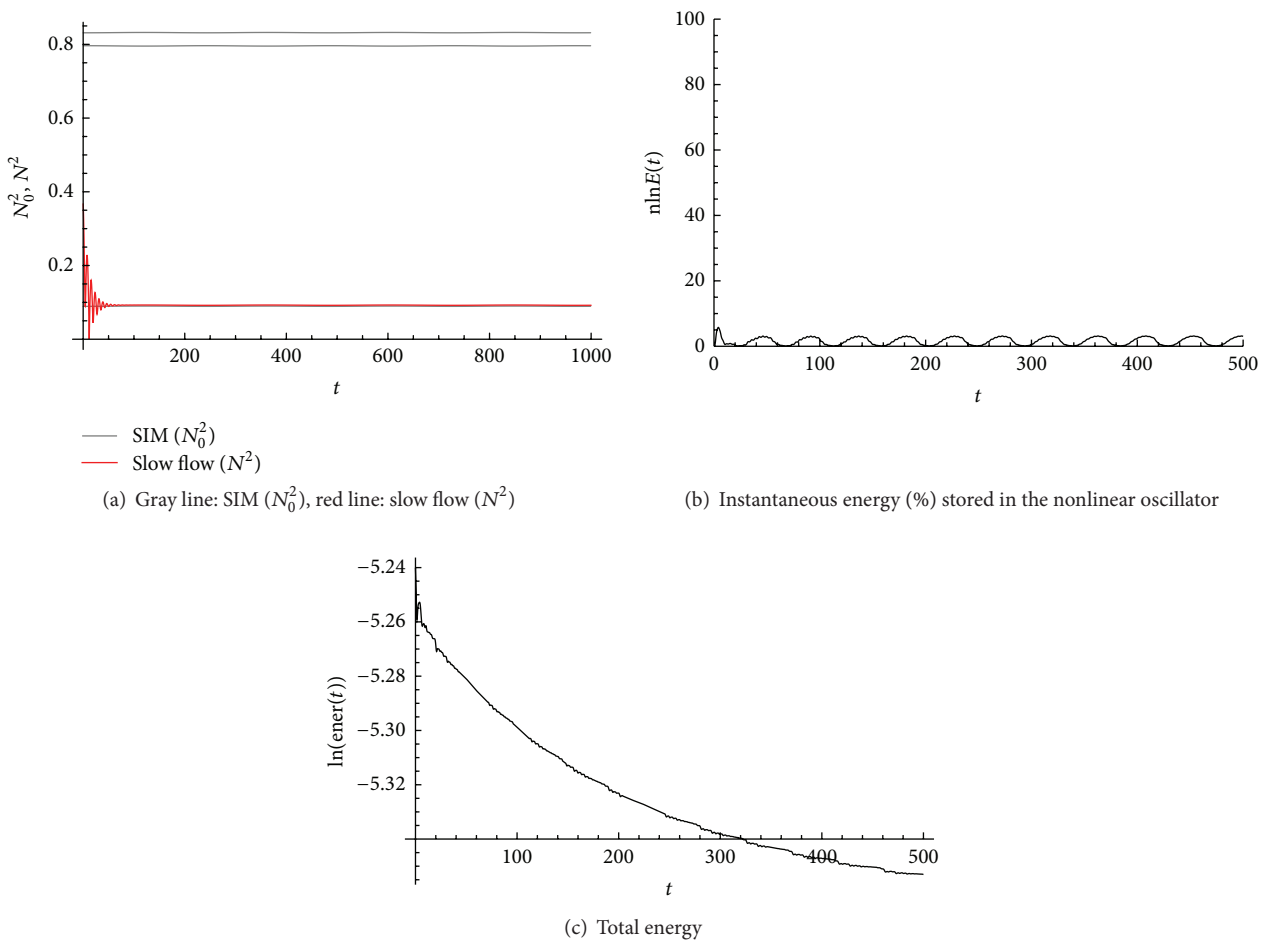
$$\widehat{C} = \frac{C}{\omega_{20}^2}, \quad \widehat{\lambda} = \frac{\lambda}{\omega_{20}}, \quad \widehat{A} = \frac{A}{\omega_{20}^2}, \quad \widehat{B} = \frac{B}{\omega_{20}^2}, \quad A = (d - dK_1)\, z_{10}, \quad B = (d - dK_2)\, z_{20}. \tag{6}$$

Considering the reduced system (5) we apply the complexification-averaging technique [3, 10] by introducing the complex coefficient $\Psi = w' + jw$. By making the important additional assumption that the nonlinear oscillator is in (1 : 1) resonance with the unperturbed eigenfrequency ω_{20}, that is, $\Psi = \phi e^{jt}$, we derive the averaged complex dynamical system

$$\phi' + \left(\frac{\widehat{\lambda}}{2} + \frac{j}{2}\right)\phi - \frac{3\widehat{C} j}{8} |\phi|^2 \phi + \frac{J}{2} + \frac{j\widetilde{A}}{2} + \frac{\widehat{B} j}{2} e^{j(\epsilon \overline{B}/\omega_{20})\widehat{t}} + O(\epsilon) = 0, \tag{7}$$

where

$$J = \frac{\widehat{A}\omega_{20}\omega_1}{\pi(\omega_1^2 - \omega_{20}^2)} \left(\cos\left(\frac{\omega_1}{\omega_{20}} 2\pi\right) - 1\right), \qquad \widetilde{A} = \frac{\widehat{A}\omega_{20}^2}{\pi(\omega_1^2 - \omega_{20}^2)} \sin\left(\frac{\omega_1}{\omega_{20}} 2\pi\right). \tag{8}$$



(a) Gray line: SIM (N_0^2), red line: slow flow (N^2)

(b) Instantaneous energy (%) stored in the nonlinear oscillator

(c) Total energy

Figure 5: Always three branches of the SIM: $a = 0.01$, $d = 1.5$, $\widehat{\lambda} = 0.1$, $x_0 = 0.09$, and $x_1 = 0.05$.

We mention at this point that system (7) is approximate since it takes into account only a single "fast" frequency. Clearly, since the system under consideration is strongly nonlinear, this is only an approximation since higher harmonics will exist in the dynamics. However, we conjecture that there exist regimes where the harmonic components with normalized frequency unity dominate.

We could have chosen another time transformation by considering $\widehat{t} \to \omega_{10} t$ and continue the analysis in the same way with the resonance being between ω_{10} and the nonlinear oscillator.

Since we are interested in the amplitude of the oscillations of the nonlinear attachment, we use the polar representation of the complex number $\phi = N(t)e^{j\eta(t)}$. After separating the real and imaginary parts, we obtain the following two equations for N and η, representing the slow flow of the system:

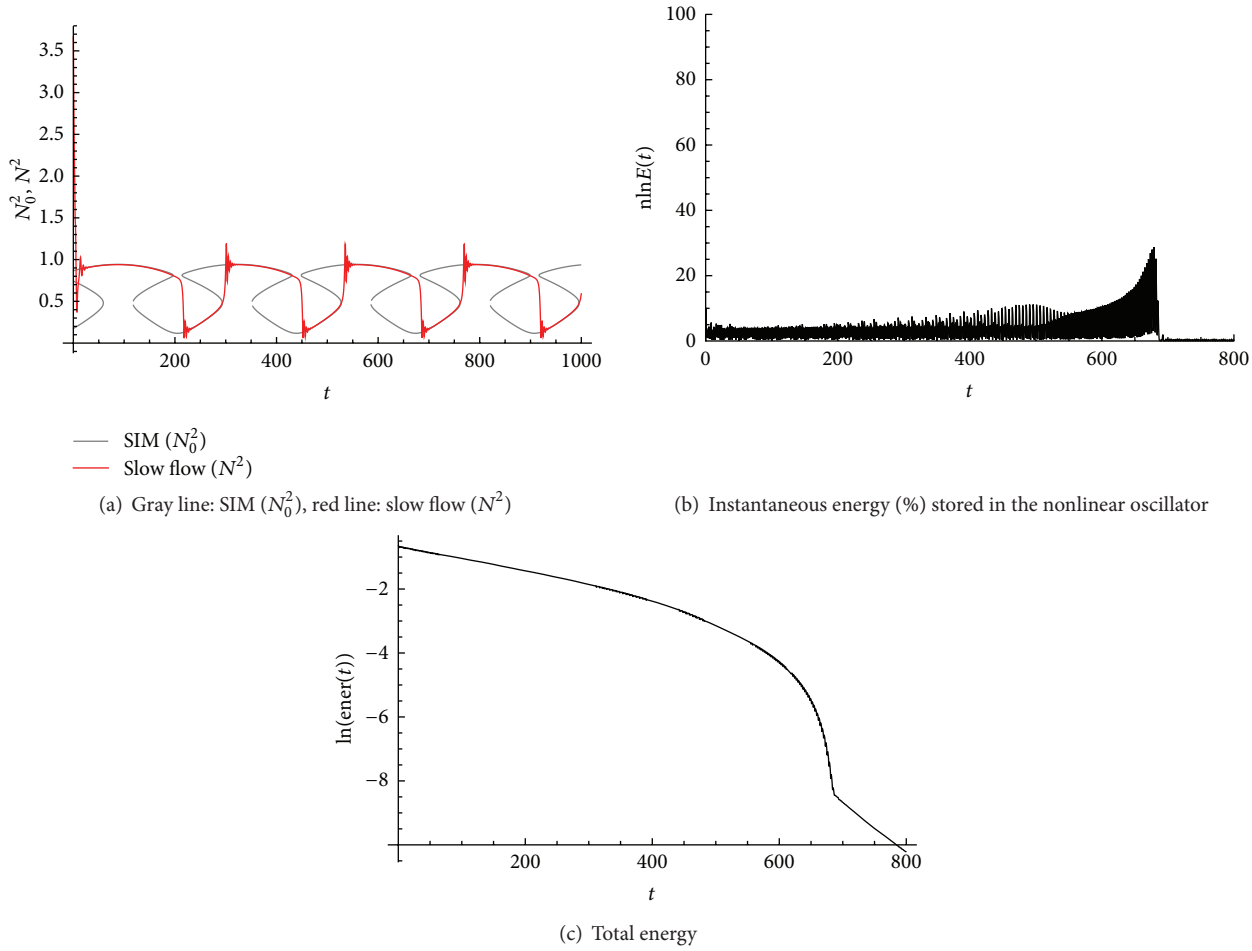
$$\begin{aligned} N' &= -\frac{\widehat{\lambda} N}{2} - \frac{J}{2}\cos(\eta) - \frac{\widetilde{A}}{2}\sin(\eta) \\ &\quad + \frac{\widehat{B}}{2}\sin\left(\frac{\epsilon \overline{B}}{\omega_{20}}\widehat{t} - \eta\right) + O(\epsilon), \\ N\eta' &= -\frac{N}{2} + \frac{3\widehat{C}N^3}{8} + \frac{J}{2}\sin(\eta) - \frac{\widetilde{A}}{2}\cos(\eta) \\ &\quad - \frac{\widehat{B}}{2}\cos\left(\frac{\epsilon \overline{B}}{\omega_{20}}\widehat{t} - \eta\right) + O(\epsilon). \end{aligned} \tag{9}$$

We use multiple scales analysis

$$\begin{aligned} N(t) &= N(t_0, t_1, \ldots) \\ &= N_0(t_0, t_1, \ldots) + \epsilon N_1(t_0, t_1, \ldots) + O(\epsilon), \\ \eta(t) &= \eta(t_0, t_1, \ldots) \\ &= \eta_0(t_0, t_1, \ldots) + \epsilon \eta_1(t_0, t_1, \ldots) + O(\epsilon), \end{aligned} \tag{10}$$

where $t_0 = \widehat{t}$ is the fast time and $t_1 = \epsilon\widehat{t}$ is the slow time. By keeping $O(1)$ terms in (9) for N and η we derive the equations

$$\begin{aligned} &\frac{\partial N_0}{\partial t_0} + \frac{\widehat{\lambda} N_0}{2} + \frac{J}{2}\cos(\eta_0) \\ &\quad + \frac{\widetilde{A}}{2}\sin(\eta_0) - \frac{\widehat{B}}{2}\sin\left(\frac{\epsilon \overline{B}\widehat{t}}{\omega_{20}} - \eta_0\right) = 0, \end{aligned}$$



(a) Gray line: SIM (N_0^2), red line: slow flow (N^2)

(b) Instantaneous energy (%) stored in the nonlinear oscillator

(c) Total energy

FIGURE 6: Relaxation oscillations of the slow flow: $a = 4$, $d = 4$, $\widehat{\lambda} = 0.15$, $x_0 = 0.9$, and $x_1 = 0.5$.

$$N_0 \frac{\partial \eta_0}{\partial t_0} + \frac{N_0}{2} - \frac{3\widehat{C}N_0^3}{8} - \frac{J}{2}\sin(\eta_0) + \frac{\widetilde{A}}{2}\cos(\eta_0) + \frac{\widehat{B}}{2}\cos\left(\frac{\epsilon\overline{B}\widehat{t}}{\omega_{20}} - \eta_0\right) = 0. \tag{11}$$

To study the steady state dynamics of the above system, in terms of the fast time scale t_0, we examine the limit of the dynamics as $t_0 \to \infty$ and impose the conditions $\partial\widehat{N}_0/\partial t_0 = 0$, $\partial n_0/\partial t_0 = 0$. This will provide us with the long-term behavior of the dynamics in the limit of large values of the fast time scale. Therefore $\widehat{t}$ is not a constant variable and its variation provide us the variation of the SIM. Then, from (11) we find

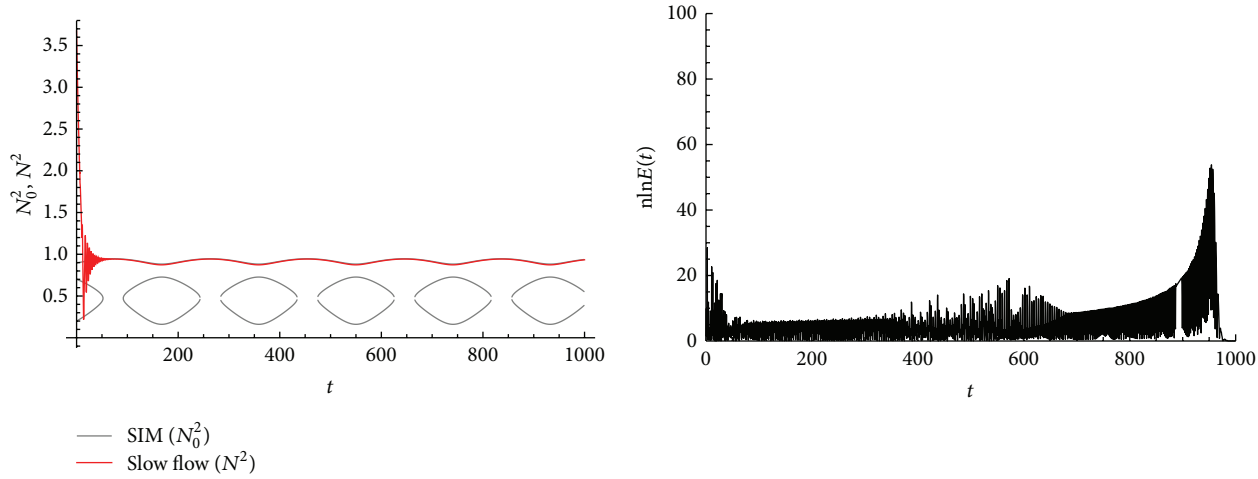
$$\begin{aligned} \frac{\widehat{\lambda}\widehat{N}_0}{2} &= -\frac{J}{2}\cos(\widehat{\eta}_0) - \frac{\widetilde{A}}{2}\sin(\widehat{\eta}_0) + \frac{\widehat{B}}{2}\sin\left(\frac{\epsilon\overline{B}\widehat{t}}{\omega_{20}} - \widehat{\eta}_0\right), \\ \frac{\widehat{N}_0}{2} - \frac{3\widehat{C}\widehat{N}_0^3}{8} &= \frac{J}{2}\sin(\widehat{\eta}_0) - \frac{\widetilde{A}}{2}\cos(\widehat{\eta}_0) \\ &\quad - \frac{\widehat{B}}{2}\cos\left(\frac{\epsilon\overline{B}\widehat{t}}{\omega_{20}} - \widehat{\eta}_0\right). \end{aligned} \tag{12}$$

Manipulating expressions (12) we derive the steady state phase

$$\begin{aligned} \cos\widehat{\eta}_0 = &\left(\left(\widetilde{A} + \widehat{B}\cos\left(\frac{\epsilon\overline{B}}{\omega_{20}}\widehat{t}\right)\right)\left(\frac{3C\widehat{N}_0^3}{4} - \widehat{N}_0\right)\right. \\ &\left. + \widehat{\lambda}\widehat{N}_0\left(\widehat{B}\sin\left(\frac{\epsilon\overline{B}}{\omega_{20}}\widehat{t}\right) - J\right)\right) \\ &\times\left(J^2 + \widetilde{A}^2 + 2\widetilde{A}\widehat{B}\cos\left(\frac{\epsilon\overline{B}}{\omega_{20}}\widehat{t}\right)\right. \\ &\left. + \widehat{B}^2 - 2J\widehat{B}\sin\left(\frac{\epsilon\overline{B}}{\omega_{20}}\widehat{t}\right)\right)^{-1}, \end{aligned} \tag{13}$$

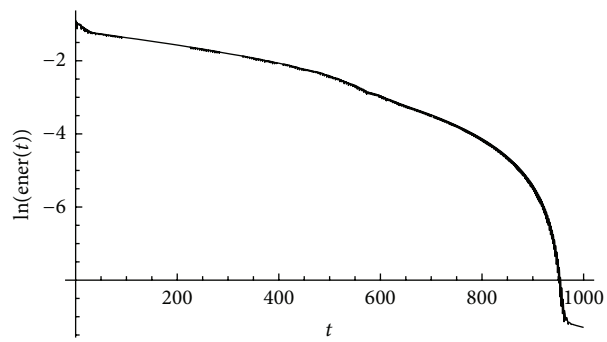
with the steady state amplitude given by

$$\begin{aligned} \widehat{N}_0^6 - \frac{8}{3\widehat{C}}\widehat{N}_0^4 + \frac{16\left(\widehat{\lambda}^2 + 1\right)}{9\widehat{C}^2}\widehat{N}_0^2 &= \frac{16}{9\widehat{C}^2}\left(\widetilde{A}^2 + 2\widetilde{A}\widehat{B}\cos\left(\frac{\epsilon\overline{B}}{\omega_{20}}\widehat{t}\right) + \widehat{B}^2\right. \\ &\quad \left. + J^2 - 2J\widehat{B}\sin\left(\frac{\epsilon\overline{B}}{\omega_{20}}\widehat{t}\right)\right). \end{aligned} \tag{14}$$



(a) Gray line: SIM (N_0^2), red line: slow flow (N^2)

(b) Instantaneous energy (%) stored in the nonlinear oscillator



(c) Total energy

Figure 7: The SIM bifurcates: $a = 6$, $d = 6$, $\widehat{\lambda} = 0.5$, $x_0 = 0.9$, and $x_1 = 0.04$.

Equations (13) and (14) represent the SIM of the dynamics of (5) [1, 3].

This is equivalent to singularity analysis of (9). Notice that if we consider $\epsilon\widehat{t} = t'$ then system (9) becomes

$$\begin{aligned} \epsilon N' &= f\left(N, n, t'\right) + O(\epsilon), \\ \epsilon N n' &= g\left(N, n, t'\right) + O(\epsilon) \end{aligned} \tag{15}$$

and in the singularity limit we get $f = g = 0$ which is exactly the SIM of the system, that is, (14).

There are many theorems for this singular manifold which allow us to determine the long-term behavior of (9) [4–6].

3. The Energy of the System

In order to study the effect of the SIM and the slow flow on the energy transfer and dissipation of our initial system we will study the rate of the total energy dissipation of the system. We use as a tool the graph of the logarithm of the total energy versus time for the different cases of the SIM.

Furthermore, in order to study the possible energy transfer from the linear to the nonlinear oscillators we calculate the instantaneous energy that is stored in the nonlinear oscillator.

The total energy of the system is calculated by the Hamiltonian of the initial system (1) without the dissipative terms and it is given by

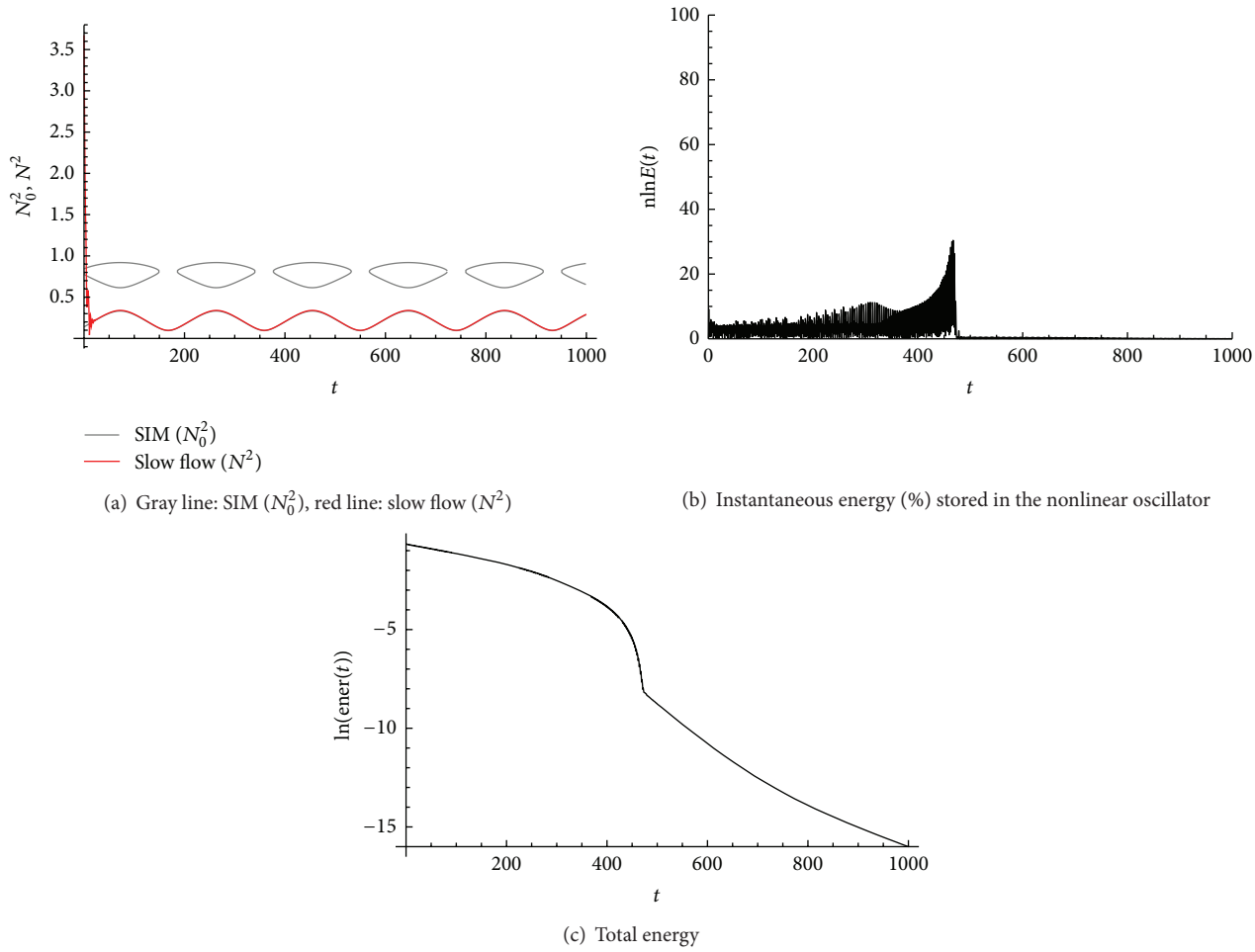
$$\begin{aligned} \text{ener}(t) = {} & \frac{1}{2}\left(\epsilon\dot{y}^2 + \dot{x}_0^2 + \dot{x}_1^2\right) + \frac{C}{4}(y - x_0)^4 \\ & + \frac{d}{2}(x_0 - x_1)^2 + \frac{a}{2}x_1^2. \end{aligned} \tag{16}$$

The energy that is stored in the nonlinear oscillator is given, as a fraction of the total energy, by

$$\begin{aligned} \text{nln}E(t) = {} & \left(\frac{\epsilon}{2}\dot{y}^2 + \frac{C}{4}(y - x_0)^4\right) \\ & \times \left(\frac{1}{2}\left(\epsilon\dot{y}^2 + \dot{x}_0^2 + \dot{x}_1^2\right) + \frac{C}{4}(y - x_0)^4\right. \\ & \left. + \frac{d}{2}(x_0 - x_1)^2 + \frac{a}{2}x_1^2\right)^{-1} \cdot 100\%. \end{aligned} \tag{17}$$

4. Numerical Simulations

The structure of the SIM depends on the parameters a, d, the initial conditions $\dot{x}_0(0)$, $\dot{x}_1(0)$ of the initial system (1), and the damping parameter $\widehat{\lambda}$. The SIM may have one branch that is

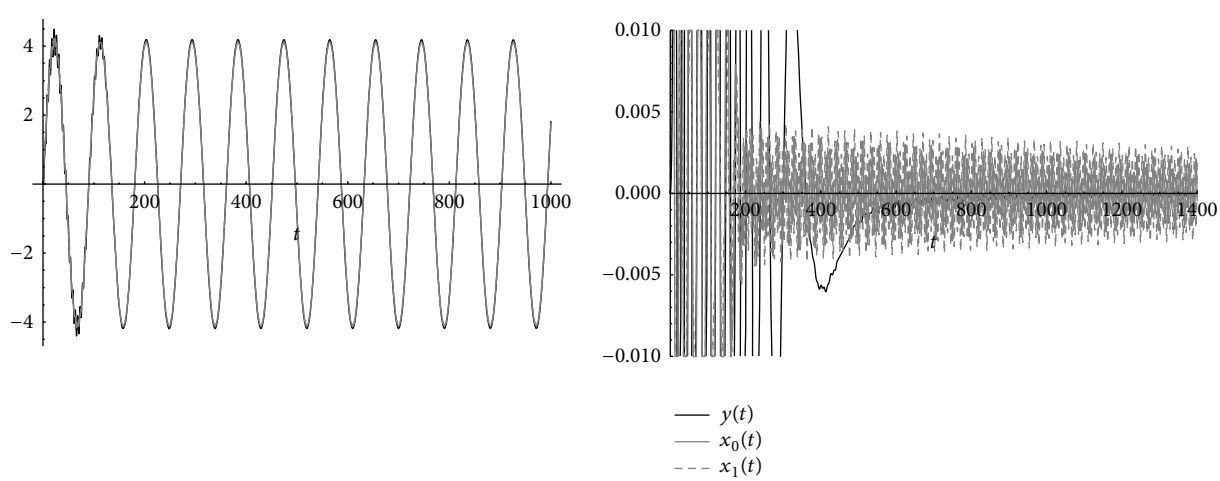


(a) Gray line: SIM (N_0^2), red line: slow flow (N^2)

(b) Instantaneous energy (%) stored in the nonlinear oscillator

(c) Total energy

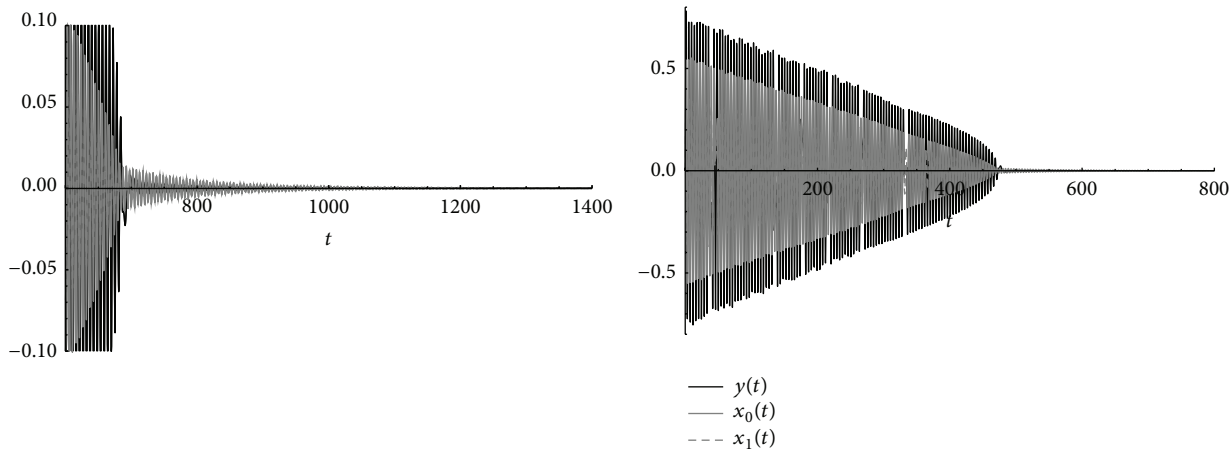
Figure 8: The SIM bifurcates: $a = 6$, $d = 6$, $\widehat{\lambda} = 0.15$, $x_0 = 0.9$, and $x_1 = 0.5$.



(a) $a = 0.01$, $d = 1$, $\widehat{\lambda} = 0.1$, $x_0 = 0.9$, and $x_1 = 0.5$

(b) $a = 0.2$, $d = 0.2$, $\widehat{\lambda} = 0.05$, $x_0 = 0.01$, and $x_1 = 0.01$

Figure 9: Oscillations of the initial system when the SIM has no bifurcations. Black line: $y(t)$, thick gray line: $x_0(t)$, and dashed gray line: $x_1(t)$.



(a) $a = 4$, $d = 4$, $\widehat{\lambda} = 0.15$, $x_0 = 0.9$, and $x_1 = 0.5$

(b) $a = 6$, $d = 6$, $\widehat{\lambda} = 0.15$, $x_0 = 0.9$, and $x_1 = 0.5$

Figure 10: Oscillations of the initial system when the SIM bifurcates. Black line: $y(t)$, thick gray line: $x_0(t)$, and dashed gray line: $x_1(t)$.

always stable, three branches, two of them stable and the one that lies between them unstable, or it may bifurcate.

In what follows we present numerical simulations for different sets of parameters and initial conditions.

For every set of the parameters a, d there are different structures of the SIM that are related to the different initial conditions $\dot{x}_0(0)$, $\dot{x}_1(0)$ and the structure of the SIM affects the dynamics of the initial system.

Since we are interested only in the amplitude of the oscillation, the SIM is computed by finding the real roots of (14) for every time $\widehat{t}$ and the figures depict the steady state N_0^2 with respect to the fast time. The slow flow is computed by solving system (9) numerically, with the use of Runge-Kutta method.

In order to compute the total energy of the system and the amount of energy that is stored in the nonlinear oscillator we solve system (1) numerically, with the use of Runge-Kutta method, calculate the total energy from (16) and the amount of the energy that is stored in the nonlinear oscillator from (17).

Our results are depicted in three types of graphs. The first one gives the SIM and the amplitude of the slow flow (N^2) of the system versus time, the second gives the amount of energy that is stored in the nonlinear oscillator versus time, and the third gives the logarithm of the total energy of the system versus time in order to follow the rate of dissipation of the total energy.

First, we study the cases where the SIM has no bifurcations and the slow flow has regular orbits, that is, when the SIM has always one branch or when the SIM has always three branches.

In Figures 1 and 2 we present some examples where the SIM has only one stable branch. In these cases the slow flow follows the SIM. The total energy of the system dissipates smoothly and, as we observe in the diagrams of the instantaneous energy that is stored in the nonlinear oscillator, there is no energy transfer from the linear to the nonlinear oscillator.

An important remark here is that, for the cases where the slow flow oscillates rapidly before it follows the SIM (Figures 3 and 4), we detect energy transfer to the nonlinear oscillator of the system in the time interval before the slow flow follows the SIM. Furthermore, in these cases, the total energy of the system dissipates faster for the time interval when the slow flow oscillates rapidly around the SIM than the period of time when the slow flow follows the SIM.

The same behavior, as in the case where the SIM has only one stable branch, is seen when the SIM has always three branches (Figure 5). The amount of energy that is transfered from the linear to the nonlinear oscillator depends on the initial conditions, that is, the initial energy that is given to the system.

The next case in our study is where the SIM bifurcates. As we observe in Figures 6, 7, and 8, the SIM has three real roots and then suddenly the two of them disappear (a bifurcation occurs), one real root remains for a period of time, and then two other real roots suddenly reappear (another bifurcation occurs). In this case, energy is transferred to the nonlinear oscillator, irrespectively whether the slow flow performs relaxation oscillations or follows other cases of bifurcations. The rate of the total energy dissipation depends on whether energy is transferred to the nonlinear oscillator or not. Specifically, the system disipates energy faster in the time intervals when there is energy transfer to the nonlinear oscillator.

We must indicate here that the behavior of the initial system, in the case where the slow flow oscillates rapidly until it follows the SIM (with no bifurcations), differs from its behavior when there are bifurcations.

In the first case energy transfer from the linear to the nonlinear oscillator occurs in the beginning of the oscillations, until the time when the system oscillates in a regular way (Figure 9). In the second case energy transfer from the linear to the nonlinear oscillator occurs at a time when the SIM bifurcates. After the energy transfer, the system dissipates its energy and the oscillations fade out (Figure 10).

5. Conclusions

From the above study we conclude that the bifurcations of the SIM and the dynamics of the slow flow play an important role in the energy transfer from the linear to the nonlinear oscillator and the dissipation of the total energy of the initial system.

When the SIM has no bifurcations, there is no energy transfer from the linear to the nonlinear oscillator and the energy dissipates smoothly. When the SIM has bifurcations, then energy transfer occurs. Furthermore, when there is energy transfer to the nonlinear oscillator, the rate of the dissipation of the total energy of the system becomes larger.

When the slow flow oscillates rapidly around the SIM, the energy is transfered to the nonlinear oscillator. The amount of energy that transfers is related to the initial energy given to the system. The difference between this case and the previous one is that when the SIM has bifurcations the system dissipates its energy and the oscillations fade out.

The damping parameter $\widehat{\lambda}$ determines whether the SIM bifurcates; that is, in order for the SIM to bifurcate, the damping parameter must satisfy the relation $\widehat{\lambda} < 1/\sqrt{3}$. Therefore, from the above analysis, we conclude that the damping parameter determines the ability of the system to transfer energy from the linear to the nonlinear oscillator and plays a role in the rate of the total energy dissipation of the system.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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